MathCastle

Pre-Algebra

Complete course booklet • 16 topics

Contents

1 Core Ideas in Plain Terms **7**

1.1 Negative Numbers and the Number Line 7

1.2 Fractions, Decimals, and Percents 7

1.3 Variables and Solving Equations 8

1.4 Going Deeper: Exponents and Number Sense 8

1.5 Problem-Solving Playbook 9

2 Introduction to Algebra **9**

2.1 Variables and What They Represent 9

2.2 Expressions vs. Equations 10

2.3 Writing Expressions from Word Phrases 11

2.4 Evaluating Expressions by Substitution 12

2.5 The Properties of Operations 12

2.6 Using the Distributive Property to Expand 13

2.7 Combining Like Terms 14

2.8 Simplifying: Distribute, Then Combine 14

2.9 Going Deeper: Toward Real Algebra 15

2.10 Why It Works 18

3 Equations & Inequalities **18**

3.1 What an Equation Is 18

3.2 Checking Whether a Value Is a Solution 19

3.3 Inverse Operations and Keeping Balance 19

3.4 Keep the Equation Balanced 20

3.5 One-Step Equations 20

3.6 Two-Step Equations 21

3.7 Combining Like Terms First 22

3.8 Writing Equations from Word Problems 23

3.9 What an Inequality Is . . . 24
3.10 Solving Inequalities . . . 25
3.11 Flipping the Inequality Sign . . . 26
3.12 Graphing Inequality Solutions . . . 26
3.13 Going Deeper: Advanced Equations & Inequalities . . . 27
3.14 Variables on Both Sides and Clearing Fractions . . . 27
3.15 Two Special Cases: No Solution and Infinitely Many . . . 29
3.16 A First Look at Systems of Two Equations . . . 29
3.17 Compound Inequalities and Interval Notation . . . 30
3.18 Why It Works . . . 30

4 The Coordinate Plane **31**
4.1 The Coordinate Plane . . . 31
4.2 The Four Quadrants . . . 32
4.3 Plotting Points and Naming Coordinates . . . 33
4.4 Tables of Values . . . 34
4.5 Graphing a Linear Relationship . . . 34
4.6 Distance on a Horizontal or Vertical Line . . . 35
4.7 Real-World Coordinates and Patterns . . . 36
4.8 Going Deeper: Advanced Coordinate Geometry . . . 37
4.9 Why It Works . . . 41

5 Geometry Basics **41**
5.1 Basic Vocabulary: Points, Lines, and Angles . . . 41
5.2 Triangles . . . 43
5.3 Perimeter of Polygons . . . 44
5.4 Area of Rectangles, Squares, Parallelograms, Triangles, and Trapezoids . . . 45
5.5 Circles . . . 47
5.6 The Pythagorean Theorem . . . 47
5.7 Surface Area and Volume of Prisms and Cylinders . . . 48
5.8 Real-World Geometry Problems . . . 49
5.9 Going Deeper: Advanced Geometry . . . 50
5.10 Why It Works . . . 53

6 Data, Statistics & Probability **54**
6.1 Mean, Median, and Mode (Measures of Center) . . . 54

6.2 Range (A Measure of Spread) . . . 56
6.3 Reading and Making Graphs . . . 56
6.4 Interpreting Data from Tables and Graphs . . . 58
6.5 Basic Probability . . . 58
6.6 Probability as a Fraction, Decimal, and Percent . . . 59
6.7 Compound Events and the Counting Principle . . . 60
6.8 Real-World Data and Probability Problems . . . 61
6.9 Going Deeper: Advanced Statistics & Probability . . . 61
6.10 Why It Works . . . 66

7 Whole Numbers & Operations **66**
7.1 Place Value & Reading Large Numbers . . . 66
7.2 Rounding & Estimation . . . 67
7.3 Adding & Subtracting Whole Numbers . . . 68
7.4 Multiplying Whole Numbers . . . 69
7.5 Dividing Whole Numbers (Long Division) . . . 69
7.6 Order of Operations (PEMDAS) . . . 70
7.7 Exponents & Perfect Squares . . . 71
7.8 Word Problems: Putting It Together . . . 72
7.9 Going Deeper: Advanced Whole-Number Ideas . . . 73
7.10 Why It Works . . . 75

8 Integers **76**
8.1 What Integers Are . . . 76
8.2 The Number Line . . . 76
8.3 Opposites . . . 76
8.4 Absolute Value . . . 77
8.5 Comparing and Ordering Integers . . . 78
8.6 Adding Integers . . . 79
8.7 Subtracting Integers . . . 79
8.8 Multiplying Integers . . . 80
8.9 Dividing Integers . . . 81
8.10 Order of Operations with Integers . . . 81
8.11 Word Problems with Integers . . . 82
8.12 Going Deeper: Advanced Integer Ideas . . . 83
8.13 Why It Works . . . 86

9 Factors & Multiples **86**

9.1 Factors of a Number . . . 86
9.2 Multiples of a Number . . . 87
9.3 Prime and Composite Numbers . . . 88
9.4 Divisibility Rules . . . 89
9.5 Prime Factorization . . . 90
9.6 Greatest Common Factor (GCF) . . . 90
9.7 Least Common Multiple (LCM) . . . 91
9.8 Word Problems: GCF or LCM? . . . 92
9.9 Going Deeper: Advanced Number Theory . . . 93
9.10 Why It Works . . . 96

10 Fractions **96**

10.1 Fraction Vocabulary & Equivalent Fractions . . . 96
10.2 Simplifying Fractions to Lowest Terms . . . 97
10.3 Comparing & Ordering Fractions . . . 99
10.4 Improper Fractions & Mixed Numbers . . . 100
10.5 Adding & Subtracting: LIKE Denominators . . . 101
10.6 Adding & Subtracting: UNLIKE Denominators (LCD) . . . 102
10.7 Adding & Subtracting Mixed Numbers . . . 103
10.8 Multiplying Fractions & Mixed Numbers . . . 104
10.9 Dividing Fractions & Mixed Numbers . . . 104
10.10 Word Problems with Fractions . . . 105
10.11 Going Deeper: Advanced Fraction Ideas . . . 106
10.12 Why It Works . . . 109

11 Decimals **109**

11.1 Decimal Place Value; Reading & Writing Decimals . . . 109
11.2 Comparing & Ordering Decimals . . . 110
11.3 Rounding Decimals . . . 111
11.4 Adding & Subtracting Decimals . . . 112
11.5 Multiplying Decimals . . . 113
11.6 Dividing Decimals . . . 114
11.7 Converting Between Fractions & Decimals . . . 115
11.8 Word Problems with Decimals (Money & Measurement) . . . 116
11.9 Going Deeper: Advanced Decimal Ideas . . . 117

11.10Why It Works 119

12 Ratios, Rates & Proportions **120**

12.1 What a Ratio Is 120
12.2 Equivalent Ratios & Ratio Tables 121
12.3 Rates & Unit Rates 122
12.4 Writing & Solving Proportions 123
12.5 Using Proportions to Solve Word Problems 124
12.6 Scale Drawings, Maps & Models 125
12.7 Similar Figures & Missing Sides 126
12.8 Going Deeper: Advanced Ratio & Proportion Ideas 127
12.9 Why It Works 130

13 Percents **131**

13.1 What Percent Means (Per Hundred) 131
13.2 Converting Among Fractions, Decimals, and Percents 132
13.3 Finding the Percent of a Number 133
13.4 Finding What Percent One Number Is of Another 134
13.5 Finding the Whole (Given a Part and a Percent) 135
13.6 Percent Increase and Percent Decrease 136
13.7 Real-World Percents: Discount, Tax, Tip, Commission 137
13.8 Simple Interest ($I = Prt$) 138
13.9 Going Deeper: Advanced Percent Ideas 139
13.10Why It Works 142

14 Rational Numbers **142**

14.1 What a Rational Number Is 142
14.2 Classifying Numbers 143
14.3 Rational Numbers on the Number Line 144
14.4 Absolute Value of Rational Numbers 144
14.5 Adding and Subtracting Fractions 145
14.6 Adding and Subtracting Decimals 146
14.7 Multiplying and Dividing Rational Numbers 147
14.8 Terminating vs. Repeating Decimals 148
14.9 Order of Operations with Rational Numbers 149
14.10Word Problems with Rational Numbers 150

14.11Going Deeper: Advanced Rational-Number Ideas 150
14.12Why It Works 153

15 Exponents & Roots **153**

15.1 Exponent Notation: Base & Exponent 153
15.2 Powers of Whole Numbers & Powers of 10 154
15.3 Exponent Rules: Product, Quotient & Power of a Power 155
15.4 The Zero Exponent 156
15.5 Scientific Notation 157
15.6 Square Roots & Perfect Squares 158
15.7 Estimating Square Roots of Non-Perfect Squares 158
15.8 Order of Operations with Exponents & Roots 159
15.9 Word Problems: Areas & Powers of Ten 160
15.10Going Deeper: Advanced Exponent & Root Ideas 160
15.11Why It Works 163

16 Course Review **163**

16.1 Whole Numbers & Operations 163
16.2 Integers 164
16.3 Factors & Multiples 165
16.4 Fractions 166
16.5 Decimals 166
16.6 Ratios, Rates & Proportions 167
16.7 Percents 168
16.8 Rational Numbers 168
16.9 Exponents & Roots 169
16.10Introduction to Algebra 170
16.11Equations & Inequalities 170
16.12The Coordinate Plane 171
16.13Geometry Basics 172
16.14Data, Statistics & Probability 173

1 Core Ideas in Plain Terms

The foundations, in everyday language

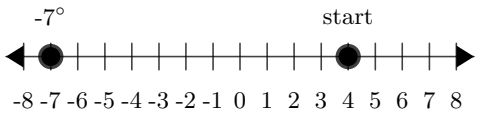
1.1 Negative Numbers and the Number Line

Negatives are just "below zero"

Numbers live on a line: zero in the middle, positives to the right, negatives to the left. Adding moves you right, subtracting moves you left. **Subtracting a negative** is like removing a debt — it moves you right, so $5-(-3)=5+3=8$. Two negatives multiplied make a positive. The rule at work is $a-b=a+(-b)$: subtracting is adding the opposite.

A temperature drop

It is 4° and drops 11 degrees. Start at 4, move 11 left, and you land on $-7°$.



1.2 Fractions, Decimals, and Percents

Three ways to write the same amount

A **fraction** like $\frac{3}{4}$, the **decimal** 0.75, and the **percent** 75% are the same number in different clothes. "Percent" means "per hundred," so 75% is $\frac{75}{100}$. To add fractions you need a common bottom number; to find a percent of something you multiply, using part $=\dfrac{p}{100}\times$ whole.



$3/4 = 0.75 = 75\%$

25% of 60

$25\% = \frac{25}{100} = \frac{1}{4}$, and a quarter of 60 is 15.

1.3 Variables and Solving Equations

A letter is a placeholder for a number

A **variable** like x just stands for a number you don't know yet. An equation is a balance scale: whatever you do to one side you must do to the other. To solve, peel away operations in reverse with **inverse operations** — undo addition with subtraction, undo multiplication with division. The licence to do this is the **properties of equality**: if $a = b$, then $a+c = b+c$ and $ac = bc$.

Solve 2x + 3 = 11

Subtract 3 from both sides: $2x = 8$. Divide both sides by 2: $x = 4$.

1.4 Going Deeper: Exponents and Number Sense

Exponent shortcuts

The **product rule for exponents**, $x^a \cdot x^b = x^{a+b}$: three 2s multiplied together, times four more 2s, is seven 2s — $2^3 \cdot 2^4 = 2^7 = 128$. Dividing subtracts exponents the same way, and anything (nonzero) to the power 0 is 1.

Divisibility tricks

A number is divisible by 3 exactly when its digit sum is, and by 9 when its digit sum is divisible by 9. Try 4,527: digits add to $4+5+2+7 = 18$, so it is divisible by 9 (indeed $4{,}527 = 9 \times 503$). Divisible by 5? Just look at the last digit.

Estimate before you compute

49×21 is about $50 \times 20 = 1000$. The exact answer is 1029 — close to the estimate, so no slip. If you had gotten 10,290, the estimate would catch the dropped decimal instantly.

Successive percents (multi-step)

A \$40 jacket is marked up 25%, then sold at 10% off. Step 1 — markup, using part $= \frac{p}{100} \times$ whole: $40 \times 1.25 = 50$. Step 2 — discount: $50 \times 0.90 = 45$. The final price is \$45 — *not* \$46: percents apply to the **current** price, not the original.

1.5 Problem-Solving Playbook

Translate, then undo in reverse

Turn the words into an equation phrase by phrase ("three more than twice a number" $\rightarrow 2n{+}3$). Then undo operations in **reverse order** with inverse operations — and estimate first so a slip cannot hide.

Worked: words to equation

"Three more than twice a number is 17." Step 1 — translate: $2n + 3 = 17$. Step 2 — undo the $+3$: $2n = 14$. Step 3 — undo the $\times 2$: $n = 7$. (Check: $2 \cdot 7 + 3 = 17$.)

2 Introduction to Algebra

Variables, expressions, properties, and simplifying

2.1 Variables and What They Represent

The Big Idea

A **variable** is a letter (like x, n, or y) that stands for a number we do not know yet, or a number that can *change*. Instead of writing "some number," we just write a letter. The same letter always stands for the same value within one problem.

Algebra is arithmetic with a placeholder. A variable stands for a number, an expression is a calculation written with it, and evaluating means putting a number in. Translating words into symbols is the skill: "three more than twice a number" is $2n + 3$, and the order of the words is not always the order of the symbols.

Reading a Variable

If a bag holds n marbles and you add 3 more, the bag now holds $n + 3$ marbles. Here n could

be 5 (giving 8) or 10 (giving 13)—the letter lets us describe every case at once.

Where it goes wrong

Translating "5 less than x" as $5 - x$. It is $x - 5$: start with x and take 5 away. "Less than" reverses the order.

Tip: A number written right next to a variable means *multiply*. So $4x$ means "4 times x," never "forty-something."

Try it: Quick check

A box holds n pencils. Write an expression for the number of pencils after you add 5 more.

Work it on paper before revealing the answer.

Answer. Answer: $n + 5$

2.2 Expressions vs. Equations

Two Different Things

An **expression** is a combination of numbers, variables, and operations with *no* equals sign, such as $3x + 7$. An **equation** says two expressions are equal and *does* have an equals sign, such as $3x + 7 = 19$. You *simplify or evaluate* an expression; you *solve* an equation.

Vocabulary of an Expression

Consider $5x + 2y - 8$.

- A **term** is a piece separated by $+$ or $-$ signs: the terms are $5x$, $2y$, and -8.
- A **coefficient** is the number multiplied by a variable: 5 is the coefficient of x, and 2 is the coefficient of y.
- A **constant** is a term that is just a number, with no variable: here the constant is -8.

Naming the Parts

In $7a - 3 + a$: the terms are $7a$, -3, and a. The coefficient of the first term is 7; the coefficient of a (the last term) is 1, because a means $1a$. The constant is -3.

Tip: A lone variable like a has a coefficient of 1, and $-x$ has a coefficient of -1, even though the 1 is invisible.

Try it: Quick check

Sara has x stickers and gives 2 away. Write an expression for how many she has left.

Work it on paper before revealing the answer.

Answer. Answer: $x - 2$

2.3 Writing Expressions from Word Phrases

Key Words

Certain words signal each operation:

- **Addition:** sum, plus, more than, increased by, total
- **Subtraction:** difference, minus, less than, decreased by, fewer
- **Multiplication:** product, times, of, twice, double, triple
- **Division:** quotient, divided by, per, split equally

Phrase to Expression

"The sum of a number and 6" $\rightarrow x + 6$.
"Three times a number" $\rightarrow 3x$.
"The quotient of a number and 4" $\rightarrow \frac{x}{4}$.
"5 more than twice a number" $\rightarrow 2x + 5$.

The "Less Than" Order Trap

"7 less than a number" means you start with the number and take 7 away: $x - 7$, *not* $7 - x$. The number you subtract comes *after* the minus sign. Same idea with "more than": "4 more than x" is $x + 4$.

Order trap: "less than" and "subtracted from" *reverse* the order. "3 less than n" is $n - 3$. Read carefully!

Try it: Quick check

In the term $6y$, what operation is happening between the 6 and the y?

Work it on paper before revealing the answer.

Answer. Answer: Multiplication

2.4 Evaluating Expressions by Substitution

Substitute, Then Compute

To **evaluate** an expression, replace each variable with its given value and follow the order of operations (PEMDAS). Wrap negative values in parentheses so you do not lose a sign.

Evaluating with a Positive Value

Evaluate $3x + 5$ when $x = 4$.

$$\begin{aligned} 3x + 5 &= 3(4) + 5 && \text{substitute} \\ &= 12 + 5 = 17 && \text{multiply, then add} \end{aligned}$$

Evaluating with an Integer

Evaluate $2a - b$ when $a = -3$ and $b = 5$.

$$\begin{aligned} 2a - b &= 2(-3) - 5 && \text{substitute with parentheses} \\ &= -6 - 5 = -11 && \text{multiply, then subtract} \end{aligned}$$

Tip: Always put a substituted number in parentheses, especially a negative one. Writing $2(-3)$ keeps you from misreading $2 - 3$.

Try it: Quick check

Is $4x + 1$ an expression or an equation?

Work it on paper before revealing the answer.

Answer. Answer: Expression

2.5 The Properties of Operations

Four Properties to Know

- **Commutative** (order can swap): $a + b = b + a$ and $ab = ba$.
- **Associative** (regroup with parentheses): $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **Distributive** (multiply across a sum): $a(b + c) = ab + ac$.
- **Identity** (leaves a number unchanged): $a + 0 = a$ and $a \times 1 = a$.

Spotting the Property

$4 + 9 = 9 + 4$ shows the **commutative** property of addition.
$(2 \times 5) \times 3 = 2 \times (5 \times 3)$ shows the **associative** property of multiplication.
$6(x + 2) = 6x + 12$ shows the **distributive** property.
$8 \times 1 = 8$ shows the **identity** property of multiplication.

Watch out: subtraction and division are *not* commutative or associative. $8 - 3 \neq 3 - 8$, and $12 \div 4 \neq 4 \div 12$.

Try it: Quick check

Is $2x = 10$ an expression or an equation?

Work it on paper before revealing the answer.

Answer. Answer: Equation

2.6 Using the Distributive Property to Expand

Multiply Every Term Inside

To **expand**, multiply the number outside the parentheses by *each* term inside, keeping track of signs:

$$a(b + c) = ab + ac \qquad a(b - c) = ab - ac.$$

Expanding

$5(x + 3) = 5 \cdot x + 5 \cdot 3 = 5x + 15$.
$4(2y - 7) = 4 \cdot 2y - 4 \cdot 7 = 8y - 28$.
$-3(a + 6) = -3 \cdot a + (-3) \cdot 6 = -3a - 18$. (The negative multiplies *both* terms.)

Tip: A negative in front distributes to *every* term. Forgetting the second sign is the most common mistake—$-2(x-4) = -2x+8$, not $-2x-8$.

Try it: Quick check

In $7x+3$, name the coefficient and name the constant.

Work it on paper before revealing the answer.

Answer. Answer: Coefficient 7; constant 3

2.7 Combining Like Terms

Only Like Terms Combine

Like terms have the exact same variable part (same letter, same power). You combine them by adding or subtracting their coefficients. Constants are like terms with each other. You may *not* combine terms with different variables.

Combining

$3x+5x = 8x$ (like terms: both have x).
$7a-2a+4 = 5a+4$ (combine the a-terms; the 4 stays).
$6x+3y$ *cannot* be combined—x and y are different variables.

Key idea: Only *like terms* combine. Add the coefficients but keep the variable the same: $2n+5n = 7n$, never $7n^2$.

Try it: Quick check

Write an expression for: the sum of a number and 8. (Use x.)

Work it on paper before revealing the answer.

Answer. Answer: $x+8$

2.8 Simplifying: Distribute, Then Combine

The Full Strategy

To fully **simplify** an expression: first use the distributive property to remove any parentheses, then combine all like terms. Work carefully with signs at each step.

Distribute Then Combine

Simplify $2(x+3)+4x$.

$$\begin{aligned} 2(x+3)+4x &= 2x+6+4x && \text{distribute} \\ &= 6x+6 && \text{combine } 2x+4x \end{aligned}$$

A Trickier One

Simplify $5(2a-1)-3(a+4)$.

$$\begin{aligned} 5(2a-1)-3(a+4) &= 10a-5-3a-12 && \text{distribute both} \\ &= 7a-17 && \text{combine like terms} \end{aligned}$$

Notice -3 multiplied *both* a and 4, giving $-3a-12$.

Tip: Distribute *first*, then combine. Line up like terms and watch every sign, especially after a minus sign in front of parentheses.

Try it: Quick check

Write an expression for: 5 times a number.

Work it on paper before revealing the answer.

Answer. Answer: $5x$

2.9 Going Deeper: Toward Real Algebra

From Expression to Function

A **function** is a rule that takes an input and produces exactly one output. Instead of naming the rule "the expression $3x+5$," we give it a name and write **function notation**: $f(x)=3x+5$. Read this as "f of x." The letter inside the parentheses is the input; the whole symbol $f(x)$ stands for the output. Note that $f(x)$ does *not* mean f times x—it is the single value the rule returns.

Evaluating a Function

Let $f(x)=3x+5$. To find $f(4)$, substitute 4 for every x and compute.

$$\begin{aligned} f(4) &= 3(4)+5 && \text{substitute } x=4 \\ &= 12+5=17 && \text{simplify} \end{aligned}$$

So $f(4) = 17$. Likewise $f(-2) = 3(-2) + 5 = -6 + 5 = -1$. Each input gives exactly one output.

Solving Literal Equations (Rearranging Formulas)

A **literal equation** has several letters, and we solve for one of them in terms of the others. The strategy is the same as solving a numerical equation: use inverse operations to isolate the wanted letter, treating every other letter as if it were a known number.

Isolating a Variable in a Formula

The perimeter of a rectangle is $P = 2\ell + 2w$. Solve for the width w.

$P = 2\ell + 2w$	start
$P - 2\ell = 2w$	subtract 2ℓ from both sides
$\dfrac{P - 2\ell}{2} = w$	divide both sides by 2

So $w = \dfrac{P - 2\ell}{2}$. Now any values of P and ℓ give w directly, no re-solving needed.

Inequalities and the Sign-Flip Rule

An **inequality** uses $<$, $>$, $\leq$, or $\geq$ instead of an equals sign, and describes a whole *range* of values. You solve it almost exactly like an equation—add, subtract, multiply, and divide on both sides—with one crucial exception: **whenever you multiply or divide both sides by a negative number, you must reverse the inequality symbol.**

Solving an Inequality with a Flip

Solve $-2x + 1 < 9$.

$-2x + 1 < 9$	start
$-2x < 8$	subtract 1 from both sides
$x > -4$	divide by -2: *flip* the sign

The solution is every number greater than -4. Check with $x = 0$: $-2(0) + 1 = 1 < 9$. ✓

Sign-flip trap: Adding or subtracting never changes the direction. Only multiplying or dividing by a *negative* flips $<$ to $>$ (or $\geq$ to $\leq$). Dividing by a positive leaves it alone.

Factoring: The Distributive Property in Reverse

The distributive property $a(b+c) = ab + ac$ can be run *backward*. **Factoring** means pulling out a common factor shared by every term: $ab + ac = a(b+c)$. To factor, find the **greatest common factor (GCF)** of all the terms, write it outside the parentheses, and put what remains inside.

Expanding and Factoring as Opposites

Expanding removes parentheses; factoring restores them.

$$6(x+2) = 6x + 12 \qquad \longleftrightarrow \qquad 6x + 12 = 6(x+2)$$

To factor $8y - 20$, the GCF of 8 and 20 is 4, so $8y - 20 = 4(2y - 5)$. Check by re-distributing: $4(2y - 5) = 8y - 20$. ✓

Modeling Multi-Step Word Problems

Real problems become algebra in three moves: (1) let a letter stand for the unknown quantity; (2) translate each sentence into an expression or equation; (3) solve, then check the answer against the words. When two quantities are related, define *both* in terms of the *same* variable.

Building and Solving a Model

A shirt costs \$8 more than a hat. Together they cost \$34. Find each price. Let h be the hat's price; then the shirt costs $h + 8$.

$$\begin{aligned} h + (h+8) &= 34 && \text{total cost} \\ 2h + 8 &= 34 && \text{combine like terms} \\ 2h &= 26 && \text{subtract 8} \\ h &= 13 && \text{divide by 2} \end{aligned}$$

The hat is \$13 and the shirt is $13 + 8 = \$21$. Check: $13 + 21 = 34$. ✓

Turning a Pattern into a Formula

Many number patterns grow by the same amount each step. If a sequence starts at a first value and adds a fixed amount d every term, the nth term has a **closed-form** rule: (first term) $+ (n-1) \cdot d$. A closed form lets you jump straight to any term without listing them all.

From Pattern to Closed Form

The pattern $5, 8, 11, 14, \ldots$ starts at 5 and adds 3 each step. Its rule is

$$a_n = 5 + (n-1) \cdot 3 = 3n + 2.$$

Check: $n = 1$ gives $3(1)+2 = 5$; $n = 4$ gives $3(4)+2 = 14$. ✓ The 20th term is $3(20)+2 = 62$—found instantly, with no counting.

Big picture: A formula, a function, and a pattern's closed form are all the *same idea*—a compact rule that turns any input into its output. Learning to build these rules is what algebra is really about.

2.10 Why It Works

Expressions obey the same laws as numbers because a variable IS a number — just one we have not named. So $x + x = 2x$ for the same reason $7 + 7 = 2 \cdot 7$, and $2(x + 3) = 2x + 6$ by the distributive property. Every algebraic move must remain true when a specific number is substituted for the variable, which is why substituting a value is a reliable check on any simplification.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

3 Equations & Inequalities

Solving one-step and two-step equations and inequalities

3.1 What an Equation Is

The Big Idea

An **equation** is a math sentence that says two things are equal. It always has an equals sign, =, with an expression on each side, like $x + 5 = 12$. A letter such as x is a **variable**, an unknown number we want to find. To **solve** an equation means to find the value of the variable that makes the sentence true. That value is called the **solution**.

Solving an equation is peeling operations off the variable in reverse order, doing the same thing to both sides each time. Two-step equations undo the addition first and the multiplication second — the reverse of the order in which they were applied. Inequalities are solved the same way, with one twist: multiplying or dividing by a negative flips the sign.

3.2 Checking Whether a Value Is a Solution

You can check a possible solution by **substituting** it in for the variable and seeing if both sides come out equal. If they match, it is a solution. If they do not match, it is not.

Is $x = 4$ a solution of $x + 5 = 12$?

Substitute 4 for x and simplify the left side:

$$\begin{aligned} x + 5 &= 12 \\ 4 + 5 &= 12 \quad ? \\ 9 &\neq 12 \end{aligned}$$

The two sides are *not* equal, so $x = 4$ is **not** a solution. (The real solution is $x = 7$, since $7 + 5 = 12$.)

Where it goes wrong

Forgetting to flip the inequality when dividing by a negative: from $-2x < 6$ you get $x > -3$, not $x < -3$. Test a number to be sure.

Tip: Checking is never wasted work. After you solve any equation, substitute your answer back in. If both sides are equal, you know you are right.

Try it: Quick check

Is $x = 6$ a solution of $x + 4 = 10$? (Substitute and check.)

Work it on paper before revealing the answer.

Answer. Answer: Yes ($6 + 4 = 10$)

3.3 Inverse Operations and Keeping Balance

Undo with the Opposite

Every operation has an **inverse** (an opposite) that undoes it:

- Addition and subtraction undo each other.
- Multiplication and division undo each other.

To get the variable alone, use the inverse of whatever is happening to it.

3.4 Keep the Equation Balanced

Think of an equation like a balanced scale. The = sign is the center. **Whatever you do to one side, you must do to the other side**, or the scale tips and the equation is no longer true. Do the same operation to both sides every time.

Balancing in Action

To undo "+5" in $x + 5 = 12$, subtract 5 from *both* sides:

$$\begin{aligned} x + 5 &= 12 \\ x + 5 - 5 &= 12 - 5 \\ x &= 7 \end{aligned}$$

Subtracting 5 on both sides keeps the scale balanced and leaves x alone.

Golden rule: Do the same thing to both sides. Always. This one idea is the heart of solving every equation and inequality.

Try it: Quick check

Is $x = 3$ a solution of $2x = 8$?

Work it on paper before revealing the answer.

Answer. Answer: No $(2 \cdot 3 = 6 \neq 8)$

3.5 One-Step Equations

One Move to Solve

A **one-step equation** needs just a single inverse operation to get the variable alone. Look at what is being done to the variable, then do the opposite to both sides.

Addition and Subtraction Equations

Solve $x - 8 = 3$ (the 8 is subtracted, so add 8):

$$\begin{aligned} x - 8 &= 3 \\ x - 8 + 8 &= 3 + 8 \\ x &= 11 \end{aligned}$$

Solve $n + 6 = 15$ (the 6 is added, so subtract 6):

$$n + 6 = 15$$
$$n + 6 - 6 = 15 - 6$$
$$n = 9$$

Multiplication and Division Equations

Solve $4x = 20$ (the x is multiplied by 4, so divide by 4):

$$4x = 20$$
$$\frac{4x}{4} = \frac{20}{4}$$
$$x = 5$$

Solve $\frac{y}{3} = 6$ (the y is divided by 3, so multiply by 3):

$$\frac{y}{3} = 6$$
$$\frac{y}{3} \cdot 3 = 6 \cdot 3$$
$$y = 18$$

Tip: $4x$ means "4 times x." To undo multiplication, divide. Never subtract the 4; that will not free the variable.

Try it: Quick check

Is $n = 5$ a solution of $n - 2 = 3$?

Work it on paper before revealing the answer.

Answer. Answer: Yes ($5 - 2 = 3$)

3.6 Two-Step Equations

Undo in Reverse Order

A **two-step equation** has two operations, like $2x + 3 = 11$. Undo them in the *reverse* of the order of operations: first undo addition or subtraction, *then* undo multiplication or division. Peel the variable free from the outside in.

Solve $2x + 3 = 11$

First subtract 3 from both sides, then divide both sides by 2:

$$\begin{aligned} 2x + 3 &= 11 \\ 2x + 3 - 3 &= 11 - 3 \\ 2x &= 8 \\ \frac{2x}{2} &= \frac{8}{2} \\ x &= 4 \end{aligned}$$

Check: $2(4) + 3 = 8 + 3 = 11$. ✓

Solve $\frac{x}{5} - 2 = 4$

First add 2 to both sides, then multiply both sides by 5:

$$\begin{aligned} \frac{x}{5} - 2 &= 4 \\ \frac{x}{5} &= 6 \\ \frac{x}{5} \cdot 5 &= 6 \cdot 5 \\ x &= 30 \end{aligned}$$

Check: $\frac{30}{5} - 2 = 6 - 2 = 4$. ✓

Tip: Order of operations *builds* an expression (multiply, then add). To take it apart, run backwards: undo the adding first, undo the multiplying last.

Try it: Quick check

$x + 7 = 12$

Work it on paper before revealing the answer.

Answer. Answer: $x = 5$

3.7 Combining Like Terms First

Tidy Up, Then Solve

Sometimes a side has more than one term with the same variable. **Like terms** have the same variable, so you can combine them: $3x + 2x = 5x$. Simplify each side *first*, then solve as a

normal equation.

Solve $3x + 2x = 20$

Combine the like terms on the left, then divide:

$$\begin{aligned} 3x + 2x &= 20 \\ 5x &= 20 \\ \frac{5x}{5} &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

Solve $4x + 3 - x = 12$

Combine $4x$ and $-x$ into $3x$, then solve the two-step equation:

$$\begin{aligned} 4x + 3 - x &= 12 \\ 3x + 3 &= 12 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Tip: x by itself means $1x$, so $4x - x = 3x$ (not 4). Only combine terms that share the same variable.

Try it: Quick check

$x - 4 = 9$

Work it on paper before revealing the answer.

Answer. Answer: $x = 13$

3.8 Writing Equations from Word Problems

Turn Words into Math

To solve a word problem, let a variable stand for the unknown, translate the words into an equation, then solve. Watch for signal words: “more than” or “increased by” means add; “less than” means subtract; “times” or “product” means multiply; “split” or “per” often means divide.

A Two-Step Word Problem

A taxi charges a \$3 flat fee plus \$2 per mile. A ride cost \$17. How many miles was it? Let m be the number of miles. The cost is the fee plus \$2 times the miles:

$$\begin{aligned} 2m + 3 &= 17 \\ 2m &= 14 \\ m &= 7 \end{aligned}$$

The ride was 7 miles. **Check:** $2(7) + 3 = 17$. ✓

Tip: Always say in words what your variable stands for ("let m = miles"). Then answer the question in a full sentence with units.

Try it: Quick check

$6x = 30$

Work it on paper before revealing the answer.

Answer. Answer: $x = 5$

3.9 What an Inequality Is

More Than One Answer

An **inequality** compares two expressions that are *not* necessarily equal. Instead of one solution, it usually has *many*. The four symbols are:

- $<$ means "less than"
- $>$ means "greater than"
- $\leq$ means "less than or equal to"
- $\geq$ means "greater than or equal to"

For example, $x > 5$ means every number bigger than 5 is a solution.

Reading an Inequality

The inequality $x \leq 3$ means "x is less than or equal to 3." Solutions include 3, 2, 1, 0, -1, and every number below 3. The number 3 itself *is* allowed because of the "or equal to" part.

Tip: The inequality sign always "opens" toward the larger side. In $7 > 2$, the wide opening faces the bigger number, 7.

Try it: Quick check

$\frac{x}{4} = 3$

Work it on paper before revealing the answer.

Answer. Answer: $x = 12$

3.10 Solving Inequalities

Solve Like an Equation, with One Warning

You solve inequalities almost exactly like equations: use inverse operations and do the same thing to both sides. One-step and two-step inequalities work the same way. There is just *one* special rule, coming next.

One-Step Inequality: Solve $x + 4 > 9$

Subtract 4 from both sides:

$$\begin{aligned} x + 4 &> 9 \\ x &> 5 \end{aligned}$$

Every number greater than 5 is a solution.

Two-Step Inequality: Solve $3x - 2 \leq 10$

Add 2 to both sides, then divide both sides by 3 (a *positive* number, so the sign does not change):

$$\begin{aligned} 3x - 2 &\leq 10 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

The solution is $x \leq 4$.

Tip: Adding or subtracting any number, or multiplying/dividing by a *positive* number, keeps the inequality sign pointing the same direction.

Try it: Quick check

$n + 9 = 15$

Work it on paper before revealing the answer.

Answer. Answer: $n = 6$

3.11 Flipping the Inequality Sign

The One Big Rule

When you **multiply or divide both sides of an inequality by a negative number, you must flip the inequality sign**. A "$<$" becomes "$>$," and a "$\leq$" becomes "$\geq$" (and the other way around). This is the single most important rule that makes inequalities different from equations.

Dividing by a Negative: Solve $-2x < 8$

Divide both sides by -2 and flip $<$ to $>$:

$$\begin{aligned} -2x &< 8 \\ \frac{-2x}{-2} &> \frac{8}{-2} \\ x &> -4 \end{aligned}$$

Why? Test $x = 0$: $-2(0) = 0 < 8$ is true, and $0 > -4$ is true. The flip keeps the answer correct.

Two-Step with a Flip: Solve $-3x + 1 \geq 7$

Subtract 1 first (no flip), then divide by -3 and flip $\geq$ to $\leq$:

$$\begin{aligned} -3x + 1 &\geq 7 \\ -3x &\geq 6 \\ \frac{-3x}{-3} &\leq \frac{6}{-3} \\ x &\leq -2 \end{aligned}$$

Remember: Only *multiplying or dividing by a negative* flips the sign. Adding or subtracting a negative does *not* flip it.

3.12 Graphing Inequality Solutions

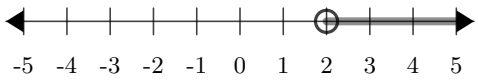
Show the Answer on a Number Line

Because an inequality has many solutions, we picture them on a **number line**. Put a circle at the boundary number and shade the direction of all the solutions:

- Use an **open circle** (hollow) for $<$ or $>$, because the boundary number is *not* included.
- Use a **closed circle** (filled in) for $\leq$ or $\geq$, because the boundary number *is* included.

Then shade to the **right** for greater-than, or to the **left** for less-than.

An open circle means the endpoint is not included.



Graphing $x > 2$

Draw a number line. Place an **open circle** on 2 (since 2 is not a solution), and shade the arrow to the **right** to show every number greater than 2.

Graphing $x \leq -1$

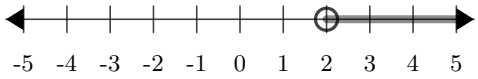
Place a **closed circle** on -1 (since -1 *is* a solution), and shade to the **left** for all numbers less than -1.

Tip: Open circle = "not included" ($<$ or $>$). Closed circle = "included" ($\leq$ or $\geq$). The "or equal to" line under the symbol is your reminder to fill the circle in.

3.13 Going Deeper: Advanced Equations & Inequalities

3.14 Variables on Both Sides and Clearing Fractions

An open circle means the endpoint is not included.



Get the Variable onto One Side

When the variable appears on *both* sides, like $5x - 3 = 2x + 9$, first gather all the variable terms onto one side and all the plain numbers onto the other. Add or subtract a variable term from both sides to move it, then finish with the usual two-step method. If the equation has **fractions**, you can **clear the denominators** first: multiply *every* term on both sides by the least common denominator (LCD). This turns a messy fraction equation into a clean whole-number one.

Variables on Both Sides: Solve $5x - 3 = 2x + 9$

Subtract $2x$ from both sides to collect the variable on the left, then finish:

$$\begin{aligned} 5x - 3 &= 2x + 9 \\ 5x - 3 - 2x &= 2x + 9 - 2x \\ 3x - 3 &= 9 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Check: left $= 5(4) - 3 = 17$; right $= 2(4) + 9 = 17$. Both sides equal 17. ✓

Clearing Denominators: Solve $\frac{x}{2} + 1 = \frac{x}{3} + 3$

The denominators are 2 and 3, so the LCD is 6. Multiply *every* term by 6:

$$\begin{aligned} \frac{x}{2} + 1 &= \frac{x}{3} + 3 \\ 6 \cdot \frac{x}{2} + 6 \cdot 1 &= 6 \cdot \frac{x}{3} + 6 \cdot 3 \\ 3x + 6 &= 2x + 18 \\ x + 6 &= 18 \\ x &= 12 \end{aligned}$$

Check: left $= \frac{12}{2} + 1 = 7$; right $= \frac{12}{3} + 3 = 7$. ✓

3.15 Two Special Cases: No Solution and Infinitely Many

When the Variable Disappears

Sometimes, while solving, the variable cancels out completely and you are left with a statement of pure numbers. What that statement says tells you the answer:

- If you get a **false** statement like $5 = -3$, there is **no solution**. No number could ever make the original equation true.
- If you get a **true** statement like $8 = 8$, there are **infinitely many solutions**. *Every* number works, because the two sides were really the same expression in disguise (this is called an **identity**).

For example, $2x + 5 = 2x - 3$ gives $5 = -3$ after subtracting $2x$: no solution. But $4x + 8 = 4(x + 2)$ becomes $4x + 8 = 4x + 8$, then $8 = 8$: infinitely many solutions.

Tip: A normal equation lands on "$x =$ (a number)." If instead the variable vanishes, stop and read what remains: false means no solution, true means all numbers.

3.16 A First Look at Systems of Two Equations

Two Equations, Two Unknowns

A **system** is two equations that must *both* be true at the same time, usually with two variables like x and y. The solution is the pair of values that satisfies both. One reliable method is **substitution**: if one equation already tells you what a variable equals (for example $y = x+1$), plug that expression into the *other* equation. This leaves a single equation in one variable, which you already know how to solve. Then back-substitute to find the second variable.

Substitution: Solve the System $y = x + 1$ and $2x + y = 7$

The first equation says y *is* $x + 1$. Substitute $x + 1$ in place of y in the second equation:

$$\begin{aligned} 2x + y &= 7 \\ 2x + (x + 1) &= 7 \\ 3x + 1 &= 7 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Now find y using $y = x + 1 = 2 + 1 = 3$. The solution is the pair $(x, y) = (2, 3)$.

Check: $y = x + 1 \Rightarrow 3 = 2 + 1$ ✓ and $2x + y = 2(2) + 3 = 7$ ✓

3.17 Compound Inequalities and Interval Notation

Two Conditions at Once

A **compound inequality** traps the variable between two bounds, like $-1 < 2x + 3 \leq 9$. It really means two inequalities joined by "and": the middle is greater than -1 *and* at most 9. To solve, do the same operation to *all three* parts at once. **Interval notation** is a compact way to write the answer: a *round* bracket ( ) excludes an endpoint (matches $<$ or $>$), and a *square* bracket [ ] includes it (matches $\leq$ or $\geq$). So $-2 < x \leq 3$ is written $(-2,\ 3]$.

Solve $-1 < 2x + 3 \leq 9$ and Write the Interval

Subtract 3 from all three parts, then divide all three parts by 2 (positive, so no flip):

$$\begin{aligned} -1 < 2x + 3 &\leq 9 \\ -1 - 3 < 2x &\leq 9 - 3 \\ -4 < 2x &\leq 6 \\ -2 < x &\leq 3 \end{aligned}$$

The solution is every number greater than -2 and up to and including 3. In interval notation: $(-2,\ 3]$.
Check: $x = 3$ gives $2(3) + 3 = 9 \leq 9$ ✓; $x = -2$ gives $2(-2) + 3 = -1$, which is *not* > -1, so -2 is correctly excluded. ✓

Remember: If you ever multiply or divide a compound inequality by a *negative* number, flip *both* signs at once. And match your brackets to your symbols: round for strict $(<, >)$, square for "or equal" $(\leq, \geq)$.

3.18 Why It Works

Adding the same number to both sides of a true equation keeps it true, and so does multiplying by a nonzero number; that is why the balance method is sound. For inequalities, multiplying by -1 reverses order because it reflects the number line: $2 < 3$ but $-2 > -3$. Every other step (adding, multiplying by a positive) preserves order, so only the negative-multiplier step needs the flip.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

4 The Coordinate Plane

Points, quadrants, tables, and graphing lines

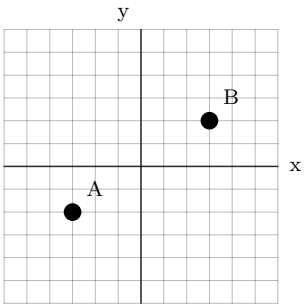
4.1 The Coordinate Plane

The Big Idea

The **coordinate plane** is made from two number lines that cross at right angles. The horizontal number line is the x**-axis**. The vertical number line is the y**-axis**. The point where they cross is the **origin**, written $(0, 0)$.

Every location on the plane is named by an **ordered pair** (x, y). The first number tells you how far to move *left or right* along the x-axis. The second number tells you how far to move *up or down* parallel to the y-axis. Because the order matters, we call it an *ordered* pair: $(3, 5)$ is a different point than $(5, 3)$.

Distance and midpoint come from the coordinates.



A line on the coordinate plane is the set of points whose coordinates satisfy an equation like $y = 2x + 1$. Its slope, $\frac{\text{rise}}{\text{run}}$, says how steep it is and in which direction, and the y-intercept is where it crosses the vertical axis. Two points determine the slope; the slope and one point determine the line.

Reading an Ordered Pair

The point $(4, 2)$ means: start at the origin, move **right** 4 (because 4 is positive), then move **up** 2 (because 2 is positive). You land at a spot that is 4 units right and 2 units above the origin.

The point $(-3, 1)$ means: move **left** 3, then **up** 1.

Where it goes wrong

Computing slope as run over rise, or mixing the order of the points: $\frac{y_2 - y_1}{x_2 - x_1}$ with the SAME point first in both numerator and denominator.

Tip: Always read the coordinates in the order (x, y) — x **before** y, just like the alphabet. A helpful memory trick: "walk before you climb" (move sideways first, then up or down).

Try it: Quick check

What is the name of the *horizontal* number line in the coordinate plane?

Work it on paper before revealing the answer.

Answer. Answer: The x-axis.

4.2 The Four Quadrants

Dividing the Plane

The two axes split the plane into four regions called **quadrants**. We number them with Roman numerals, starting at the top-right and going *counter-clockwise*:

- **Quadrant I** — top right — x positive, y positive: $(+, +)$
- **Quadrant II** — top left — x negative, y positive: $(-, +)$
- **Quadrant III** — bottom left — x negative, y negative: $(-, -)$
- **Quadrant IV** — bottom right — x positive, y negative: $(+, -)$

Naming the Quadrant

Where is $(-5, -2)$? Both coordinates are negative, which matches $(-, -)$, so the point is in **Quadrant III**.

Where is $(6, -1)$? Here x is positive and y is negative, matching $(+, -)$, so the point is in **Quadrant IV**.

Points on the Axes

If a point sits *on* an axis, it is not in any quadrant.

- A point on the x-axis has the form $(x, 0)$ — its y-coordinate is 0. Example: $(4, 0)$.
- A point on the y-axis has the form $(0, y)$ — its x-coordinate is 0. Example: $(0, -3)$.
- The **origin** $(0, 0)$ sits on both axes.

Tip: A zero tells you the point is on an axis. Zero in the *second* spot means the x-axis; zero in the *first* spot means the y-axis.

Try it: Quick check

What is the special name of the point $(0, 0)$ where the two axes cross?

Work it on paper before revealing the answer.

Answer. Answer: The origin.

4.3 Plotting Points and Naming Coordinates

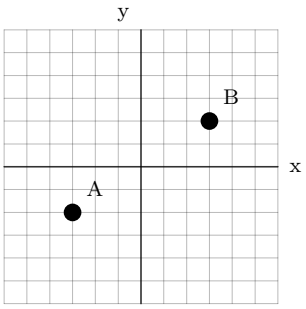
Plotting a Point

To **plot** the point (x, y):

1. Start at the origin $(0, 0)$.
2. Move along the x-axis: right if $x > 0$, left if $x < 0$.
3. From there move parallel to the y-axis: up if $y > 0$, down if $y < 0$.
4. Mark a dot where you stop.

To **name** the coordinates of a point you already see, do the reverse: count how far it is left/right, then up/down, from the origin.

Distance and midpoint come from the coordinates.



Plotting $(-2, 3)$

Start at the origin. Because $x = -2$, move **left** 2. Because $y = 3$, move **up** 3. The dot lands in the top-left region, so this point is in Quadrant II.

Naming a Point

Suppose a dot is 3 units to the right of the origin and 4 units below it. Moving right 3 gives $x = 3$; moving down 4 gives $y = -4$. So the point is $(3, -4)$, which is in Quadrant IV.

Try it: Quick check

In an ordered pair (x, y), which coordinate is written *first*?

Work it on paper before revealing the answer.

Answer. Answer: The x-coordinate.

4.4 Tables of Values

Building a Table

A **table of values** is a neat list of ordered pairs. We usually pick a few x-values, apply a **rule**, and record the matching y-values. Each row of the table (x, y) becomes one point on the plane.

Table for $y = x + 2$

Choose $x = 0, 1, 2, 3$ and add 2 to each to find y:

The ordered pairs are $(0, 2)$, $(1, 3)$, $(2, 4)$, and $(3, 5)$.

Tip: To read a table, take the top number and bottom number in the same column (or the two numbers in the same row) and pair them as (x, y).

Try it: Quick check

Write the ordered pair for a point that is 3 units to the right of the origin and 4 units up.

Work it on paper before revealing the answer.

Answer. Answer: $(3, 4)$

4.5 Graphing a Linear Relationship

From Rule to Graph

A rule like $y = x + 2$ or $y = 2x$ describes how y depends on x. When you plot all the ordered

pairs from its table, the points fall in a perfectly straight line — this is a **linear relationship**. Connecting the dots gives the graph of the rule.

Graphing $y = 2x$

Make a table by doubling each x:

The points $(0, 0)$, $(1, 2)$, $(2, 4)$, $(3, 6)$ all lie on one straight line that climbs steeply through the origin. Each time x goes up by 1, y goes up by 2.

Tip: Plot at least three points from the table. If they do not line up straight, check your arithmetic — for a linear rule they always should.

Try it: Quick check

Which quadrant contains the point $(2, 5)$?

Work it on paper before revealing the answer.

Answer. Answer: Quadrant I

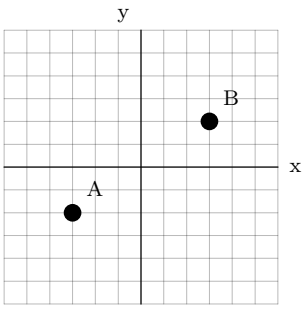
4.6 Distance on a Horizontal or Vertical Line

Same Row or Same Column

Two points lie on the same **horizontal line** when they share the same y-coordinate. Two points lie on the same **vertical line** when they share the same x-coordinate. In either case the distance between them is easy to find:

- **Horizontal:** subtract the x-coordinates and take the positive difference.
- **Vertical:** subtract the y-coordinates and take the positive difference.

Distance and midpoint come from the coordinates.



Horizontal Distance

Find the distance from $(2, 5)$ to $(9, 5)$. They share $y = 5$, so this is horizontal. Subtract the x-values: $9 - 2 = 7$. The distance is 7 **units**.

Vertical Distance with a Negative

Find the distance from $(-3, 4)$ to $(-3, -2)$. They share $x = -3$, so this is vertical. The distance is the gap between $y = 4$ and $y = -2$. Counting from 4 down to -2: that is $4 + 2 = 6$ units. (In general, take the positive difference $|4 - (-2)| = 6$.) The distance is 6 **units**.

Tip: Distance is never negative. If subtracting gives a negative number, just drop the sign — distance is the *positive* difference.

Try it: Quick check

Which quadrant contains the point $(-4, 6)$?

Work it on paper before revealing the answer.

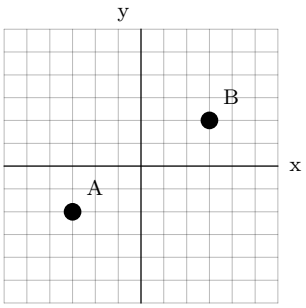
Answer. Answer: Quadrant II

4.7 Real-World Coordinates and Patterns

Maps and Grids

Coordinates appear everywhere: on maps, game boards, and city grids. A treasure map might place a chest at $(5, 3)$, meaning 5 blocks east and 3 blocks north of a starting corner. Counting blocks between two spots on a grid is exactly the horizontal or vertical distance from the last section.

Distance and midpoint come from the coordinates.



A City Grid

A school is at $(1, 2)$ and a library is at $(1, 7)$ on a city map where each unit is one block. Because both have $x = 1$, they are on the same vertical street. The walking distance is $7 - 2 = 5$ blocks.

A Growing Pattern

A plant is 1 cm tall on day 0 and grows 3 cm each day. The rule is $y = 3x + 1$, where x is the day and y is the height. On day 4: $y = 3(4) + 1 = 13$ cm. The points $(0, 1)$, $(1, 4)$, $(2, 7)$, $(3, 10)$, $(4, 13)$ climb in a straight line.

Tip: In word problems, decide what x and y stand for *first*. Then a "starting amount plus a steady change" almost always turns into a straight-line rule.

Try it: Quick check

Which quadrant contains the point $(-3, -1)$?

Work it on paper before revealing the answer.

Answer. Answer: Quadrant III

4.8 Going Deeper: Advanced Coordinate Geometry

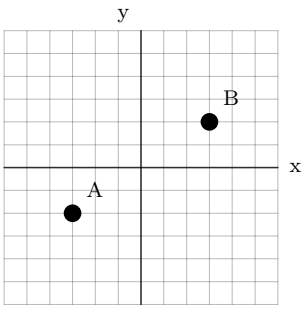
The Distance Formula

For two points *not* on the same horizontal or vertical line, the straight-line distance still comes from a right triangle. Between (x_1, y_1) and (x_2, y_2), the horizontal leg has length $|x_2 - x_1|$ and the vertical leg has length $|y_2 - y_1|$. The distance is the **hypotenuse**, so by the Pythagorean theorem

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Because each difference is squared, the sign of the subtraction never matters — squaring erases it.

Distance and midpoint come from the coordinates.



Distance from $(1, 2)$ to $(5, 5)$

The horizontal change is $x_2 - x_1 = 5 - 1 = 4$ and the vertical change is $y_2 - y_1 = 5 - 2 = 3$. These are the legs of a right triangle:

Then

$$d = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \text{ units.}$$

This is the famous 3–4–5 right triangle.

The Midpoint Formula

The **midpoint** of the segment joining (x_1, y_1) and (x_2, y_2) is the point exactly halfway between them. You simply *average* the x-coordinates and *average* the y-coordinates:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

The average of two numbers always lands in the middle, so this works even when the coordinates are negative.

Midpoint of $(-2, 3)$ and $(4, -1)$

Average the x-values, then the y-values:

$$x \text{ of } M = \frac{-2 + 4}{2} = \frac{2}{2} = 1,$$
$$y \text{ of } M = \frac{3 + (-1)}{2} = \frac{2}{2} = 1.$$

So the midpoint is $M = (1, 1)$. Check: it is 3 right and 2 down from $(-2, 3)$, and also 3 left and 2 up from $(4, -1)$ — perfectly centered.

Slope: Rate of Change

The **slope** of a line measures how steeply it rises. It is the ratio of vertical change to horizontal

change between any two points on the line:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A positive slope climbs left-to-right; a negative slope falls. A larger $|m|$ means a steeper line. In a real setting, slope is a **rate of change** — for the plant rule $y = 3x + 1$, the slope 3 means "3 cm of growth per day."

Slope–Intercept Form: $y = mx + b$ (Preview)

Every non-vertical line can be written as

$$y = mx + b,$$

where m is the **slope** and b is the **y-intercept** — the y-value where the line crosses the y-axis (that is, where $x = 0$). For $y = 2x + 1$: the slope is $m = 2$, and the line crosses the y-axis at $(0, 1)$. Each step right of 1 raises y by 2. This is the same "starting amount plus steady change" idea from the tables, written in one tidy formula.

Watch out: slope is $\dfrac{\text{change in } y}{\text{change in } x}$, *not* the other way around. Subtract the coordinates in the *same order* on top and bottom — if you start with point 2 in the numerator, start with point 2 in the denominator too.

Transformations as Coordinate Rules

Moving or flipping a shape changes each point by a simple rule:

- **Translation** (slide) by a right and b up: $(x, y) \to (x + a,\ y + b)$.
- **Reflection over the x-axis:** $(x, y) \to (x,\ -y)$.
- **Reflection over the y-axis:** $(x, y) \to (-x,\ y)$.
- **Reflection over the origin:** $(x, y) \to (-x,\ -y)$.

Apply the rule to every vertex, then reconnect the new points to get the image.

Translating a Triangle

A triangle has vertices $A(1, 2)$, $B(4, 2)$, and $C(2, 5)$. Slide it 3 left and 1 down, using the rule

$(x, y) \to (x - 3,\ y - 1)$:

$$A(1,2) \to (1-3,\ 2-1) = (-2,1),$$
$$B(4,2) \to (4-3,\ 2-1) = (1,1),$$
$$C(2,5) \to (2-3,\ 5-1) = (-1,4).$$

The new triangle $A'B'C'$ has the same size and shape — only its position changed.

Area from Vertices: The Shoelace Idea

You can find the area of a triangle straight from its corner coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) — no height needed:

$$\text{Area} = \tfrac{1}{2}\left|\, x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \,\right|.$$

The absolute-value bars keep the area positive no matter the order of the vertices. This trick is called the **shoelace formula** because of the criss-cross pattern of multiplications.

Area of a Triangle with Vertices $(1,1)$, $(5,2)$, $(3,6)$

Label $(x_1, y_1) = (1,1)$, $(x_2, y_2) = (5,2)$, $(x_3, y_3) = (3,6)$ and substitute:

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\left|\, 1(2-6) + 5(6-1) + 3(1-2) \,\right| \\ &= \tfrac{1}{2}\left|\, 1(-4) + 5(5) + 3(-1) \,\right| \\ &= \tfrac{1}{2}\left|\, -4 + 25 - 3 \,\right| \\ &= \tfrac{1}{2}\,(18) = 9 \text{ square units.}\end{aligned}$$

Lattice Points and Taxicab Distance

A **lattice point** is a point whose coordinates are both integers, like $(3, -2)$ — exactly where grid lines cross. On a city grid you cannot cut diagonally through buildings, so the travel distance is the **taxicab distance**: add the horizontal and vertical blocks,

$$d_{\text{taxi}} = |x_2 - x_1| + |y_2 - y_1|.$$

From $(1,2)$ to $(5,5)$ this is $|5-1| + |5-2| = 4 + 3 = 7$ blocks, even though the straight-line distance is only 5. The taxicab distance is never shorter than the straight-line distance.

Big picture: coordinates turn geometry into arithmetic. Distance, midpoints, slope, area, and even flips and slides all become formulas you can *compute* from the numbers (x, y) — the foundation of the algebra and geometry still ahead.

Try it: Quick check

Which quadrant contains the point $(5, -2)$?

Work it on paper before revealing the answer.

Answer. Answer: Quadrant IV

4.9 Why It Works

Why is the slope the same between any two points on a line? Any two "slope triangles" drawn on the same line are similar (their angles match because the line makes the same angle with every horizontal), so the ratio rise/run is the same for all of them. This is also why $y = mx + b$ describes exactly the lines: from the point $(0, b)$, moving x units right always changes y by mx.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

5 Geometry Basics

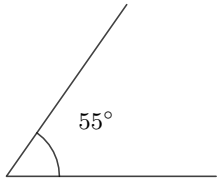
Points, angles, shapes, and the formulas that measure them

5.1 Basic Vocabulary: Points, Lines, and Angles

The Building Blocks

- A **point** is an exact location. It has no size, just position.
- A **line** goes straight forever in both directions (two arrowheads).
- A **segment** is part of a line with *two endpoints*.
- A **ray** starts at one endpoint and goes forever in one direction.
- An **angle** is formed by two rays that share an endpoint, called the **vertex**. We measure angles in **degrees** (°).

An angle measures a turn.



Perimeter is the distance around; area is the space inside. Angle facts do most of the work in geometry: angles on a straight line add to 180°, angles around a point to 360°, and the three angles of any triangle add to 180°. Formulas for rectangles, triangles, and circles follow from cutting and rearranging.

Types of Angles

- **Acute:** greater than 0° and less than 90°.
- **Right:** exactly 90° (marked with a small square).
- **Obtuse:** greater than 90° and less than 180°.
- **Straight:** exactly 180° (a straight line).

Complementary and Supplementary Angles

Two angles are **complementary** if their measures add to 90°.
Two angles are **supplementary** if their measures add to 180°.

Finding a Complement and a Supplement

The **complement** of a 35° angle is $90° - 35° = 55°$.
The **supplement** of a 35° angle is $180° - 35° = 145°$.

Where it goes wrong

Confusing perimeter (linear units) with area (square units), and using the diameter where the radius belongs in πr^2.

Classifying by Measure

An angle of 118° is **obtuse** (between 90° and 180°).
An angle of 72° is **acute** (less than 90°).

Tip: **C**omplementary comes before **S**upplementary in the alphabet, just as 90 comes before 180. So C goes with $90°$ and S goes with $180°$.

Try it: Quick check

Is a $40°$ angle acute, right, obtuse, or straight?

Work it on paper before revealing the answer.

Answer. Answer: Acute

5.2 Triangles

Classifying Triangles

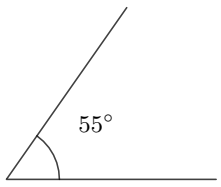
By sides:

- **Equilateral:** all three sides equal.
- **Isosceles:** exactly two sides equal.
- **Scalene:** no sides equal.

By angles:

- **Acute:** all three angles less than $90°$.
- **Right:** one angle exactly $90°$.
- **Obtuse:** one angle greater than $90°$.

An angle measures a turn.



The Angle Sum Rule

The three angles of *any* triangle always add up to 180°:

$$\angle A + \angle B + \angle C = 180°.$$

Finding a Missing Angle

Two angles of a triangle are 55° and 65°. Find the third.
Add the known angles: $55° + 65° = 120°$.
Subtract from 180°: $180° - 120° = 60°$. The third angle is **60°**.

Classifying a Triangle

A triangle has sides 6, 6, and 9, and angles 50°, 50°, 80°.
Two sides are equal, so it is **isosceles**. All angles are below 90°, so it is also **acute**.

Tip: A triangle can have *at most one* right or obtuse angle. If it had two, they would already use up 180° or more with nothing left for the third angle.

Try it: Quick check

Is a 125° angle acute, right, obtuse, or straight?

Work it on paper before revealing the answer.

Answer. Answer: Obtuse

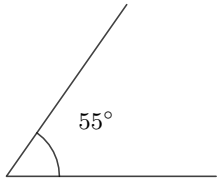
5.3 Perimeter of Polygons

What Perimeter Means

The **perimeter** is the total distance *around* a shape: just add up the lengths of all the sides. Perimeter is a **length**, so it uses plain units (cm, m, in).

$$P_{\text{rectangle}} = 2l + 2w \qquad P_{\text{square}} = 4s$$

An angle measures a turn.



Perimeter of a Rectangle

A rectangle is 9 cm long and 4 cm wide.

$$P = 2(9) + 2(4) = 18 + 8 = 26 \text{ cm}.$$

Perimeter of a Triangle

A triangle has sides 5 in, 8 in, and 11 in.

$$P = 5 + 8 + 11 = 24 \text{ in}.$$

Tip: Perimeter is measured in single units (like cm). *Area* is measured in **square** units (like cm^2). Don't mix them up!

Try it: Quick check

Find the complement of a 25° angle.

Work it on paper before revealing the answer.

Answer. Answer: 65°

5.4 Area of Rectangles, Squares, Parallelograms, Triangles, and Trapezoids

The Area Formulas

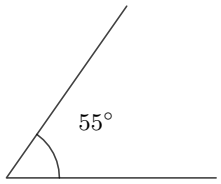
Area is the amount of surface a flat shape covers, measured in **square units**.

- Rectangle: $A = l \times w$
- Square: $A = s^2$
- Parallelogram: $A = b \times h$

- Triangle: $A = \dfrac{1}{2}bh$
- Trapezoid: $A = \dfrac{1}{2}(b_1 + b_2)\,h$

Here h is always the **perpendicular height** (straight up from the base), not a slanted side.

An angle measures a turn.



Triangle and Parallelogram

Triangle, base 10 cm, height 6 cm:

$$A = \tfrac{1}{2}(10)(6) = \tfrac{1}{2}(60) = 30 \text{ cm}^2.$$

Parallelogram, base 12 m, height 5 m:

$$A = (12)(5) = 60 \text{ m}^2.$$

Trapezoid

A trapezoid has parallel sides $b_1 = 7$ cm and $b_2 = 11$ cm and height $h = 4$ cm.

$$A = \tfrac{1}{2}(7 + 11)(4) = \tfrac{1}{2}(18)(4) = \tfrac{1}{2}(72) = 36 \text{ cm}^2.$$

Tip: For a triangle, remember the $\frac{1}{2}$! A triangle is exactly half of a rectangle (or parallelogram) with the same base and height.

Try it: Quick check

Find the supplement of a 70° angle.

Work it on paper before revealing the answer.

Answer. Answer: 110°

5.5 Circles

Parts of a Circle

- The **radius** r is the distance from the center to the edge.
- The **diameter** d goes all the way across through the center: $d = 2r$, so $r = \dfrac{d}{2}$.
- **Circumference** (distance around): $C = \pi d$, which is the same as $C = 2\pi r$.
- **Area:** $A = \pi r^2$.

We use $\pi \approx 3.14$.

Circumference

A circle has diameter $d = 10$ cm.

$$C = \pi d \approx 3.14 \times 10 = 31.4 \text{ cm}.$$

Area of a Circle

A circle has radius $r = 6$ m.

$$A = \pi r^2 \approx 3.14 \times 6^2 = 3.14 \times 36 = 113.04 \text{ m}^2.$$

Tip: For *area* you square the radius *first*, then multiply by π. Compute r^2 before multiplying by 3.14. And remember: area answers get square units.

Try it: Quick check

What is the geometric term for a part of a line that has two endpoints?

Work it on paper before revealing the answer.

Answer. Answer: A segment (line segment)

5.6 The Pythagorean Theorem

The Rule for Right Triangles

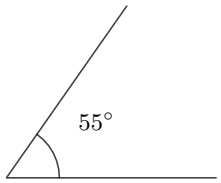
In a **right triangle**, the two shorter sides (the **legs**, a and b) and the longest side (the

hypotenuse, c, opposite the right angle) are related by

$$a^2 + b^2 = c^2.$$

The hypotenuse c is always the longest side and sits across from the $90°$ angle.

An angle measures a turn.



Finding the Hypotenuse

A right triangle has legs $a = 6$ and $b = 8$.

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100, \qquad c = \sqrt{100} = 10.$$

Finding a Leg

A right triangle has hypotenuse $c = 13$ and one leg $a = 5$. Find leg b.

$$5^2 + b^2 = 13^2 \;\Rightarrow\; 25 + b^2 = 169 \;\Rightarrow\; b^2 = 144 \;\Rightarrow\; b = 12.$$

Tip: Only the hypotenuse sits alone on one side of the equation. If you are missing a *leg*, subtract; if you are missing the *hypotenuse*, add.

Try it: Quick check

Two angles of a triangle are $45°$ and $55°$. Find the third angle.

Work it on paper before revealing the answer.

Answer. Answer: $80°$

5.7 Surface Area and Volume of Prisms and Cylinders

Rectangular Prisms

For a box with length l, width w, and height h:

$$V = l \times w \times h \qquad SA = 2(lw + lh + wh).$$

Volume is measured in **cubic** units (cm^3); **surface area** in **square** units (cm^2).

Cylinders

For a cylinder with radius r and height h:

$$V = \pi r^2 h.$$

(The base is a circle of area πr^2; multiply by the height.)

Volume and Surface Area of a Box

A box is $l = 5$ cm, $w = 4$ cm, $h = 3$ cm.

$$V = 5 \times 4 \times 3 = 60 \text{ cm}^3.$$

$$SA = 2(5 \cdot 4 + 5 \cdot 3 + 4 \cdot 3) = 2(20 + 15 + 12) = 2(47) = 94 \text{ cm}^2.$$

Volume of a Cylinder

A cylinder has radius $r = 3$ cm and height $h = 10$ cm.

$$V = \pi r^2 h \approx 3.14 \times 3^2 \times 10 = 3.14 \times 9 \times 10 = 282.6 \text{ cm}^3.$$

Tip: Surface area is the total of all the *outside faces* (like the paper needed to wrap a gift). Volume is how much *fills the inside*. Square units for surface, cubic units for volume.

Try it: Quick check

Classify a triangle with side lengths 7, 7, and 7 by its sides.

Work it on paper before revealing the answer.

Answer. Answer: Equilateral

5.8 Real-World Geometry Problems

A Plan of Attack

1. Decide what the question is really asking: distance around (**perimeter/circumference**), surface covered (**area**), or space filled (**volume**).
2. Pick the matching formula and substitute the numbers.
3. Keep the units, and label the answer with the correct kind of unit.

Fencing and Sod

A rectangular garden is 12 ft by 8 ft.
Fence around it = perimeter $= 2(12) + 2(8) = 24 + 16 = 40$ ft.
Grass to cover it = area $= 12 \times 8 = 96$ ft^2.

Cost of Carpet

A floor is 10 ft by 9 ft, and carpet costs \$4 per square foot.
Area $= 10 \times 9 = 90$ ft^2. Cost $= 90 \times \$4 = \$360$.

Tip: "Around" problems (fencing, trim, edging) use perimeter. "Cover" problems (paint, carpet, sod) use area. "Fill" problems (water, sand) use volume.

Try it: Quick check

A triangle has angles 90°, 45°, and 45°. Classify it by its angles.

Work it on paper before revealing the answer.

Answer. Answer: Right triangle

5.9 Going Deeper: Advanced Geometry

Why the Pythagorean Theorem Is True (A Proof)

Take four copies of a right triangle with legs a, b and hypotenuse c, and arrange them inside a big square whose side is $a + b$. The four hypotenuses form a tilted square of side c in the middle.

Compute the big square's area two ways:

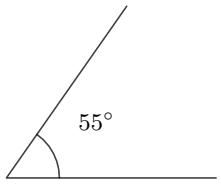
$$\underbrace{(a+b)^2}_{\text{outer square}} = \underbrace{4 \cdot \tfrac{1}{2}ab}_{\text{four triangles}} + \underbrace{c^2}_{\text{tilted square}}.$$

Expand the left side and simplify:

$$a^2 + 2ab + b^2 = 2ab + c^2 \implies a^2 + b^2 = c^2.$$

The $2ab$ cancels from both sides, leaving the theorem. This shows the rule is not a coincidence: it follows from just adding up areas.

An angle measures a turn.



Heron's Formula: Area From Three Sides

You can find a triangle's area from its *side lengths alone*—no height needed. First compute the **semiperimeter** (half the perimeter):

$$s = \frac{a+b+c}{2}.$$

Then

$$A = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

This is the tool to reach for when you know all three sides but cannot easily measure a perpendicular height.

Heron's Formula in Action

Find the area of a triangle with sides $a = 13$, $b = 14$, $c = 15$.
Semiperimeter: $s = \dfrac{13+14+15}{2} = \dfrac{42}{2} = 21$.
Now the differences: $s - a = 8$, $s - b = 7$, $s - c = 6$.

$$A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = \mathbf{84} \text{ square units.}$$

(Check the product step by step: $21 \cdot 8 = 168$, then $168 \cdot 7 = 1176$, then $1176 \cdot 6 = 7056$.)

Similar Triangles and Proportional Reasoning

Two triangles are **similar** if they have the same shape but not necessarily the same size: their corresponding angles are equal, and their corresponding sides are in the same **ratio** (scale

factor). We write $\triangle ABC \sim \triangle DEF$.

The key fact is that matching sides form equal fractions:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

This lets you find a distance you *cannot* measure directly (like the height of a tree) by comparing it to one you can.

Measuring a Tree With Its Shadow

At the same time of day, a 6-ft person casts a 4-ft shadow, and a tree casts a 24-ft shadow. The sun's rays make similar triangles, so heights and shadows are proportional:

$$\frac{\text{tree height}}{\text{tree shadow}} = \frac{\text{person height}}{\text{person shadow}} \implies \frac{h}{24} = \frac{6}{4}.$$

Cross-multiply: $4h = 24 \times 6 = 144$, so $h = \dfrac{144}{4} = \mathbf{36}$ ft. The tree is 36 feet tall.

Angle Chasing: The Exterior Angle Rule

Because a triangle's angles sum to $180°$ and a straight line also measures $180°$, an **exterior angle** (formed by extending one side) equals the *sum of the two remote interior angles*:

Here the exterior angle at B equals $x + y$. *Why:* the interior angle at B is $180° - x - y$ (angle sum), and the exterior angle is its supplement, $180° - (180° - x - y) = x + y$.

Chasing a Missing Angle

A triangle has interior angles $x = 50°$ and $y = 70°$ at two vertices. The exterior angle at the third vertex is

$$50° + 70° = 120°,$$

and a quick check: the third interior angle is $180° - 50° - 70° = 60°$, whose supplement is $180° - 60° = 120°$. The two methods agree.

A Circle Fact: The Inscribed Angle (Preview)

An **inscribed angle** has its vertex *on* the circle; a **central angle** has its vertex *at the center*. If both open onto the same arc, then

$$\text{inscribed angle} = \tfrac{1}{2}\,(\text{central angle}).$$

A famous special case (*Thales' theorem*): any angle inscribed in a **semicircle** is a right angle,

because it opens onto a 180° arc, and $\frac{1}{2}(180°) = 90°$. So if one side of an inscribed triangle is a diameter, the triangle is a right triangle.

Deriving the Surface Area of a Cylinder

Imagine unwrapping a can. You get *two circles* (the top and bottom) plus *one rectangle* (the curved side, or "label"). Each circle has area πr^2. When you unroll the side, its width is the circumference $2\pi r$ and its height is h, so the rectangle's area is $2\pi r \cdot h$. Add the pieces:

$$SA = \underbrace{2\pi r^2}_{\text{two circles}} + \underbrace{2\pi r h}_{\text{unrolled side}} .$$

This is why the formula looks the way it does—it is just areas of shapes you already know, glued together.

Composite Figure: Rectangle Plus a Semicircle

A window is a 10-cm-wide by 6-cm-tall rectangle topped by a semicircle whose diameter is the rectangle's width (10 cm, so radius $r = 5$ cm). Find its total area by **decomposing** it into familiar pieces.

$$A_{\text{rectangle}} = 10 \times 6 = 60 \text{ cm}^2.$$

$$A_{\text{semicircle}} = \tfrac{1}{2}\pi r^2 \approx \tfrac{1}{2}(3.14)(5^2) = \tfrac{1}{2}(3.14)(25) = 39.25 \text{ cm}^2.$$

Total: $60+39.25 = \mathbf{99.25}$ cm^2. Break any odd shape into rectangles, triangles, and circle-parts, then add (or subtract) the areas.

Tip: Most "advanced" geometry is really *old rules used together.* Angle chasing leans on the 180° angle sum and the straight-line 180°; the Pythagorean proof and composite figures are just areas added up; similar triangles are proportions in disguise. When a problem looks new, ask: *which basic facts can I combine?*

5.10 Why It Works

The angles of a triangle add to 180°: draw the line through one vertex parallel to the opposite side. The two angles it makes with the triangle's sides equal the two far angles of the triangle (alternate interior angles), and together with the vertex angle they fill a straight line. The area of a circle is πr^2 because cutting it into many thin wedges and laying them alternately point-up and point-down makes a near-rectangle of height r and width half the circumference, πr.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line

forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

6 Data, Statistics & Probability

Measures of center and spread, reading and making graphs, interpreting data, and finding the probability of events as fractions, decimals, and percents

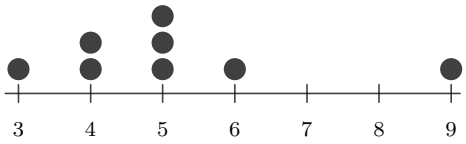
6.1 Mean, Median, and Mode (Measures of Center)

The Rule

A **measure of center** is one number that summarizes a whole list of numbers. There are three:

- **Mean** (the average): add up all the values, then divide by how many values there are.
- **Median**: the **middle** value once the numbers are put *in order*. If there are two middle values, take their mean (add them and divide by 2).
- **Mode**: the value that appears **most often**. A data set can have one mode, several modes, or no mode at all.

One outlier drags the mean but not the median.



Statistics summarizes data with a center (mean, median, mode) and a spread (range). Probability measures how likely an event is, as a fraction of equally likely outcomes: favorable over total. Listing the outcomes — a table for two dice, a tree for coin flips — is the reliable way to count them.

Worked Example: all three centers

Find the mean, median, and mode of 7, 3, 9, 3, 8.

Order first: 3, 3, 7, 8, 9.

Mean: $\dfrac{3+3+7+8+9}{5} = \dfrac{30}{5} = \mathbf{6}$.

Median: the middle of five ordered numbers is the 3rd one, so median = **7**.

Mode: 3 appears twice, more than any other, so mode = **3**.

Where it goes wrong

Counting outcomes that are not equally likely as if they were. Two coins have four outcomes (HH, HT, TH, TT), not three; "one head" happens two ways.

Worked Example: even number of values

Find the median of 10, 4, 6, 2. Order them: 2, 4, 6, 10. There are two middle values, 4 and 6, so the median is their mean: $\frac{4+6}{2} = \mathbf{5}$. Notice the median need not be a value in the list.

When Is Each One Most Useful?

- Use the **mean** when the data are fairly even, with no extreme values.
- Use the **median** when there is an **outlier** (a value far from the rest), because the median is not pulled toward it.
- Use the **mode** for things you *count* or categories, like the most common shoe size or favorite color.

Worked Example: effect of an outlier

Four friends have \$5, \$6, \$7, and \$8. Mean $= \frac{26}{4} = \$6.50$; median $= \frac{6+7}{2} = \$6.50$. Now a fifth friend joins with \$100 (an outlier). New mean $= \frac{126}{5} = \$25.20$, but the ordered list $5, 6, 7, 8, 100$ has median = \$**7**. The single big value pulled the **mean** way up, while the **median** barely moved. That is why the median better describes a "typical" friend here.

Tip: *Always put the numbers in order before you find the median.* Forgetting to order the data is the most common mistake in this whole topic.

Try it: Quick check

Find the mean (average) of 4, 6, 8.

Work it on paper before revealing the answer.

Answer. Answer: $\dfrac{4+6+8}{3} = \dfrac{18}{3} = 6$

6.2 Range (A Measure of Spread)

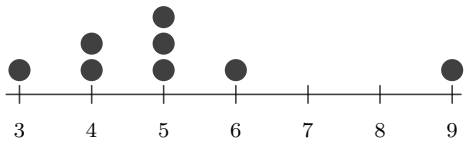
The Rule

The **range** tells how spread out the data are. It is the largest value minus the smallest value:

$$\text{range} = \text{maximum} - \text{minimum}.$$

A small range means the values are close together; a large range means they are spread far apart.

One outlier drags the mean but not the median.



Worked Example: finding the range

Test scores were 88, 95, 72, 90, 81. The maximum is 95 and the minimum is 72, so the range $= 95 - 72 =$ **23**.

Tip: Range is a *single number*, not an interval. Say "the range is 23," not "the range is 72 to 95."

Try it: Quick check

Find the mean of 3, 5, 7, 5.

Work it on paper before revealing the answer.

Answer. Answer: $\dfrac{3+5+7+5}{4} = \dfrac{20}{4} = 5$

6.3 Reading and Making Graphs

The Rule

Different graphs are good for different jobs:

- **Bar graph**: compares separate categories (favorite fruits, sports). Taller bar means larger amount.
- **Line graph**: shows how something **changes over time** (temperature through the day). Look at whether the line rises, falls, or stays flat.
- **Circle (pie) graph**: shows **parts of one whole**. All the slices together make 100% of the total.
- **Dot plot / line plot**: an $\times$ or dot stacked above a number line for each data value; great for seeing repeats and the shape of the data.
- **Frequency table**: lists each value or category next to how many times it occurs (its **frequency**).

Worked Example: frequency table and dot plot

A class recorded how many pets each student has:

The frequency column tells us $4 + 6 + 3 + 2 = 15$ students answered. The **mode** is 1 pet (highest frequency, 6). A dot plot would stack 4 dots above 0, 6 dots above 1, and so on. To find the total number of pets, multiply and add: $0(4) + 1(6) + 2(3) + 3(2) = 0 + 6 + 6 + 6 = 18$ pets, so the mean is $\dfrac{18}{15} = 1.2$ pets.

Worked Example: reading a circle graph

A circle graph of how 60 students get to school shows Walk = 25%, Bus = 50%, Car = 25%. Because the whole circle is 100% of 60 students, the bus slice stands for 50% of 60 = 30 students, and each 25% slice stands for 15 students. The slices $25\% + 50\% + 25\% = 100\%$, as they must.

Tip: On a circle graph the percents *always add to* 100%. On any graph, first read the **title** and **labels** so you know what the numbers mean before you compare them.

Try it: Quick check

Find the median of 9, 3, 7, 1, 5.

Work it on paper before revealing the answer.

Answer. Answer: Order $1, 3, 5, 7, 9$; median $= 5$

6.4 Interpreting Data from Tables and Graphs

The Rule

To **interpret** data, read carefully and answer the exact question asked. Good questions to ask yourself:

- Which category is *largest* or *smallest*?
- What is the *total*, or the *difference* between two values?
- Is the data *increasing*, *decreasing*, or staying the same over time?

Worked Example: interpreting a line graph

A plant's height was measured each week: Week 1: 2 cm, Week 2: 5 cm, Week 3: 6 cm, Week 4: 10 cm.

The plant **grew every week**, so the line rises the whole time. The greatest growth was between Weeks 3 and 4: $10 - 6 = 4$ cm. Total growth over the whole period was $10 - 2 = 8$ cm.

Tip: Watch the **scale**. If a bar graph counts by 5s, a bar reaching the third line means 15, not 3.

Try it: Quick check

Find the median of $10,\ 4,\ 8,\ 2$.

Work it on paper before revealing the answer.

Answer. Answer: Order $2, 4, 8, 10$; median $= \dfrac{4+8}{2} = 6$

6.5 Basic Probability

The Rule

Probability measures how likely an event is. An **outcome** is one possible result; a **favorable outcome** is a result you are hoping for. When every outcome is equally likely,

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}.$$

Worked Example: rolling a number cube

A number cube has faces $1, 2, 3, 4, 5, 6$, so there are 6 equally likely outcomes. What is $P(\text{even})$? The favorable outcomes are $2, 4, 6$, so 3 of them.

$$P(\text{even}) = \frac{3}{6} = \frac{1}{2}.$$

What is $P(\text{a } 5)$? Only one face works, so $P = \frac{1}{6}$.

Tip: Every probability is a number **between** 0 **and** 1. If your answer is negative or bigger than 1, you made a mistake. Count the total *before* you count the favorable outcomes.

Try it: Quick check

Find the mode of 2, 4, 4, 7, 9.

Work it on paper before revealing the answer.

Answer. Answer: Mode = 4 (it appears twice)

6.6 Probability as a Fraction, Decimal, and Percent

The Rule

A probability can be written three equivalent ways. Start with the fraction, divide to get the decimal, then multiply by 100 to get the percent. On the "likelihood" line:

- $P = 0$ means the event is **impossible**.
- $P = 1$ (that is, 100%) means it is **certain**.
- $P = \frac{1}{2}$ (50%) means it is **equally likely** to happen or not. Larger than $\frac{1}{2}$ is **likely**; smaller is **unlikely**.

Worked Example: three forms

A bag has 20 marbles, and 5 are red. Then

$$P(\text{red}) = \frac{5}{20} = \frac{1}{4} = 0.25 = 25\%.$$

Since 25% is less than 50%, drawing red is **unlikely**. Drawing "a marble" at all is **certain**: $P = \frac{20}{20} = 1 = 100\%$. Drawing a green marble is **impossible** (there are none): $P = 0$.

Tip: To turn a fraction into a percent, divide top by bottom, then move the decimal point two places right. $\frac{3}{4} = 0.75 = 75\%$.

Try it: Quick check

For the data 4, 8, 6, 4, 3, find the mean, the median, and the mode.

Work it on paper before revealing the answer.

Answer. Answer: Order $3, 4, 4, 6, 8$: mean $= \frac{25}{5} = 5$, median $= 4$, mode $= 4$

6.7 Compound Events and the Counting Principle

The Rule

A **compound event** involves more than one thing happening (such as flipping a coin *and* rolling a cube). The **Counting Principle** says: if one choice can happen in m ways and a second choice in n ways, then together they can happen in

$$m \times n \text{ ways.}$$

This gives the total number of outcomes, which you can then use in a probability fraction.

Worked Example: counting outfits

You have 3 shirts and 4 pairs of shorts. How many different outfits? By the Counting Principle, $3 \times 4 = \mathbf{12}$ outfits.

Worked Example: coin and cube together

Flip a coin (2 outcomes: H, T) and roll a cube (6 outcomes). Total outcomes $= 2 \times 6 = 12$. What is $P(\text{heads and a 3})$? Only one pair works out of 12, so

$$P(\text{H and 3}) = \frac{1}{12}.$$

Tip: For "and" with the Counting Principle you **multiply** the numbers of ways. A tree diagram is a great way to list every outcome when the numbers are small.

Try it: Quick check

Find the range of 3, 8, 5, 12, 6.

Work it on paper before revealing the answer.

Answer. Answer: Range $= 12 - 3 = 9$

6.8 Real-World Data and Probability Problems

The Rule

Word problems combine these ideas. Read slowly and decide what is being asked: a **center** (mean, median, mode), a **spread** (range), a reading from a **graph**, or a **probability**. Then set up the matching calculation and check that your answer makes sense.

Worked Example: putting it together

In five games a player scored 12, 8, 15, 12, 3 points.

Mean: $\dfrac{12+8+15+12+3}{5} = \dfrac{50}{5} = 10$ points.

Median (order $3, 8, 12, 12, 15$): the middle value is 12.

Mode: 12 (it appears twice). **Range:** $15 - 3 = 12$.

If the coach picks one of these five games at random to rewatch, the probability it is a game where the player scored more than 10 points (the 12, 15, and 12 games = 3 games) is $\frac{3}{5} = 0.6 = 60\%$ — **likely**.

Tip: After solving, reread the question. Answers should have the right **units** (points, students, %) and land in a sensible range — probabilities between 0 and 1, and a mean somewhere between the smallest and largest data values.

Try it: Quick check

Find the range of 22, 15, 19, 27, 20.

Work it on paper before revealing the answer.

Answer. Answer: Range $= 27 - 15 = 12$

6.9 Going Deeper: Advanced Statistics & Probability

The Weighted Mean

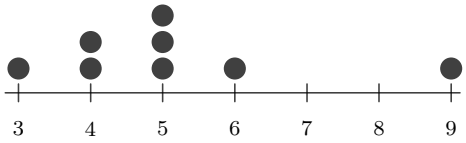
A plain mean treats every value equally. A **weighted mean** lets some values count more than

others. If each value x has a **weight** w (how many times it counts), then

$$\text{weighted mean} = \frac{\sum(x \cdot w)}{\sum w} = \frac{x_1 w_1 + x_2 w_2 + \cdots}{w_1 + w_2 + \cdots}.$$

This is exactly how a course grade works when tests count more than homework.

One outlier drags the mean but not the median.



Worked Example: a weighted class grade

Homework is worth 20%, quizzes 30%, and the final exam 50%. A student earns 90, 80, and 70 on these. Multiply each score by its weight (as a decimal) and add:

$$\text{grade} = 90(0.20) + 80(0.30) + 70(0.50) = 18 + 24 + 35 = \mathbf{77}.$$

The plain average would be $\dfrac{90+80+70}{3} = 80$, but because the heavily weighted final was lowest, the weighted grade drops to 77. The weights $0.20 + 0.30 + 0.50 = 1$ must add to 1 (that is 100%).

Measuring Spread: MAD and Standard Deviation

Range uses only two values. Two better measures of spread use *every* value's distance from the mean.

- **Mean Absolute Deviation (MAD):** average how far each value is from the mean, using positive distances.

$$\text{MAD} = \frac{\sum |x - \bar{x}|}{n}.$$

- **Standard deviation (σ):** instead of absolute values, *square* each distance, average the squares (this average is the **variance**), then take the square root.

$$\sigma = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}.$$

For both, a **larger** value means the data are more spread out. Standard deviation is the one you will meet most in later courses.

Worked Example: MAD and standard deviation

Find the MAD and standard deviation of 2, 4, 6, 8, 10. First the mean: $\bar{x} = \dfrac{2+4+6+8+10}{5} = \dfrac{30}{5} = 6$. Now each distance from 6:

MAD: $\dfrac{4+2+0+2+4}{5} = \dfrac{12}{5} = \mathbf{2.4}$.

Standard deviation: the squared distances sum to $16+4+0+4+16 = 40$, so the variance is $\dfrac{40}{5} = 8$ and $\sigma = \sqrt{8} \approx \mathbf{2.83}$. Both numbers describe the "typical" distance of a value from the mean.

Quartiles, IQR, and Outliers

The **median** splits ordered data in half. The **quartiles** split it into quarters:

- Q_1 (lower quartile) is the median of the **lower half**.
- Q_2 is the overall median.
- Q_3 (upper quartile) is the median of the **upper half**.

The **interquartile range** is $\text{IQR} = Q_3 - Q_1$; it measures the spread of the middle 50% of the data and ignores extremes. A value is usually called an **outlier** if it lies below $Q_1 - 1.5(\text{IQR})$ or above $Q_3 + 1.5(\text{IQR})$.

Worked Example: quartiles, IQR, and an outlier check

For 4, 7, 8, 9, 11, 12, 20 (already in order, $n = 7$), the median is the 4th value, $Q_2 = 9$. The lower half is $4, 7, 8$, so $Q_1 = 7$; the upper half is $11, 12, 20$, so $Q_3 = 12$. Then

$$\text{IQR} = Q_3 - Q_1 = 12 - 7 = 5.$$

Outlier fences: $Q_1 - 1.5(5) = 7 - 7.5 = -0.5$ and $Q_3 + 1.5(5) = 12 + 7.5 = 19.5$. Since $20 > 19.5$, the value **20** is an **outlier**.

Permutations and Combinations

When the Counting Principle is used to arrange or choose from one group, two special cases appear. Write $n! = n(n-1)(n-2)\cdots 1$ (read "n factorial").

- A **permutation** counts arrangements where **order matters** (like finishing 1st, 2nd,

3rd):

$$ {}_nP_r = \frac{n!}{(n-r)!}. $$

- A **combination** counts selections where **order does not matter** (like choosing a team):

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!}. $$

Because order matters in more cases, ${}_nP_r$ is always at least as large as ${}_nC_r$.

Worked Example: order matters vs. order does not

From 5 runners, how many ways can they take gold, silver, and bronze? Order matters, so

$$ {}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = \mathbf{60}. $$

From those same 5 runners, how many ways can we pick 3 to form a relay team (no ordering)? Now

$$ {}_5C_3 = \frac{5!}{3!\,2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = \mathbf{10}. $$

Each unordered team of 3 can be ordered in $3! = 6$ ways, and indeed $10 \times 6 = 60$, matching the permutation count.

Expected Value

The **expected value** is the long-run average outcome of a random situation. Multiply each outcome's value by its probability, then add:

$$ E = \sum (\text{value} \times \text{probability}). $$

It need not be a possible single outcome; it is what you would average over many, many repeats.

Worked Example: is the game fair?

A game costs \$2 to play. You roll a number cube: rolling a 6 wins you \$10; anything else wins nothing. The expected winnings are

$$ E = 10 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} = \frac{10}{6} \approx \$1.67. $$

Since the expected win \$1.67 is less than the \$2 cost, the game favors the house: on average you **lose** about \$0.33 each play.

Independent vs. Dependent Events and Conditional Probability

Two events are **independent** if one happening does not change the other's probability (like two separate coin flips). They are **dependent** if it does (like drawing cards *without* replacing them).

- Independent "and": $P(A \text{ and } B) = P(A) \cdot P(B)$.
- Dependent "and": $P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$, where $P(B \text{ given } A)$ is the **conditional probability** of B once A has happened.

Worked Example: drawing without replacement

A bag has 3 red and 2 blue marbles (5 total). You draw two marbles *without* putting the first back. Find $P(\text{both red})$. The first draw is red with probability $\frac{3}{5}$. Now only 2 red remain out of 4 marbles, so the second is red with probability $\frac{2}{4}$. These are **dependent**, so multiply:

$$P(\text{both red}) = \frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20} = \frac{3}{10}.$$

If instead you *replaced* the first marble, the draws would be independent and $P = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$.

Tip: The word "**and**" tells you to *multiply* probabilities; the word "**or**" (for events that cannot both happen) tells you to *add* them. Always ask whether the second event's probability changed — if it did, the events are dependent.

"At Least One" via the Complement

The **complement** of an event is everything *except* that event, and $P(\text{not } A) = 1 - P(A)$. Problems that ask for the probability of "**at least one**" success are usually far easier through the complement, because "at least one" is the opposite of "**none**":

$$P(\text{at least one}) = 1 - P(\text{none}).$$

Worked Example: at least one head

Flip a fair coin 3 times. Find $P(\text{at least one head})$. Listing all the winning cases is tedious, so use the complement. The only way to get *no* heads is all tails: $P(\text{TTT}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Therefore

$$P(\text{at least one head}) = 1 - \frac{1}{8} = \frac{7}{8}.$$

Tip: Whenever a probability question contains the phrase "at least one," try the complement first: find $P(\text{none})$ and subtract from 1. It almost always saves work.

6.10 Why It Works

Probability as favorable/total only works when every outcome is equally likely, which is why organized listing matters: it makes the outcomes genuinely equal. The mean balances the data (deviations sum to zero), so an outlier shifts it; the median only counts positions, so an outlier cannot move it more than one place. Choosing the right summary depends on which kind of question you are answering.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

7 Whole Numbers & Operations

Place value, rounding, the four operations, order of operations, and exponents

7.1 Place Value & Reading Large Numbers

The Rule

In our number system, the **place** of a digit tells you its value. Moving left, each place is worth **ten times** more than the one to its right. Reading right to left the places are: ones, tens, hundreds, then thousands, ten-thousands, hundred-thousands, then millions, ten-millions, hundred-millions.

We group digits into **periods** of three (ones, thousands, millions) separated by commas. Read each period, then say its group name.

The four operations come with laws that let you rearrange work: addition and multiplication are commutative ($a + b = b + a$) and associative (grouping does not matter), and multiplication distributes over addition. Mental math is these laws in action — 17×5 is $(10 + 7) \times 5 = 50 + 35$.

Worked Example: reading a large number

Read 6,308,071.

- Millions period: 6 → "six million"
- Thousands period: 308 → "three hundred eight thousand"

- Ones period: 071 → "seventy-one"

Answer: **six million, three hundred eight thousand, seventy-one**. Notice the 0 in the ten-thousands place is a placeholder: it holds the spot so the other digits keep their value.

Where it goes wrong

Assuming subtraction and division are commutative. $10-3 \neq 3-10$ and $10 \div 2 \neq 2 \div 10$; order matters for them.

Worked Example: value of a digit

In 4,752,900, what is the value of the digit 7? The 7 sits in the **hundred-thousands** place, so its value is $7 \times 100{,}000 = 700{,}000$.

Tip: *Expanded form* shows each digit's value added together. For 3,405: $3000 + 400 + 0 + 5$. The zero shows there are no tens.

Try it: Quick check

In the number 5,432, what is the *value* of the digit 4?

Work it on paper before revealing the answer.

Answer. Answer: 400 (hundreds place)

7.2 Rounding & Estimation

The Rule

To **round** to a place value: look at the digit *just to the right* of that place.

- If it is 5 or more, round **up** (add one to the rounding digit).
- If it is 4 or less, round **down** (keep the rounding digit).

Then replace every digit to the right with zeros. **Estimation** means rounding first, then computing, to get a quick, close answer.

Worked Example: rounding

Round 47,381 to the nearest thousand. The thousands digit is 7. The digit to its right is 3

(which is 4 or less), so round down and keep the 7. Zero out the rest: **47,000**.

Worked Example: estimating a sum

Estimate $612 + 289$. Round each to the nearest hundred: $612 \approx 600$ and $289 \approx 300$. Estimate: $600 + 300 = \mathbf{900}$. (The exact answer is 901, so our estimate is very close.)

Tip: A digit of exactly 5 always rounds up. And rounding never changes digits to the *left* of your rounding place unless a carry ripples through, e.g. rounding 1,960 to the nearest hundred gives 2,000.

Try it: Quick check

Write the number 3,072 in words.

Work it on paper before revealing the answer.

Answer. Answer: three thousand, seventy-two

7.3 Adding & Subtracting Whole Numbers

The Rule

Line up numbers by place value (ones under ones). **Add** from right to left; when a column totals 10 or more, **carry** the extra ten to the next column. **Subtract** from right to left; when the top digit is too small, **borrow** (regroup) one unit from the next place to the left.

Worked Example: addition with carrying

$487 + 356$. Ones: $7 + 6 = 13$, write 3 carry 1. Tens: $8 + 5 + 1 = 14$, write 4 carry 1. Hundreds: $4 + 3 + 1 = 8$. Answer: **843**.

Worked Example: subtraction with borrowing

$803 - 547$. Ones: $3 - 7$ can't be done, borrow: $13 - 7 = 6$. Tens: after borrowing the 0 became 9 (we borrowed across the zero), $9 - 4 = 5$. Hundreds: $7 - 5 = 2$. Answer: **256**. Check: $256 + 547 = 803$.

Tip: Addition and subtraction undo each other. Always check a subtraction by adding your answer back to the number you subtracted.

Try it: Quick check

Which digit is in the *thousands* place in 68,415?

Work it on paper before revealing the answer.

Answer. Answer: 8

7.4 Multiplying Whole Numbers

The Rule

To multiply multi-digit numbers, multiply by each digit of the bottom number one at a time, shifting each partial product one place left (because each new digit is worth ten times more). Then add the partial products.

Worked Example: two-digit multiplication

34×26.

- $34 \times 6 = 204$
- $34 \times 20 = 680$ (shifted one place left)
- Add: $204 + 680 = \mathbf{884}$

Worked Example: multiplying by a round number

250×40. Multiply the non-zero parts: $25 \times 4 = 100$. Then attach the total number of trailing zeros ($1 + 1 = 2$ zeros plus the zero already in 100): $250 \times 40 = \mathbf{10{,}000}$.

Tip: Multiplication is repeated addition, and order does not matter: $34 \times 26 = 26 \times 34$. To multiply by 10, 100, or 1000, just attach that many zeros.

Try it: Quick check

Round 47 to the nearest ten.

Work it on paper before revealing the answer.

Answer. Answer: 50

7.5 Dividing Whole Numbers (Long Division)

The Rule

Division splits a total into equal groups. In $a \div b$, a is the **dividend**, b is the **divisor**, and the result is the **quotient**. Long division repeats four steps: **Divide, Multiply, Subtract, Bring down**. Whatever is left at the end that will not divide evenly is the **remainder**.

Worked Example: long division with a remainder

$947 \div 4$.

- $9 \div 4 = 2$ remainder 1; write 2, subtract 8, bring down 4 to make 14.
- $14 \div 4 = 3$ remainder 2; write 3, subtract 12, bring down 7 to make 27.
- $27 \div 4 = 6$ remainder 3; write 6.

Answer: **236** R **3**. Check: $236 \times 4 = 944$, and $944 + 3 = 947$.

Worked Example: exact division

$156 \div 6$. $15 \div 6 = 2$ R 3; bring down 6 to make 36; $36 \div 6 = 6$. Answer: **26**, no remainder.

Tip: Check any division with: (quotient $\times$ divisor) + remainder = dividend. The remainder must always be *smaller* than the divisor.

Try it: Quick check

Round 830 to the nearest hundred.

Work it on paper before revealing the answer.

Answer. Answer: 800

7.6 Order of Operations (PEMDAS)

The Rule

When an expression has several operations, do them in this order:

1. **P**arentheses (grouping symbols) first.
2. **E**xponents next.
3. **M**ultiplication and **D**ivision, left to right.
4. **A**ddition and **S**ubtraction, left to right.

Multiplication and division rank equally, so you handle them in reading order. The same is true of addition and subtraction.

Worked Example: PEMDAS step by step

Evaluate $5 + 3 \times (8 - 6)^2$.

- Parentheses: $8 - 6 = 2$, giving $5 + 3 \times 2^2$.
- Exponent: $2^2 = 4$, giving $5 + 3 \times 4$.
- Multiply: $3 \times 4 = 12$, giving $5 + 12$.
- Add: $\mathbf{17}$.

Worked Example: left-to-right

Evaluate $20 \div 4 \times 2$. Division and multiplication tie, so go left to right: $20 \div 4 = 5$, then $5 \times 2 = \mathbf{10}$. (Doing the multiply first would wrongly give 2.)

Tip: PEMDAS does not mean "all multiplication before all division." M and D share a rank; A and S share a rank. Ties are always broken left to right.

Try it: Quick check

Estimate $58 + 41$ by first rounding each number to the nearest ten.

Work it on paper before revealing the answer.

Answer. Answer: $60 + 40 = 100$

7.7 Exponents & Perfect Squares

The Rule

An **exponent** tells how many times to use the **base** as a factor. In b^n, b is the base and n is the exponent: $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$. We read 5^2 as "five squared" and 5^3 as "five cubed." A **perfect square** is a whole number times itself, like $1, 4, 9, 16, 25, \ldots$

Worked Example: evaluating powers

$2^4 = 2 \times 2 \times 2 \times 2 = \mathbf{16}$. $10^3 = 10 \times 10 \times 10 = \mathbf{1000}$. $7^2 = 7 \times 7 = \mathbf{49}$.

Worked Example: perfect squares

The first several perfect squares: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$, $10^2 = 100$. Memorizing these speeds up a lot of later math.

Tip: An exponent is *not* multiplication of the base by the exponent. $3^2 = 9$, *not* $3 \times 2 = 6$. And any base to the first power is itself: $8^1 = 8$.

Try it: Quick check

$245 + 132$

Work it on paper before revealing the answer.

Answer. Answer: 377

7.8 Word Problems: Putting It Together

The Rule

To solve a word problem: (1) read carefully and find what is asked, (2) decide which operation(s) fit — combining suggests addition, taking away suggests subtraction, equal groups suggest multiplication, and splitting into equal parts suggests division, (3) compute in the right order, and (4) check that your answer makes sense.

Worked Example: multi-step problem

A school orders 8 boxes of markers with 24 markers each, then hands out 150 markers. How many are left?

- Total ordered: $8 \times 24 = 192$ markers.
- After handing out 150: $192 - 150 = \mathbf{42}$ markers left.

Tip: Underline the numbers and the question. A quick estimate first tells you whether your final answer is reasonable.

Try it: Quick check

$68 + 57$

Work it on paper before revealing the answer.

Answer. Answer: 125

7.9 Going Deeper: Advanced Whole-Number Ideas

The Division Algorithm (the exact rule behind remainders)

Every long-division problem is really a statement about whole numbers. For any dividend a and any divisor $b > 0$, there is **exactly one** pair of whole numbers — a quotient q and a remainder r — such that

$$a = bq + r, \qquad \text{where } 0 \le r < b.$$

This is called the **Division Algorithm**. The condition $0 \le r < b$ is what forces a *single* correct answer: the remainder is never negative and never as large as the divisor. This one equation is the foundation for fractions, modular arithmetic, and much of the number theory you will meet later.

Modular Arithmetic & Clock Arithmetic

Often we care only about the **remainder** of a division, not the quotient. We write

$$a \bmod b = r,$$

meaning r is the remainder when a is divided by b. A clock is the everyday example: it works "mod 12." If it is 9 o'clock and 7 hours pass, we do not say 16 o'clock — we compute $16 \bmod 12 = 4$, so it is 4 o'clock. Numbers that share the same remainder are called **congruent**; for example 16 and 4 are congruent mod 12 because both leave remainder 4.

Worked Example: clock arithmetic

A 24-hour trip starts at 10 o'clock. On a 12-hour clock, what time does it end?

- Add the hours: $10 + 24 = 34$.
- Reduce mod 12: $34 \div 12 = 2$ remainder 10, so $34 \bmod 12 = 10$.

The trip ends at **10** o'clock. This makes sense: 24 hours is exactly two full loops of the clock, so the hand returns to where it started.

Why the Divisibility Rules Work

The quick tests for divisibility are not magic — each one comes from place value.

- **By 2, 5, or 10:** only the ones digit matters, because 10, 100, 1000, ... are all divisible by 2, 5, and 10. So the whole number is divisible exactly when its last digit is.

- **By 3 or 9:** add the digits. Since $10 = 9 + 1$, $100 = 99 + 1$, and so on, each place is a multiple of 9 plus 1. Subtracting off the multiples of 9 leaves just the *sum of the digits*, so the number is divisible by 9 (or 3) exactly when that digit sum is.
- **By 4:** check the last *two* digits, because 100 is divisible by 4, so everything above the hundreds place already is.

Worked Example: testing divisibility by 9

Is 6,308,071 divisible by 9? Add the digits: $6+3+0+8+0+7+1 = 25$. Add again: $2+5 = 7$. Since 7 is not divisible by 9, **neither is the original number**. (Its remainder mod 9 is even predictable: it is 7.)

Number Bases: Counting in Binary

Our usual system is **base ten**: each place is a power of 10 $(\ldots, 1000, 100, 10, 1)$. Computers use **base two** (binary), where each place is a power of 2 $(\ldots, 8, 4, 2, 1)$ and the only digits are 0 and 1. To convert a whole number to binary, repeatedly divide by 2 and record the remainders from *bottom to top*.

Worked Example: converting 13 to binary

Convert 13 to base two by dividing by 2 over and over:

- $13 \div 2 = 6$ remainder **1**
- $6 \div 2 = 3$ remainder **0**
- $3 \div 2 = 1$ remainder **1**
- $1 \div 2 = 0$ remainder **1**

Read the remainders bottom-up: $13 = \mathbf{1101_2}$. Check by expanding the places: $1\times8 + 1\times4 + 0\times2 + 1\times1 = 8 + 4 + 1 = 13$. ✓

Triangular Numbers & the Sum $1 + 2 + \cdots + n$

The **triangular numbers** count dots stacked in a growing triangle: $1, 3, 6, 10, 15, \ldots$, where the nth one is the sum $1+2+3+\cdots+n$. Instead of adding term by term, there is a beautiful shortcut:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Why it works (Gauss's trick): write the sum forwards and backwards and add the two lines in columns:

$$\begin{array}{ccccccc} 1 & + & 2 & + & \cdots & + & n \\ n & + & (n-1) & + & \cdots & + & 1 \end{array}$$

Every column adds to $n+1$, and there are n columns, giving $n(n+1)$ for *twice* the sum — so the sum itself is $\dfrac{n(n+1)}{2}$.

Worked Example: adding 1 through 100

Legend says the young Gauss found $1+2+\cdots+100$ in seconds. Use the formula with $n = 100$:

$$\frac{100 \times 101}{2} = \frac{10{,}100}{2} = \mathbf{5{,}050}.$$

No need to add one hundred numbers by hand.

Fast Mental-Math & Estimation Tricks

A few reliable shortcuts:

- **Multiply by 5:** multiply by 10 and halve. So $48 \times 5 = 480 \div 2 = 240$.
- **Square a number ending in 5:** for a two-digit number $\underline{d}5$, compute $d \times (d{+}1)$ and attach "25." So 35^2: $3 \times 4 = 12$, giving **1225**.
- **Front-end estimation:** add just the leading digits for a fast lower estimate, then refine. For $612 + 289$: $600 + 200 = 800$, plus the tens gives about 900.

Where this leads: The Division Algorithm grows into fractions and the study of prime numbers; modular arithmetic underlies calendars, music, and the encryption that protects online banking; number bases run every computer; and the sum $1 + 2 + \cdots + n$ is your first taste of the *series* you will meet in algebra and calculus. Advanced math is mostly these whole-number ideas, pushed further.

7.10 Why It Works

The distributive property is a fact about rectangles: a rectangle of width $b + c$ and height a splits into two rectangles of areas ab and ac, so $a(b + c) = ab + ac$. Commutativity of multiplication is the same rectangle turned on its side. These are not rules to memorize but pictures — and every algorithm for multiplication, from the area model to long multiplication, is the distributive property organized on paper.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

8 Integers

Everything you need to know about negative numbers

8.1 What Integers Are

The Big Idea

The **integers** are the whole numbers together with their negatives and zero:

$$\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots$$

Integers do *not* include fractions or decimals. A **positive** integer is greater than 0, a **negative** integer is less than 0, and 0 itself is neither positive nor negative.

Opposites sit the same distance from zero.



Integers are the whole numbers and their negatives. Adding integers is combining moves on the number line; subtracting is adding the opposite; multiplying and dividing follow the sign rule (same signs positive, different signs negative). Absolute value gives the size of an integer without its sign.

8.2 The Number Line

Every integer has a home on the number line. Positive numbers sit to the right of 0, negative numbers sit to the left. As you move **right**, numbers get **bigger**; as you move **left**, numbers get **smaller**.

8.3 Opposites

Two numbers are **opposites** if they are the same distance from 0 but on opposite sides. The opposite of 5 is -5, and the opposite of -8 is 8. The opposite of 0 is 0.

Finding Opposites

The opposite of -7 is 7, because both are 7 steps from 0 (one left, one right).

Writing a negative sign in front means "the opposite of": $-(-3) = 3$. Two opposites in a row cancel out.

Where it goes wrong

Applying the multiplication sign rule to addition. $-4 + (-6) = -10$; there is no "two negatives make a positive" in addition.

Tip: A negative sign means "opposite of." So $-(-4)$ reads as "the opposite of -4," which is 4.

Try it: Quick check

Write the opposite of 9.

Work it on paper before revealing the answer.

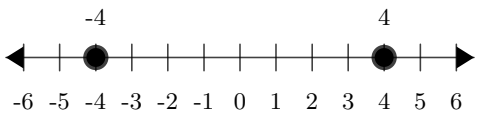
Answer. Answer: -9

8.4 Absolute Value

Definition

The **absolute value** of a number is its distance from 0 on the number line. Distance is never negative, so absolute value is always 0 or positive. We write it with bars: $|-6| = 6$ and $|6| = 6$.

Opposites sit the same distance from zero.



Working with Absolute Value

$|-9| = 9$ $|12| = 12$ $|0| = 0$

Absolute value tells you *how far*, not *which direction*. Both -9 and 9 are 9 units from zero, so both have absolute value 9.

Tip: To find $|x|$, just drop the sign. But be careful: a negative sign *outside* the bars stays, so $-|-5| = -(5) = -5$.

Try it: Quick check

Write the opposite of -15.

Work it on paper before revealing the answer.

Answer. Answer: 15

8.5 Comparing and Ordering Integers

The Rule

On the number line, the number farther to the **right** is **greater**. So any positive number is greater than any negative number, and for two negatives, the one closer to 0 is greater. That means $-2 > -7$, because -2 is to the right of -7.

Ordering from Least to Greatest

Order $3,\ -5,\ 0,\ -1,\ 4$ from least to greatest.
Picture them on the number line and read left to right:

$$-5 < -1 < 0 < 3 < 4.$$

Real-World Context: Temperature

On a cold day the temperatures were $-3°$, $5°$, $-8°$, and $0°$. Which is coldest?
Coldest means smallest (farthest left): $-8°$. Warmest is $5°$. In order from coldest to warmest: $-8°,\ -3°,\ 0°,\ 5°$.

Tip: With negatives, "bigger digits" can be *smaller* numbers. -100 is much less than -2. Always think about position on the line.

Try it: Quick check

Simplify $-(-6)$.

Work it on paper before revealing the answer.

Answer. Answer: 6

8.6 Adding Integers

Two Cases

Same signs: add the absolute values and keep the common sign.
Different signs: subtract the smaller absolute value from the larger, and keep the sign of the number with the larger absolute value.

Same Signs

$-6 + (-4)$: both negative, so add $6 + 4 = 10$ and keep the negative sign: -10.
$7 + 5 = 12$: both positive, answer stays positive.

Different Signs

$-9 + 4$: subtract $9 - 4 = 5$; since $9 > 4$, keep the negative sign: -5.
$8 + (-3)$: subtract $8 - 3 = 5$; since $8 > 3$, keep the positive sign: 5.

Sign rule for adding: same signs → add and keep the sign; different signs → subtract and take the sign of the "bigger" number.

Try it: Quick check

Find $|-8|$.

Work it on paper before revealing the answer.

Answer. Answer: 8

8.7 Subtracting Integers

Add the Opposite

To subtract an integer, **add its opposite**. Change the subtraction to addition and flip the sign of the number being subtracted:

$$a - b = a + (-b).$$

Then follow the rules for adding integers.

Opposites sit the same distance from zero.



Subtract Means Add the Opposite

$5 - 8 = 5 + (-8) = -3.$
$-4 - 6 = -4 + (-6) = -10.$
$3 - (-7) = 3 + (+7) = 10.$ (Subtracting a negative adds!)

Key idea: *Subtracting means adding the opposite.* A minus sign in front of a negative, like $-(-7)$, becomes a plus.

Try it: Quick check

Find $|5|$.

Work it on paper before revealing the answer.

Answer. Answer: 5

8.8 Multiplying Integers

Sign Rules for Products

Multiply the absolute values, then decide the sign:

- positive $\times$ positive = positive
- negative $\times$ negative = positive
- positive $\times$ negative = negative (either order)

Same signs give a **positive**; different signs give a **negative**.

Multiplying

$(-6)(-4) = 24$ (same signs $\rightarrow$ positive)

$(-7)(3) = -21$ (different signs → negative)
$(8)(-5) = -40$ (different signs → negative)

Sign rule: an *even* number of negative factors gives a positive answer; an *odd* number of negative factors gives a negative answer.

Try it: Quick check

Find $|-13|$.

Work it on paper before revealing the answer.

Answer. Answer: 13

8.9 Dividing Integers

Same Sign Rules as Multiplying

Division follows the exact same sign rules as multiplication. Divide the absolute values, then apply: same signs → positive, different signs → negative.

Dividing

$\frac{-20}{-4} = 5$ (same signs → positive)
$\frac{36}{-9} = -4$ (different signs → negative)
$\frac{-30}{6} = -5$ (different signs → negative)

Tip: Multiplication and division share one motto: *same signs → positive, different signs → negative.*

Try it: Quick check

Compare using $<$ or $>$: -4 _____ 3.

Work it on paper before revealing the answer.

Answer. Answer: $-4 < 3$

8.10 Order of Operations with Integers

PEMDAS Still Rules

Follow the usual order: **P**arentheses, **E**xponents, **M**ultiply/**D**ivide (left to right), **A**dd/**S**ubtract (left to right). The only new thing is keeping careful track of *signs* at every step.

A Multi-Step Problem

Evaluate $-3 + 2 \times (-4)$.

$$\begin{aligned} -3 + 2 \times (-4) &= -3 + (-8) && \text{multiply first} \\ &= -11 && \text{then add} \end{aligned}$$

With Parentheses and Exponents

Evaluate $(-2)^2 - 5(3 - 7)$.

$$\begin{aligned} (-2)^2 - 5(3 - 7) &= 4 - 5(-4) && \text{exponent and parentheses} \\ &= 4 + 20 && \text{multiply, then add the opposite} \\ &= 24 \end{aligned}$$

Note $(-2)^2 = (-2)(-2) = 4$, but $-2^2 = -(2 \times 2) = -4$. Parentheses matter!

Tip: A negative in parentheses raised to a power, like $(-2)^2$, is different from -2^2. Watch where the parentheses are.

Try it: Quick check

Compare using $<$ or $>$: $\quad -7 _____ -2$.

Work it on paper before revealing the answer.

Answer. Answer: $-7 < -2$

8.11 Word Problems with Integers

Translating Situations

Real situations often use negatives: temperatures below zero, money *owed* (debt), and elevation *below* sea level. Rising, gaining, and depositing are positive; falling, losing, and spending are negative.

Temperature Change

At dawn it was $-6°$C. By noon the temperature rose 9 degrees. What was the noon temperature?
Rising means adding: $-6 + 9 = 3$. The noon temperature was $3°$C.

Money Owed and Elevation

Money: Maya owes her brother \$12 (that is -12) and pays back \$5. Her balance is $-12+5 = -7$, so she still owes \$7.

Elevation: A diver is 18 m below sea level (-18) and descends 7 more meters. New depth: $-18 + (-7) = -25$, so 25 m below sea level.

Tip: Choose a positive direction (up, gain, deposit) and make the opposite negative. Then just add the signed amounts.

8.12 Going Deeper: Advanced Integer Ideas

Why $(-)(-) = (+)$: A Real Proof

The rule "a negative times a negative is positive" is not arbitrary—it is *forced* by the distributive property, $a(b + c) = ab + ac$. Watch:

$$(-1)\big(1 + (-1)\big) = (-1)(0) = 0.$$

But distributing the same expression gives

$$(-1)(1) + (-1)(-1) = -1 + (-1)(-1).$$

Both equal 0, so $-1 + (-1)(-1) = 0$, which means $(-1)(-1)$ must be $+1$. Any other value would break arithmetic. Two negatives *must* make a positive.

Opposites sit the same distance from zero.



Closure and Properties of the Integers

A set is **closed** under an operation if combining any two members always lands you back inside the set. The integers are closed under **addition**, **subtraction**, and **multiplication**: adding, subtracting, or multiplying two integers always yields another integer. They are *not* closed under **division**, since $3 \div 2 = 1.5$ is not an integer. Other key properties for all integers a, b, c:

- **Commutative** (add, multiply): $a + b = b + a$ and $ab = ba$.
- **Associative** (add, multiply): $(a + b) + c = a + (b + c)$.
- **Distributive:** $a(b + c) = ab + ac$.
- **Identities:** $a + 0 = a$ and $a \times 1 = a$.
- **Additive inverse:** every integer a has an opposite $-a$ with $a + (-a) = 0$.

Subtraction is *not* commutative or associative: $5 - 3 \neq 3 - 5$.

Modular Arithmetic with Negatives

When we divide, the **remainder** is what is left over. Mathematicians agree that the remainder should always be *non-negative*, even when the number is negative. So we find $a \bmod n$ by adding or subtracting copies of n until we land in the range 0 to $n - 1$.

For example, $-7 \bmod 5$: add 5 twice, $-7 + 5 + 5 = 3$, so $-7 \equiv 3 \pmod 5$. Check: $-7 = (-2)(5) + 3$, and the remainder 3 sits in $\{0, 1, 2, 3, 4\}$. This is exactly how a clock works—"3 hours before 2 o'clock" wraps around to 11.

Worked Example: Remainders of Negatives

Find $-23 \bmod 7$.

$$\begin{aligned} -23 &= q \cdot 7 + r, \quad 0 \le r < 7 && \text{want non-negative remainder} \\ -23 + 7 + 7 + 7 + 7 &= 5 && \text{add 7 four times} \\ -23 &= (-4)(7) + 5 && \text{so } q = -4,\ r = 5 \end{aligned}$$

Therefore $-23 \equiv 5 \pmod 7$. Notice $q = -4$, *not* -3: we round the quotient **down** so the remainder stays non-negative.

Tip: For a negative number mod n, keep adding n until you reach a value between 0 and $n - 1$. That value is the true remainder.

Absolute-Value Equations and Inequalities (Preview)

Because $|x|$ measures distance from 0, an equation like $|x| = 5$ has *two* answers: $x = 5$ or

$x = -5$ (both are 5 units away). More generally:

$$|x| = k \;\Rightarrow\; x = k \;\text{ or }\; x = -k \quad (k \geq 0).$$

Inequalities split into two shapes:

- $|x| < k$ means x is *within* k of zero: $-k < x < k$ (a single band).
- $|x| > k$ means x is *farther* than k from zero: $x < -k$ or $x > k$ (two pieces).

And $|x| = -3$ has **no solution**: distance can never be negative.

Worked Example: Solving $|x - 2| = 6$

The bars say the distance from x to 2 is 6. So $x - 2$ is either 6 or -6:

$$\begin{aligned} x - 2 &= 6 \quad &\Rightarrow \quad x &= 8 \\ x - 2 &= -6 \quad &\Rightarrow \quad x &= -4 \end{aligned}$$

The two solutions are $x = 8$ and $x = -4$. Check: $|8 - 2| = |6| = 6$ and $|-4 - 2| = |-6| = 6$. Both work.

Sums of Consecutive Integers

The sum of the integers from 1 to n has a famous closed form (Gauss's trick: pair the first with the last):

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

For example, $1 + 2 + \cdots + 100 = \dfrac{100 \cdot 101}{2} = 5050$. This also reveals a neat fact: the sum of any run of consecutive integers that is *symmetric* about 0, such as $-4 + (-3) + \cdots + 3 + 4$, is 0, because every term cancels its opposite.

Telescoping Sums

A **telescoping** sum collapses because inner terms cancel in pairs. Consider

$$(1 - 2) + (2 - 3) + (3 - 4) + (4 - 5).$$

Rewrite without the grouping and cancel adjacent opposites:

$$1 \underbrace{-2 + 2}_{0} \underbrace{-3 + 3}_{0} \underbrace{-4 + 4}_{0} - 5 = 1 - 5 = -4.$$

Only the very first and very last numbers survive. Telescoping turns a long chain of signed integers into a quick subtraction.

Big picture: the integer sign rules, closure, and Gauss's formula are the foundation for algebra. Master signed arithmetic now and equations, sequences, and modular patterns all become far easier later.

8.13 Why It Works

The sign rules are forced by the distributive property. Since $(-2) \times 0 = 0$ and $0 = 3 + (-3)$, we get $(-2)(3) + (-2)(-3) = 0$, i.e. $-6 + (-2)(-3) = 0$, so $(-2)(-3) = 6$. Any other value would break the law $a(b+c) = ab + ac$ that all of arithmetic relies on. Subtraction as "add the opposite" follows from $a - b$ being the number that added to b gives a.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

9 Factors & Multiples

Everything about factors, multiples, primes, GCF, and LCM

9.1 Factors of a Number

What a Factor Is

A **factor** of a number divides into it *evenly*, leaving no remainder. If $a \times b = n$, then both a and b are factors of n.

To find *all* the factors of a number, work in **pairs**. Start at 1 and count upward: for each number that divides evenly, write down both partners. Stop once the pairs start to repeat.

Every whole number greater than 1 factors into primes in exactly one way — the fundamental theorem of arithmetic — and that factorization answers most questions about it. The greatest common factor takes the smallest power of each shared prime; the least common multiple takes the largest power of every prime that appears.

Finding every factor of 36

Test each number in order and record the factor pairs:

$$1 \times 36, \quad 2 \times 18, \quad 3 \times 12, \quad 4 \times 9, \quad 6 \times 6.$$

(Note 5 does not divide 36, so it is skipped.) Collecting both partners in order:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36.$$

So 36 has 9 factors.

Where it goes wrong

Mixing up GCF and LCM. The GCF is never larger than either number; the LCM is never smaller. If your "GCF" of 12 and 18 is 36, you found the LCM.

Every whole number has 1 and *itself* as factors. When a factor pair is a number times itself (like $6 \times 6 = 36$), that number is listed only *once* — this happens for perfect squares.

Try it: Quick check

List *all* the factors of 12.

Work it on paper before revealing the answer.

Answer. Answer: $1, 2, 3, 4, 6, 12$.

9.2 Multiples of a Number

What a Multiple Is

A **multiple** of a number is what you get by multiplying it by $1, 2, 3, 4, \ldots$ In other words, it is any number in that number's "times table." Multiples go on forever, while factors are a short, finite list.

$$\text{Multiples of } 6: \quad 6,\ 12,\ 18,\ 24,\ 30,\ 36,\ \ldots$$

Listing multiples and checking membership

List the first five multiples of 8, then decide whether 54 is a multiple of 8.

$$\begin{aligned} \text{Multiples of } 8: &\quad 8,\ 16,\ 24,\ 32,\ 40. \\ \text{Is 54 a multiple?} &\quad 54 \div 8 = 6 \text{ remainder } 6 \ \Rightarrow\ \text{No.} \end{aligned}$$

Because the division leaves a remainder, 54 is *not* a multiple of 8.

"Factor" and "multiple" are opposite viewpoints of the same fact: since $4 \times 5 = 20$, we can say 4 is a **factor** of 20 *and* 20 is a **multiple** of 4. Factors are smaller (or equal); multiples are larger (or equal).

Try it: Quick check

List *all* the factors of 18.

Work it on paper before revealing the answer.

Answer. Answer: $1, 2, 3, 6, 9, 18$.

9.3 Prime and Composite Numbers

Two Kinds of Whole Numbers

- A **prime number** has *exactly two* factors: 1 and itself. The first primes are $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$
- A **composite number** has *more than two* factors — it can be broken into smaller factors.
- The number 1 is **neither** prime nor composite: it has only one factor.
- 2 is the *only even prime*; every other even number has 2 as an extra factor.

Prime or composite?

Classify 23 and 39.

23 : no number from 2 to 4 divides it evenly $\Rightarrow$ **prime**.

39 : $39 = 3 \times 13 \Rightarrow$ **composite**.

To test a number, try dividing by the small primes $2, 3, 5, 7, \ldots$ You only need to check primes up to its square root.

To check whether a number is prime, you do not have to test every smaller number — only primes up to its square root. For 53, checking $2, 3, 5, 7$ is enough (since $7^2 = 49 < 53 < 64 = 8^2$), and none divide it, so 53 is prime.

Try it: Quick check

List *all* the factors of 20.

Multiples of a Number

Work it on paper before revealing the answer.

Answer. Answer: $1, 2, 4, 5, 10, 20$.

9.4 Divisibility Rules

Quick Tests for Even Division

These shortcuts tell you if a number divides evenly *without* doing the long division.

- **2:** the last digit is even $(0, 2, 4, 6, 8)$.
- **3:** the sum of the digits is divisible by 3.
- **4:** the number formed by the last *two* digits is divisible by 4.
- **5:** the last digit is 0 or 5.
- **6:** it passes the tests for *both* 2 and 3.
- **9:** the sum of the digits is divisible by 9.
- **10:** the last digit is 0.

Testing 780 against every rule

Which of $2, 3, 4, 5, 6, 9, 10$ divide 780?

- **2:** last digit 0 is even ✓
- **3:** digits $7 + 8 + 0 = 15$, divisible by 3 ✓
- **4:** last two digits $80 \div 4 = 20$ ✓
- **5:** ends in 0 ✓
- **6:** passes 2 and 3 ✓
- **9:** digit sum 15 is *not* divisible by 9 — **no**
- **10:** ends in 0 ✓

So 780 is divisible by $2, 3, 4, 5, 6$, and 10, but not 9.

The rule for 6 is really the rules for 2 *and* 3 working together. In the same way, a number divisible by 9 is automatically divisible by 3 — but not the other way around.

Try it: Quick check

List the first five multiples of 4.

Work it on paper before revealing the answer.

Answer. Answer: $4, 8, 12, 16, 20$.

9.5 Prime Factorization

Breaking a Number into Primes

The **prime factorization** of a number writes it as a product of prime numbers only. Every composite number has exactly one prime factorization (in some order). A **factor tree** is the easiest way to find it: split the number into any two factors, then keep splitting until every branch ends in a prime. Finally, write repeated primes using **exponents**.

Factor tree for 180

Split 180 step by step:

$$180 = 18 \times 10 = (2 \times 9) \times (2 \times 5) = 2 \times 3 \times 3 \times 2 \times 5.$$

Group and order the primes, then use exponents:

$$180 = 2^2 \times 3^2 \times 5.$$

Check: $4 \times 9 \times 5 = 180$. ✓

No matter which factors you split off first, an honest factor tree always ends with the *same* set of primes. Starting $180 = 4 \times 45$ or $180 = 6 \times 30$ leads to the identical answer $2^2 \times 3^2 \times 5$.

Try it: Quick check

List the first five multiples of 7.

Work it on paper before revealing the answer.

Answer. Answer: $7, 14, 21, 28, 35$.

9.6 Greatest Common Factor (GCF)

The Largest Shared Factor

The **greatest common factor** of two numbers is the biggest number that divides *both*. There are two reliable methods:

- **Listing:** list the factors of each number, then pick the largest one they share.

- **Prime factorization:** factor each number into primes, then multiply the primes they have *in common* (using the *smaller* exponent of each shared prime).

GCF of 24 and 36, two ways

Listing method:

$$24 : 1, 2, 3, 4, 6, 8, 12, 24$$
$$36 : 1, 2, 3, 4, 6, 9, 12, 18, 36$$

Shared factors: $1, 2, 3, 4, 6, 12$. The greatest is $\boxed{12}$.

Prime-factorization method:

$$24 = 2^3 \times 3, \qquad 36 = 2^2 \times 3^2.$$

Take the shared primes at their smaller powers: $2^2 \times 3 = 12$.

The prime-factorization method wins for big numbers where listing every factor is slow. For the GCF, keep only primes that appear in *both* numbers, each raised to the *lower* power.

Try it: Quick check

Is 24 a multiple of 6? Explain in one sentence.

Prime and Composite Numbers

Work it on paper before revealing the answer.

Answer. Answer: Yes; $6 \times 4 = 24$, so 24 is in the times table of 6.

9.7 Least Common Multiple (LCM)

The Smallest Shared Multiple

The **least common multiple** of two numbers is the smallest number that both divide into — the first multiple they share. Again, two methods:

- **Listing:** list the multiples of each number until one appears in both lists.
- **Prime factorization:** factor each number, then multiply *every* prime that appears in either number, using the *larger* exponent of each.

LCM of 8 and 12, two ways

Listing method:

$$8 : 8, 16, 24, 32, \ldots$$
$$12 : 12, 24, 36, \ldots$$

The first shared multiple is $\boxed{24}$.

Prime-factorization method:

$$8 = 2^3, \qquad 12 = 2^2 \times 3.$$

Take every prime at its larger power: $2^3 \times 3 = 24$.

For the LCM, use the *higher* power of every prime; for the GCF, use the *lower* power of only the *shared* primes. A handy check: GCF $\times$ LCM = the product of the two numbers. Here $4 \times 24 = 96 = 8 \times 12$. ✓

Try it: Quick check

Is 17 prime or composite?

Work it on paper before revealing the answer.

Answer. Answer: Prime (17 has only the factors 1 and 17).

9.8 Word Problems: GCF or LCM?

Deciding Which to Use

The hardest part of a word problem is choosing the right tool. Watch for these signals:

- **Use the GCF** when you *split* or *group* things into the *largest* equal piles, rows, or teams — the numbers get *smaller*. Clues: "greatest," "largest," "equal groups," "split evenly."
- **Use the LCM** when events *repeat* and you want the *next time* they line up, or the *smallest* amount that works for both — the numbers get *bigger*. Clues: "at the same time," "again," "least," "smallest," "every."

A GCF problem

A teacher has 24 pencils and 36 erasers. She wants to make identical goody bags using *all* the supplies, with the *greatest* number of bags. How many bags, and what is in each?

We are *splitting* into the largest equal groups ⇒ **GCF**.

$$\mathrm{GCF}(24, 36) = 12.$$

So 12 bags, each with $24 \div 12 = 2$ pencils and $36 \div 12 = 3$ erasers.

An LCM problem

One lighthouse flashes every 8 seconds and another every 12 seconds. They just flashed together. When will they *next* flash at the same time?

Repeating events lining up again ⇒ **LCM**.

$$\mathrm{LCM}(8, 12) = 24.$$

They flash together again after 24 seconds.

Quick gut-check: if the answer should be *smaller* than the numbers in the problem (fewer, larger groups), it is a **GCF**. If the answer should be *larger* (a later time, a bigger shared total), it is an **LCM**.

Try it: Quick check

Is 21 prime or composite?

Work it on paper before revealing the answer.

Answer. Answer: Composite ($21 = 3 \times 7$).

9.9 Going Deeper: Advanced Number Theory

Counting Divisors from the Prime Factorization

Once a number is written in prime-power form, you can count *how many* factors it has **without listing a single one**. If

$$n = p^a \times q^b \times r^c \times \cdots,$$

then the number of divisors is

$$d(n) = (a+1)(b+1)(c+1)\cdots$$

The idea: when building a divisor, the prime p can appear $0, 1, 2, \ldots,$ or a times — that is $a+1$ independent choices. Multiplying the choices for each prime counts every divisor exactly once.

How many divisors does 360 have?

First factor into primes:

$$360 = 8 \times 45 = 2^3 \times 3^2 \times 5.$$

The exponents are $3, 2$, and 1. Add one to each and multiply:

$$d(360) = (3+1)(2+1)(1+1) = 4 \times 3 \times 2 = 24.$$

So 360 has exactly $\boxed{24}$ divisors — far quicker than hunting for all 24 factor pairs by hand.

A number is a **perfect square** exactly when *every* exponent in its prime factorization is even — which forces every $(a+1)$ to be odd, so $d(n)$ is odd. In fact, perfect squares are the *only* numbers with an *odd* number of divisors, because the "middle" pair $\sqrt{n} \times \sqrt{n}$ is counted just once.

The Sum of All Divisors

You can also add up *all* the divisors straight from the prime factorization. For each prime power, sum the powers from p^0 up to p^a, then multiply those sums together:

$$\sigma(n) = \left(1 + p + \cdots + p^a\right)\left(1 + q + \cdots + q^b\right) \cdots$$

Multiplying the parentheses out reproduces every divisor of n exactly once, so their grand total falls out automatically.

Adding up the divisors of 360

Using $360 = 2^3 \times 3^2 \times 5$, sum each prime's powers separately:

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15,$$
$$3^0 + 3^1 + 3^2 = 1 + 3 + 9 = 13,$$
$$5^0 + 5^1 = 1 + 5 = 6.$$

Multiply the three totals:

$$\sigma(360) = 15 \times 13 \times 6 = 1170.$$

All 24 divisors of 360 add up to $\boxed{1170}$.

The GCF–LCM Product Theorem

For *any* two whole numbers a and b, the greatest common factor and least common multiple are locked together by

$$\mathrm{GCF}(a,b) \times \mathrm{LCM}(a,b) = a \times b.$$

This makes finding the LCM effortless once you know the GCF (or vice versa): just rearrange.

$$\mathrm{LCM}(a,b) = \frac{a \times b}{\mathrm{GCF}(a,b)}.$$

Why it works: for each prime, the GCF takes the *smaller* exponent and the LCM takes the *larger* one — together they use up exactly the two exponents that appear in a and b.

Using the theorem on 18 and 24

We have $\mathrm{GCF}(18,24) = 6$. Instead of listing multiples, use the theorem:

$$\mathrm{LCM}(18,24) = \frac{18 \times 24}{6} = \frac{432}{6} = 72.$$

Check: $18 = 2 \times 3^2$ and $24 = 2^3 \times 3$, so the LCM is $2^3 \times 3^2 = 72$. ✓

The Euclidean Algorithm

For large numbers, factoring can be slow. The **Euclidean algorithm** finds the GCF using only repeated division with remainder. Repeatedly replace the larger number by the *remainder* when it is divided by the smaller:

$$\mathrm{GCF}(a,b) = \mathrm{GCF}(b,\ a \bmod b),$$

and stop when the remainder hits 0. The *last nonzero remainder* is the GCF. It works because any common factor of a and b is also a factor of their remainder.

Euclidean algorithm for GCF(252, 105)

Divide, keep the remainder, repeat:

$$\begin{aligned} 252 &= 2 \times 105 + 42, \\ 105 &= 2 \times 42 + 21, \\ 42 &= 2 \times 21 + 0. \end{aligned}$$

The last nonzero remainder is $\boxed{21}$, so $\mathrm{GCF}(252,105) = 21$. *Check:* $252 = 21 \times 12$ and $105 = 21 \times 5$. ✓

Perfect, Abundant, and Deficient Numbers

Compare a number to the sum of its **proper divisors** (all its divisors *except* itself):

- **Perfect:** the proper divisors add up to the number exactly. The smallest is $6 = 1+2+3$; the next is $28 = 1+2+4+7+14$.

- **Abundant:** the proper divisors sum to *more* than the number. Example: 12, since $1+2+3+4+6=16>12$.
- **Deficient:** the proper divisors sum to *less* than the number. Example: 10, since $1+2+5=8<10$. Every prime is deficient (its only proper divisor is 1).

Two numbers are **relatively prime** (or *coprime*) when their only common factor is 1, i.e. GCF = 1. They need not be prime themselves: 8 and 9 are relatively prime because $8=2^3$ and $9=3^2$ share no primes. When two numbers are coprime, their LCM is simply their *product*, since GCF $\times$ LCM $= a \times b$ becomes $1 \times \text{LCM} = a \times b$.

9.10 Why It Works

For $12=2^2\cdot 3$ and $18=2\cdot 3^2$: a common factor can use at most 2^1 and 3^1 (the smaller exponents), so gcd = 6; a common multiple must contain 2^2 and 3^2 (the larger exponents), so lcm = 36. Notice $6\times 36=216=12\times 18$ — in general $\gcd(a,b)\cdot\text{lcm}(a,b)=ab$, because each prime's exponent is min + max = the sum of the two exponents.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

10 Fractions

Everything you need to master parts of a whole

10.1 Fraction Vocabulary & Equivalent Fractions

The Parts of a Fraction

A **fraction** names part of a whole (or part of a group). It is written as $\dfrac{\text{numerator}}{\text{denominator}}$.

- The **denominator** (bottom) tells how many equal pieces the whole is divided into.
- The **numerator** (top) tells how many of those pieces you have.

In $\dfrac{3}{4}$, the whole is cut into 4 equal parts and we have 3 of them. Two fractions are **equivalent** when they name the same amount, even though the numbers look different. You make an equivalent fraction by multiplying (or dividing) the top *and* bottom by the same nonzero number.

The same amount, cut differently.



3 of 4 parts shaded

Fractions add and subtract only over a common denominator, multiply straight across, and divide by multiplying by the reciprocal. Mixed numbers are improper fractions in costume; converting first avoids most errors. Comparing fractions is easiest with a common denominator or by cross-multiplying.

Building equivalent fractions

Write three fractions equivalent to $\frac{2}{3}$.

$$\frac{2}{3}=\frac{2\times 2}{3\times 2}=\frac{4}{6},\qquad \frac{2}{3}=\frac{2\times 3}{3\times 3}=\frac{6}{9},\qquad \frac{2}{3}=\frac{2\times 5}{3\times 5}=\frac{10}{15}.$$

Each time we multiplied top and bottom by the same number, so the value did not change.

Where it goes wrong

Cancelling across a plus sign: $\frac{x+2}{2}$ is not $x+1$ and $\frac{3+6}{3}$ is 3, not $1+6$. Only common FACTORS of the whole top and whole bottom cancel.

Tip: Multiplying top and bottom by the same number is really multiplying by 1 in disguise (for example $\frac{2}{2}=1$). That is why the value stays the same.

Try it: Quick check

In the fraction $\frac{5}{9}$, what is the numerator?

Work it on paper before revealing the answer.

Answer. Answer: 5.

10.2 Simplifying Fractions to Lowest Terms

Lowest Terms

A fraction is in **lowest terms** (fully simplified) when the numerator and denominator share no common factor except 1. To simplify, divide the top and bottom by their **greatest common factor** (GCF).

The same amount, cut differently.



3 of 4 parts shaded

Simplifying $\frac{12}{18}$

List factors to find the GCF of 12 and 18: it is 6.

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}.$$

Check: 2 and 3 share no common factor but 1, so $\frac{2}{3}$ is fully simplified.

Simplifying in steps

If you do not spot the GCF right away, simplify a little at a time:

$$\frac{24}{36} = \frac{24 \div 2}{36 \div 2} = \frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}.$$

You reach the same answer as long as you keep dividing until nothing but 1 divides both.

Tip: Always give final answers in lowest terms. A good habit: after every add, subtract, multiply, or divide, ask "can this simplify?"

Try it: Quick check

Fill in the missing number: $\frac{1}{2} = \frac{\quad}{6}$.

Work it on paper before revealing the answer.

Answer. Answer: 3.

10.3 Comparing & Ordering Fractions

Two Ways to Compare

To decide which of two fractions is larger:

- **Common denominators:** rewrite both with the same denominator, then compare numerators.
- **Cross-multiplying:** for $\frac{a}{b}$ and $\frac{c}{d}$, compare $a \times d$ with $b \times c$. The fraction whose numerator is part of the larger product is larger.

The same amount, cut differently.



3 of 4 parts shaded

Comparing $\frac{3}{5}$ and $\frac{5}{8}$

Cross-multiply: $3 \times 8 = 24$ and $5 \times 5 = 25$. Since $24 < 25$, the first product is smaller, so

$$\frac{3}{5} < \frac{5}{8}.$$

Check with common denominators (40): $\frac{3}{5} = \frac{24}{40}$ and $\frac{5}{8} = \frac{25}{40}$. Indeed $\frac{24}{40} < \frac{25}{40}$.

Ordering $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ from least to greatest

Use the common denominator 12:

$$\frac{1}{2} = \frac{6}{12}, \qquad \frac{2}{3} = \frac{8}{12}, \qquad \frac{3}{4} = \frac{9}{12}.$$

Compare numerators $6 < 8 < 9$, so the order is $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$.

Tip: With the *same numerator*, the fraction with the *smaller* denominator is larger (bigger pieces). So $\frac{2}{3} > \frac{2}{5}$.

Try it: Quick check

Write the fraction equivalent to $\frac{2}{5}$ that has denominator 10.

Simplifying to Lowest Terms

Work it on paper before revealing the answer.

Answer. Answer: $\frac{4}{10}$.

10.4 Improper Fractions & Mixed Numbers

Two Names for the Same Amount

An **improper fraction** has a numerator greater than or equal to its denominator (like $\frac{11}{4}$). A **mixed number** is a whole number next to a proper fraction (like $2\frac{3}{4}$). They can name the same amount.

- **Improper → mixed:** divide numerator by denominator. The quotient is the whole number; the remainder over the denominator is the fraction.
- **Mixed → improper:** multiply the whole number by the denominator, add the numerator, and keep the denominator.

Converting both ways

Improper to mixed: $\frac{11}{4}$. Since $11 \div 4 = 2$ remainder 3,

$$\frac{11}{4} = 2\frac{3}{4}.$$

Mixed to improper: $3\frac{2}{5}$. Compute $3 \times 5 + 2 = 17$,

$$3\frac{2}{5} = \frac{17}{5}.$$

Tip: For most multiplying and dividing, first turn every mixed number into an improper fraction. It keeps the arithmetic simple.

Try it: Quick check

Simplify $\frac{4}{8}$.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{1}{2}$.

10.5 Adding & Subtracting: LIKE Denominators

Same Bottom, Add the Tops

When two fractions have the **same denominator**, add or subtract the numerators and keep the denominator. Then simplify.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Like denominators

$$\frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}, \qquad \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Keep the denominator the same; only the numerators change. Always simplify the result.

Common mistake: Do *not* add the denominators. $\frac{3}{8} + \frac{1}{8}$ is $\frac{4}{8}$, not $\frac{4}{16}$.

Try it: Quick check

Simplify $\frac{6}{9}$.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{2}{3}$.

10.6 Adding & Subtracting: UNLIKE Denominators (LCD)

First Make the Denominators Match

To add or subtract fractions with different denominators, rewrite them over a **common denominator**. The easiest one to use is the **least common denominator** (LCD), which is the least common multiple of the denominators. Then add or subtract the numerators and simplify.

Adding $\frac{2}{3} + \frac{1}{4}$

The LCD of 3 and 4 is 12.

$$\frac{2}{3} + \frac{1}{4} = \frac{2 \times 4}{12} + \frac{1 \times 3}{12} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}.$$

Since 11 and 12 share no common factor, $\frac{11}{12}$ is already in lowest terms.

Subtracting $\frac{5}{6} - \frac{3}{8}$

The LCD of 6 and 8 is 24.

$$\frac{5}{6} - \frac{3}{8} = \frac{20}{24} - \frac{9}{24} = \frac{11}{24}.$$

Tip: You need common denominators to *add* or *subtract*, but *not* to multiply. Multiplying fractions does not care about denominators matching.

Try it: Quick check

Simplify $\frac{10}{15}$.

Comparing & Ordering Fractions

Work it on paper before revealing the answer.

Answer. Answer: $\frac{2}{3}$.

10.7 Adding & Subtracting Mixed Numbers

Wholes with Wholes, Fractions with Fractions

Add (or subtract) the whole-number parts and the fraction parts separately. Give the fractions a common denominator first. Sometimes subtracting forces you to **borrow** (regroup) one whole into fraction form.

Adding with regrouping

Compute $2\frac{3}{4} + 1\frac{2}{3}$. The LCD of 4 and 3 is 12.

$$2\frac{3}{4} + 1\frac{2}{3} = 2\frac{9}{12} + 1\frac{8}{12} = 3\frac{17}{12}.$$

Since $\frac{17}{12} = 1\frac{5}{12}$, regroup: $3\frac{17}{12} = 4\frac{5}{12}$.

Subtracting with borrowing

Compute $5\frac{1}{4} - 2\frac{3}{4}$. You cannot take $\frac{3}{4}$ from $\frac{1}{4}$, so borrow 1 whole from the 5:

$$5\frac{1}{4} = 4 + 1 + \frac{1}{4} = 4\frac{5}{4}.$$

Now subtract: $4\frac{5}{4} - 2\frac{3}{4} = 2\frac{2}{4} = 2\frac{1}{2}$.

Tip: When you borrow one whole, it becomes a fraction equal to $\frac{d}{d}$ (the denominator over itself). Add that to the fraction you already have.

Try it: Quick check

Fill in $<$ or $>$: $\frac{1}{3}$ _____ $\frac{1}{2}$.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{1}{3} < \frac{1}{2}$.

10.8 Multiplying Fractions & Mixed Numbers

Straight Across

To multiply fractions, multiply the numerators and multiply the denominators. Simplify at the end (or cancel common factors first).

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

For mixed numbers, first change each to an improper fraction, then multiply.

Multiplying and canceling

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12} = \frac{1}{2}.$$

You could also cancel first: the 3 on top and bottom cancel, leaving $\frac{2}{4} = \frac{1}{2}$.

Multiplying mixed numbers

Compute $1\frac{1}{2} \times 2\frac{1}{3}$. Convert first: $1\frac{1}{2} = \frac{3}{2}$ and $2\frac{1}{3} = \frac{7}{3}$.

$$\frac{3}{2} \times \frac{7}{3} = \frac{21}{6} = \frac{7}{2} = 3\frac{1}{2}.$$

Tip: "Of" usually means multiply. "$\frac{1}{2}$ of $\frac{3}{4}$" means $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

Try it: Quick check

Fill in $<$ or $>$: $\frac{3}{4}$ ____ $\frac{2}{4}$.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{3}{4} > \frac{2}{4}$.

10.9 Dividing Fractions & Mixed Numbers

Multiply by the Reciprocal

To divide by a fraction, multiply by its **reciprocal** (flip the second fraction). The reciprocal

of $\frac{c}{d}$ is $\frac{d}{c}$.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Change any mixed numbers to improper fractions before flipping.

Dividing fractions

$$\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8} = 1\frac{7}{8}.$$

Dividing mixed numbers

Compute $2\frac{1}{2} \div 1\frac{1}{4}$. Convert: $2\frac{1}{2} = \frac{5}{2}$ and $1\frac{1}{4} = \frac{5}{4}$.

$$\frac{5}{2} \div \frac{5}{4} = \frac{5}{2} \times \frac{4}{5} = \frac{20}{10} = 2.$$

Tip: Only flip the fraction *after* the division sign (the divisor). Flipping the wrong one is the most common division mistake.

10.10 Word Problems with Fractions

A Plan for Word Problems

1. Read carefully and decide what operation the story describes.
2. "Of" a quantity means multiply; "left over" or "how much more" often means subtract; "in all" means add; sharing equally means divide.
3. Do the arithmetic, then simplify and label your answer.

A recipe problem

A recipe needs $\frac{3}{4}$ cup of flour. You are making half a batch. How much flour do you need?

Half of $\frac{3}{4}$ means $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ cup of flour.

A distance problem

Maya jogged $2\frac{1}{2}$ miles on Monday and $1\frac{3}{4}$ miles on Tuesday. How far in all?

$$2\frac{1}{2} + 1\frac{3}{4} = 2\frac{2}{4} + 1\frac{3}{4} = 3\frac{5}{4} = 4\frac{1}{4} \text{ miles.}$$

Tip: Estimate first. $2\frac{1}{2} + 1\frac{3}{4}$ should be a bit more than 4, so the answer $4\frac{1}{4}$ is reasonable. Estimating catches big mistakes.

10.11 Going Deeper: Advanced Fraction Ideas

Complex Fractions (a Fraction of Fractions)

A **complex fraction** is a fraction whose numerator, denominator, or both are *themselves* fractions, like $\frac{\frac{2}{3}}{\frac{4}{5}}$. A big fraction bar means "divide," so a complex fraction is just a division problem in disguise:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Rewrite it as a division, multiply by the reciprocal, and simplify as usual.

The same amount, cut differently.



3 of 4 parts shaded

Simplifying a complex fraction

Simplify $\frac{\frac{2}{3}}{\frac{4}{5}}$. Read the main bar as "divide":

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}.$$

So a stack of two fractions collapses to the single fraction $\frac{5}{6}$.

Why Cross-Multiplication Works — and the Mediant

When you compare $\frac{a}{b}$ and $\frac{c}{d}$ (with positive b and d), multiplying *both* by bd does not change which is larger, since $bd > 0$. This turns $\frac{a}{b}$ into ad and $\frac{c}{d}$ into bc — exactly the cross-products. That is the whole justification for the cross-multiplying rule.

A surprising cousin is the **mediant** of $\frac{a}{b}$ and $\frac{c}{d}$, formed by adding tops and bottoms:

$$\text{mediant} = \frac{a+c}{b+d}.$$

For positive fractions, the mediant always lands strictly *between* the two original fractions. (Warning: this is *not* how you add fractions — it is a different, special construction.)

Unit Fractions and Egyptian Fractions

A **unit fraction** has a numerator of 1, like $\frac{1}{2}$ or $\frac{1}{7}$. The ancient Egyptians wrote every other fraction as a sum of *distinct* unit fractions, for example $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

One reliable recipe is the **greedy method**: repeatedly subtract the *largest* unit fraction that is not bigger than what remains, until nothing is left. It always terminates, and the denominators grow quickly.

An Egyptian fraction for $\frac{3}{7}$

Write $\frac{3}{7}$ as a sum of distinct unit fractions using the greedy method.

Step 1. The largest unit fraction $\le \frac{3}{7}$ is $\frac{1}{3}$ (since $\frac{1}{2} > \frac{3}{7}$). Subtract it:

$$\frac{3}{7} - \frac{1}{3} = \frac{9}{21} - \frac{7}{21} = \frac{2}{21}.$$

Step 2. The largest unit fraction $\le \frac{2}{21}$ is $\frac{1}{11}$ (since $\frac{1}{10} > \frac{2}{21}$). Subtract it:

$$\frac{2}{21} - \frac{1}{11} = \frac{22}{231} - \frac{21}{231} = \frac{1}{231}.$$

The remainder is already a unit fraction, so we stop:

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}.$$

Check: $\frac{1}{3}+\frac{1}{11}=\frac{14}{33}=\frac{98}{231}$, and $\frac{98}{231}+\frac{1}{231}=\frac{99}{231}=\frac{3}{7}$. ✓

Continued Fractions

A **continued fraction** builds a number out of a whole part plus 1 over another whole part plus 1 over …, stacking reciprocals downward. Every ordinary fraction can be rewritten this way by repeatedly separating the whole-number part and flipping the leftover. For example,

$$\frac{7}{5}=1+\frac{2}{5}=1+\frac{1}{\frac{5}{2}}=1+\cfrac{1}{2+\frac{1}{2}}.$$

Reading it back from the bottom up: $2+\frac{1}{2}=\frac{5}{2}$, then $1\div\frac{5}{2}=\frac{2}{5}$, then $1+\frac{2}{5}=\frac{7}{5}$. Continued fractions reveal the "best" simple approximations to a number.

The Harmonic Mean

The **harmonic mean** of two positive numbers a and b averages their *reciprocals* instead of the numbers themselves:

$$H=\frac{2}{\frac{1}{a}+\frac{1}{b}}=\frac{2ab}{a+b}.$$

It is the right average when the quantities are rates. If you drive one mile at 30 mph and one mile at 60 mph, your average speed is the harmonic mean

$$\frac{2\times 30\times 60}{30+60}=\frac{3600}{90}=40\text{ mph},$$

which is *less* than the plain average of 45, because you spend more time at the slower speed.

A telescoping sum

Add $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}$.
The key trick is that each term splits into a *difference* of unit fractions:

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

For instance $\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$. Writing the whole sum this way, the inside terms cancel in pairs (this is called **telescoping**):

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)=1-\frac{1}{5}=\frac{4}{5}.$$

Only the very first and very last pieces survive. In general such a sum up to $\frac{1}{N(N+1)}$ equals

$\dfrac{N}{N+1}$.

10.12 Why It Works

Cross-multiplication compares $\frac{a}{b}$ and $\frac{c}{d}$ correctly because multiplying both by the positive number bd preserves order and turns the fractions into the whole numbers ad and bc. The reciprocal rule for division works because $\frac{a}{b} \cdot \frac{b}{a} = 1$, so multiplying by $\frac{b}{a}$ undoes multiplying by $\frac{a}{b}$. Every fraction rule is a consequence of "a fraction is a division", and can be re-derived from that in a line.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

11 Decimals

Everything you need to master decimal place value, operations, and word problems

11.1 Decimal Place Value; Reading & Writing Decimals

Places to the Right of the Point

A decimal point separates the whole-number part from the fractional part. Each place to the right of the point is worth ten times *less* than the place before it:

$$\underbrace{3}_{\text{ones}} . \underbrace{4}_{\text{tenths}} \quad \underbrace{5}_{\text{hundredths}} \quad \underbrace{6}_{\text{thousandths}}$$

So $3.456 = 3 + \dfrac{4}{10} + \dfrac{5}{100} + \dfrac{6}{1000}$. The place names go **tenths, hundredths, thousandths, ten-thousandths**, ... — notice every fractional place name ends in "-ths."

Decimals are fractions with denominators $10, 100, 1000, \ldots$, so every fraction rule applies. Converting between the two forms — $\frac{3}{8} = 0.375$ by dividing, $0.4 = \frac{4}{10} = \frac{2}{5}$ by reading the place value — lets you pick whichever form is easier for the problem at hand.

Reading a decimal out loud

Read 52.07 aloud. Read the whole part, say "and" for the point, read the fractional digits as a whole number, then name the last place.

- Whole part: fifty-two.
- The digits after the point are 07, and the last digit (7) sits in the **hundredths** place.

- So 52.07 = "fifty-two **and** seven hundredths."

Where it goes wrong

Placing the decimal point in a product by aligning columns. Count decimal places instead: 1.2×0.03 has $1 + 2 = 3$ places, so $36 \to 0.036$.

Writing words as a decimal

Write "nine and thirteen thousandths" as a decimal. The word "and" marks the point, and "thousandths" means the last digit must land in the third place.

$$\text{nine} \to 9$$
$$\text{thirteen thousandths} \to 0.013 \quad \text{(fill to 3 places: 013)}$$
$$\Rightarrow 9.013$$

The word "**and**" is reserved for the decimal point. So 305 is "three hundred five," but 3.05 is "three *and* five hundredths." Never sprinkle extra "ands" into a whole number.

Try it: Quick check

What is the value of the digit 7 in 3.75? (Give it as a decimal or a fraction.)

Work it on paper before revealing the answer.

Answer. Answer: 7 tenths, that is $0.7 = \frac{7}{10}$.

11.2 Comparing & Ordering Decimals

Line Up the Points, Compare Left to Right

To compare two decimals, stack them so the decimal points line up and add trailing zeros so both have the same number of places. Then compare digit by digit from the left — the first place where they differ decides which is larger.

Which is greater, 0.6 or 0.58?

Give them the same number of decimal places by writing $0.6 = 0.60$.

$$0.60 \quad \text{vs} \quad 0.58$$
$$\text{tenths: } 6 > 5$$

The tenths digit already decides it: $0.60 > 0.58$, so $0.6 > 0.58$. A longer decimal is *not* automatically bigger!

Ordering from least to greatest

Order $0.3,\ 0.29,\ 0.305,\ 0.31$. Write each with three places:

$$0.300, \quad 0.290, \quad 0.305, \quad 0.310$$

Now compare as if they were the whole numbers $300, 290, 305, 310$. Least to greatest:

$$0.29 < 0.3 < 0.305 < 0.31.$$

Adding zeros to the *end* of a decimal never changes its value: $0.6 = 0.60 = 0.600$. This trick lets you compare, add, and subtract decimals of different lengths safely.

Try it: Quick check

Write "five and two tenths" as a decimal.

Work it on paper before revealing the answer.

Answer. Answer: 5.2.

11.3 Rounding Decimals

Look at the Next Digit

To round to a given place, find that place and look at the **single digit just to its right**:

- If that digit is 5 or more, round **up** (add one to the rounding place).
- If it is 4 or less, round **down** (leave the rounding place alone).

Then drop every digit after the rounding place.

Round 7.348 to the nearest hundredth

The hundredths digit is 4. The digit just to its right is 8.

$$\begin{aligned} 7.34\underline{8} \quad &\Rightarrow \quad 8 \geq 5, \text{ round up} \\ &\Rightarrow \ 7.35 \end{aligned}$$

Round 7.348 to the nearest tenth instead: the tenths digit is 3, the next digit is 4, so it stays: 7.3.

Rounding up a 9 carries, just like in addition: 2.98 rounded to the nearest tenth is 3.0, because the 9 tenths becomes 10 tenths. Keep the trailing zero to show the place you rounded to.

Try it: Quick check

Write 0.09 in words.

Comparing & Ordering Decimals

Work it on paper before revealing the answer.

Answer. Answer: Nine hundredths.

11.4 Adding & Subtracting Decimals

Line Up the Decimal Points

Write the numbers in a column so the decimal points sit directly above one another (this automatically lines up tenths with tenths, hundredths with hundredths). Fill empty places with zeros, then add or subtract exactly as with whole numbers. Bring the decimal point straight down into the answer.

Adding $12.5 + 3.47$

Line up the points and write 12.5 as 12.50:

$$\begin{array}{r} 12.50 \\ +\ 3.47 \\ \hline 15.97 \end{array}$$

The point in the answer sits right below the points above it: $12.5 + 3.47 = 15.97$.

Subtracting $8 - 2.65$

Write the whole number 8 as 8.00 so both have two decimal places, then subtract with borrowing:

$$\begin{array}{r} 8.00 \\ -\ 2.65 \\ \hline 5.35 \end{array}$$

So $8 - 2.65 = 5.35$.

A whole number hides a decimal point at its right end: $8 = 8.0 = 8.00$. Writing it in lets you line up the points and borrow correctly.

Try it: Quick check

Fill in $<$, $>$, or $=$: 0.5 _____ 0.45.

Work it on paper before revealing the answer.

Answer. Answer: $0.5 > 0.45$.

11.5 Multiplying Decimals

Multiply, Then Count the Places

Ignore the decimal points at first and multiply the numbers as whole numbers. Then **count the total number of decimal places** in *both* factors together, and place the decimal point in the answer so it has that many places.

Multiplying 1.2×0.4

Multiply as whole numbers: $12 \times 4 = 48$. Now count decimal places: 1.2 has 1 place and 0.4 has 1 place, for a total of 2.

$$\begin{aligned} 1.2 \times 0.4 &\Rightarrow 12 \times 4 = 48 \\ 2 \text{ decimal places} &\Rightarrow 0.48 \end{aligned}$$

So $1.2 \times 0.4 = 0.48$.

When you must add a leading zero

Compute 0.3×0.02. Multiply: $3 \times 2 = 6$. Total decimal places: $1 + 2 = 3$. The answer needs 3 places, so pad with zeros: 0.006.

$$0.3 \times 0.02 = 0.006.$$

When multiplying, you do *not* line up the decimal points. Just count total decimal places in the factors and give the product that many. Unlike addition, the answer can have more places than either factor.

Try it: Quick check

Fill in $<$, $>$, or $=$: 0.7 _____ 0.70.

Work it on paper before revealing the answer.

Answer. Answer: $0.7 = 0.70$.

11.6 Dividing Decimals

Dividing by a Whole Number

When the divisor is a whole number, place the decimal point in the quotient **directly above** the point in the dividend, then divide normally. Add zeros to the dividend if you need to keep going.

$$9.6 \div 3 = 3.2, \qquad 7 \div 4 = 1.75 \text{ (using } 7.00\text{)}.$$

Dividing by a Decimal — Move the Point

When the divisor is a decimal, slide its point to the right until it becomes a whole number, and slide the dividend's point the **same number of places** (add zeros if needed). The answer is unchanged because you multiplied both by the same power of ten.

$$2.4 \div 0.6 = 24 \div 6 = 4.$$

Dividing by a decimal, step by step

Compute $1.44 \div 0.12$. The divisor 0.12 has two decimal places, so move both points two places right:

$$\begin{aligned} 1.44 \div 0.12 &\Rightarrow 144 \div 12 \\ &= 12. \end{aligned}$$

So $1.44 \div 0.12 = 12$.

Whatever you do to the divisor's point, do the *same* to the dividend's point. Move both the same number of places — never just one of them.

Try it: Quick check

Order from least to greatest: $0.2,\ 0.18,\ 0.3,\ 0.25$.

Rounding Decimals

Work it on paper before revealing the answer.

Answer. Answer: 0.18, 0.2, 0.25, 0.3.

11.7 Converting Between Fractions & Decimals

Fraction → Decimal: Divide

To turn a fraction into a decimal, divide the numerator by the denominator. To turn a "nice" fraction into a decimal quickly, rewrite it with a denominator of 10, 100, or 1000.

$$\frac{3}{4} = 3 \div 4 = 0.75, \qquad \frac{7}{10} = 0.7, \qquad \frac{9}{20} = \frac{45}{100} = 0.45.$$

Decimal → Fraction: Use the Place Value

Write the decimal's digits over the place value of the last digit, then simplify.

$$0.6 = \frac{6}{10} = \frac{3}{5}, \qquad 0.125 = \frac{125}{1000} = \frac{1}{8}.$$

Terminating vs. Repeating

When you divide, one of two things happens:

- **Terminating:** the division ends with a remainder of 0. Example: $\frac{3}{8} = 0.375$.
- **Repeating:** a block of digits repeats forever. Mark it with a bar. Example: $\frac{1}{3} = 0.333\cdots = 0.\overline{3}$ and $\frac{2}{11} = 0.1818\cdots = 0.\overline{18}$.

Both terminating and repeating decimals are rational numbers.

A repeating-decimal conversion

Convert $\frac{5}{6}$ to a decimal. Divide $5 \div 6$:

$$\begin{aligned} 5.000 \div 6 &= 0.8333\ldots \\ &= 0.8\overline{3}. \end{aligned}$$

The 8 appears once; only the 3 repeats, so the bar goes over just the 3.

A fraction in lowest terms gives a *terminating* decimal exactly when its denominator's only prime factors are 2 and 5 (the factors of 10). So $\frac{7}{8}$ terminates ($8 = 2^3$) but $\frac{1}{6}$ repeats ($6 = 2 \times 3$).

Try it: Quick check

Round 3.68 to the nearest tenth.

Work it on paper before revealing the answer.

Answer. Answer: 3.7.

11.8 Word Problems with Decimals (Money & Measurement)

Translate, Then Compute

Decide which operation the situation calls for, line up units, and keep money to two decimal places (cents). Common signals:

- "total / altogether / sum" → add.
- "how much more / change / difference" → subtract.
- "each / per / times as much" → multiply.
- "split evenly / per item / how many fit" → divide.

Making change (money)

A book costs \$12.75 and you pay with a \$20 bill. How much change do you get? "Change" means subtract; write \$20 as \$20.00:

$$\begin{array}{r} 20.00 \\ -\ 12.75 \\ \hline 7.25 \end{array}$$

Your change is \$7.25.

Total cost then per-person (multi-step)

Four friends each buy a smoothie for \$3.50. What is the total, and if they split it evenly among 2 cars, how much does each car pay?

$$\begin{aligned} \text{Total:} \quad & 4 \times \$3.50 = \$14.00 \\ \text{Per car:} \quad & \$14.00 \div 2 = \$7.00 \end{aligned}$$

The total is \$14.00 and each car pays \$7.00.

Always attach the units to your answer and check that it makes sense. Money answers should show two decimal places (\$7.00, not \$7), and a length in meters should not suddenly turn into centimeters without converting.

Try it: Quick check

Round 5.24 to the nearest tenth.

Work it on paper before revealing the answer.

Answer. Answer: 5.2.

11.9 Going Deeper: Advanced Decimal Ideas

Why Only 2 and 5 Make a Decimal Terminate

A decimal terminates exactly when it can be written with a denominator that is a power of ten. Every power of ten factors as $10^n = 2^n \times 5^n$, so its only prime factors are 2 and 5. If a fraction in lowest terms has a denominator built only from 2s and 5s, you can multiply top and bottom by the "missing" factor to reach a power of ten:

$$\frac{7}{40} = \frac{7}{2^3 \times 5} = \frac{7 \times 5^2}{2^3 \times 5^3} = \frac{175}{1000} = 0.175.$$

Any other prime (like 3 or 7) in the reduced denominator can never be cancelled into a power of ten, so the decimal must repeat instead.

Predicting terminate vs. repeat by factoring

Without dividing, decide whether each fraction terminates. Factor each reduced denominator and check for primes other than 2 and 5.

$$\begin{aligned} \tfrac{9}{80}: &\quad 80 = 2^4 \times 5 \;\Rightarrow\; \textbf{terminates} \\ \tfrac{5}{12}: &\quad 12 = 2^2 \times 3 \;\Rightarrow\; \textbf{repeats} \text{ (the 3)} \\ \tfrac{13}{200}: &\quad 200 = 2^3 \times 5^2 \;\Rightarrow\; \textbf{terminates} \end{aligned}$$

The number of decimal places when it terminates equals the larger of the two exponents: $\frac{9}{80}$ needs 4 places (0.1125), since 2^4 beats 5^1.

The $10x - x$ Trick for Repeating Decimals

A repeating decimal is an infinite sum, but a little algebra collapses it to a fraction. Set x

equal to the decimal, then multiply by the power of ten that shifts one full repeating block past the point. Subtracting the original cancels the infinite tail, leaving an ordinary equation to solve. If the repeating block has k digits, multiply by 10^k.

Converting $0.\overline{27}$ to a fraction

The block "27" has $k = 2$ digits, so multiply by $10^2 = 100$. Let $x = 0.\overline{27}$:

$$\begin{aligned} 100x &= 27.272727\ldots \\ -\quad x &= \quad\ 0.272727\ldots \\ \hline 99x &= 27 \\ x &= \frac{27}{99} = \frac{3}{11}. \end{aligned}$$

So $0.\overline{27} = \frac{3}{11}$. Check: $3 \div 11 = 0.2727\ldots$ **Good.** The denominator 99 is always $\underbrace{9\cdots9}_{k}$ for a block of length k.

A repeating decimal with a delay: $0.16\overline{6}$

Here one digit (1) sits before the repeating part, so shift twice. Let $x = 0.16666\ldots$; multiply by 10 and by 100 so the two versions line up on the repeating 6s:

$$\begin{aligned} 100x &= 16.6666\ldots \\ -\quad 10x &= \quad 1.6666\ldots \\ \hline 90x &= 15 \\ x &= \frac{15}{90} = \frac{1}{6}. \end{aligned}$$

So $0.1\overline{6} = \frac{1}{6}$, matching the division $1 \div 6 = 0.1666\ldots$

The famous identity $0.\overline{9} = 1$ falls straight out of the trick: if $x = 0.\overline{9}$ then $10x = 9.\overline{9}$, so $10x - x = 9$, giving $9x = 9$ and $x = 1$. A repeating 9 is just another name for the whole number above it.

Scientific Notation: A Bridge for Tiny and Huge Decimals

Scientific notation writes a number as a single nonzero digit, a decimal part, and a power of ten: $a \times 10^n$ with $1 \le a < 10$. A *positive* exponent shifts the point right (big numbers); a *negative* exponent shifts it left (small decimals).

$$53{,}000 = 5.3 \times 10^4, \qquad 0.00047 = 4.7 \times 10^{-4}.$$

The exponent is just a bookkeeping count of how many places the decimal point moved — the same "move the point" idea from decimal multiplication and division.

Rounding Error and Bounds

When you round, the true value lives in a band around your rounded answer. Rounding to a place means the exact value is within *half* of that place. If x rounds to 4.7 at the tenths place, then

$$4.65 \le x < 4.75,$$

so the rounding error is at most 0.05. This is why measurements are reported with a place that reflects their precision: writing 4.70 instead of 4.7 claims a tighter band, $4.695 \le x < 4.705$.

Decimals Are Dense: Always Room in Between

Between any two different decimals there is always another decimal — in fact infinitely many. To find one, just add a place and pick a digit strictly between them, or average the two:

$$\text{between } 0.4 \text{ and } 0.5: \quad \frac{0.4 + 0.5}{2} = 0.45, \quad \text{then } 0.41,\ 0.42,\ \ldots$$

Unlike whole numbers (nothing sits between 4 and 5), the decimals have no "next" number. This **density** is a key difference between counting numbers and the numbers on the full number line.

Terminating and repeating decimals are exactly the *rational* numbers (ratios of whole numbers). A decimal that goes on forever *without* ever repeating — like $\pi = 3.14159\ldots$ or $\sqrt{2} = 1.41421\ldots$ — is **irrational** and can never be written as a fraction. Every such number still has a home on the number line.

11.10 Why It Works

Some fractions become terminating decimals and others repeat, and the denominator decides: $\frac{a}{b}$ in lowest terms terminates exactly when b has no prime factors other than 2 and 5, because then b divides some power of 10. Otherwise the long division must cycle (only b possible remainders), so the decimal repeats with period at most $b - 1$. $\frac{1}{7} = 0.\overline{142857}$ uses all six possible remainders.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

12 Ratios, Rates & Proportions

Writing and simplifying ratios, equivalent ratios, unit rates and best buys, solving proportions, scale drawings, and similar figures

12.1 What a Ratio Is

The Rule

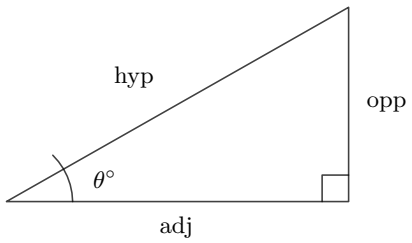
A **ratio** compares two quantities by division — it tells you how much of one thing there is *for each* amount of another. If a bowl holds 3 apples and 5 oranges, the ratio of apples to oranges is "3 to 5."

There are **three ways** to write the same ratio:

- with the word *to*: 3 to 5
- with a colon: $3 : 5$
- as a fraction: $\frac{3}{5}$

Order matters! The ratio of apples to oranges $(3 : 5)$ is different from oranges to apples $(5 : 3)$. To **simplify** a ratio, divide both numbers by their greatest common factor, just like reducing a fraction.

Opposite, adjacent and hypotenuse are named from the angle.



A proportion is two equal ratios, $\frac{a}{b} = \frac{c}{d}$, and cross-multiplying turns it into $ad = bc$, an equation you can solve. Similar figures, scale drawings, unit prices and speeds are all proportions. Keep the same quantity on top in both fractions and the setup takes care of itself.

Worked Example: three ways & simplifying

A parking lot has 12 cars and 8 trucks. Write the ratio of cars to trucks.

- Three ways: 12 to 8, $12 : 8$, $\frac{12}{8}$.
- Both numbers share a factor of 4: $12 \div 4 = 3$ and $8 \div 4 = 2$.

- Simplest form: **3 : 2** — for every 3 cars there are 2 trucks.

Where it goes wrong

Setting up the proportion with mismatched orders — miles over hours on one side and hours over miles on the other. Label every number with its unit before writing the fractions.

Worked Example: comparing with a total

In a class of 10 boys and 15 girls, the ratio of boys to girls is $10 : 15 = \mathbf{2 : 3}$. The ratio of boys to *all* students is $10 : 25 = \mathbf{2 : 5}$, because the class has $10 + 15 = 25$ students in all.

Tip: A ratio in simplest form has no common factors left, but you never turn it into a single number — keep both parts. Write 2 : 3, not "0.67." Always state the order in words so you don't flip it.

Try it: Quick check

A fruit bowl has 4 apples and 6 oranges. Write the ratio of apples to oranges in all **three** ways (with "to," a colon, and a fraction).

Work it on paper before revealing the answer.

Answer. Answer: 4 to 6, $4 : 6$, $\frac{4}{6}$

12.2 Equivalent Ratios & Ratio Tables

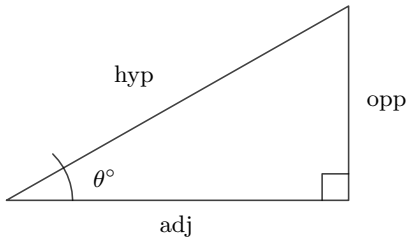
The Rule

Two ratios are **equivalent** if they simplify to the same ratio. You build an equivalent ratio by **multiplying or dividing both parts by the same number** — exactly how you make equivalent fractions.

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$

A **ratio table** lists several equivalent ratios in a row, which makes patterns easy to see and missing values easy to fill in.

Opposite, adjacent and hypotenuse are named from the angle.



Worked Example: filling a ratio table

A punch recipe mixes juice to soda in the ratio 2 : 5. Complete the table.

Each column keeps the ratio 2 : 5: multiply both parts of 2 : 5 by 2 to get 4 : 10, and by 3 to get 6 : 15.

Worked Example: are they equivalent?

Is 6 : 10 equivalent to 9 : 15? Simplify each: 6 : 10 = 3 : 5 and 9 : 15 = 3 : 5. They match, so **yes**, they are equivalent.

Tip: You must multiply (or divide) *both* numbers by the *same* amount. Changing only one part, or adding instead of multiplying, breaks the ratio.

Try it: Quick check

Simplify the ratio 8 : 12 to its lowest terms.

Work it on paper before revealing the answer.

Answer. Answer: 2 : 3

12.3 Rates & Unit Rates

The Rule

A **rate** is a ratio that compares two quantities with **different units**, such as miles and hours, or dollars and pounds. A **unit rate** is a rate with a denominator of **1**: how much of the first quantity for exactly *one* of the second, like miles *per hour* or dollars *per pound*. To find a unit rate, **divide** the first quantity by the second.

A **unit price** is a unit rate for cost (dollars per item or per unit). Comparing unit prices tells you the **best buy** — the lower unit price is the better deal.

Worked Example: finding a unit rate

A car travels 150 miles in 3 hours. The unit rate is

$$\frac{150 \text{ miles}}{3 \text{ hours}} = \frac{50 \text{ miles}}{1 \text{ hour}} = \mathbf{50} \text{ miles per hour.}$$

Divide $150 \div 3 = 50$. The word *per* means "for each one."

Worked Example: best buy by unit price

A 6-pack of juice costs \$12 and a 10-pack costs \$25. Which is the better buy?

- 6-pack: $\$12 \div 6 = \$2.00$ per bottle.
- 10-pack: $\$25 \div 10 = \$2.50$ per bottle.

The 6-pack is cheaper per bottle, so the **6-pack is the better buy**.

Tip: Keep the units consistent before you compare. If one price is per gram and another is per kilogram, convert so both use the *same* unit first, then compare.

Try it: Quick check

Simplify the ratio $10 : 15$ to its lowest terms.

Work it on paper before revealing the answer.

Answer. Answer: $2 : 3$

12.4 Writing & Solving Proportions

The Rule

A **proportion** is an equation stating that two ratios are equal, like $\frac{2}{3} = \frac{8}{12}$. When one part is unknown, we solve for it using **cross-multiplication**: in a true proportion, the two diagonal products are equal.

$$\frac{a}{b} = \frac{c}{d} \quad \Longrightarrow \quad a \times d = b \times c.$$

Multiply across the diagonals, set the products equal, then divide to isolate the unknown.

Worked Example: solving with cross-multiplication

Solve $\frac{x}{4} = \frac{6}{8}$.

- Cross-multiply: $8 \times x = 4 \times 6$, so $8x = 24$.
- Divide both sides by 8: $x = 24 \div 8 = \mathbf{3}$.

Check: $\frac{3}{4} = \frac{6}{8}$? Yes, both simplify to $\frac{3}{4}$.

Worked Example: an answer that is not a whole number

Solve $\dfrac{5}{8} = \dfrac{x}{20}$. Cross-multiply: $8x = 5 \times 20 = 100$. Divide: $x = 100 \div 8 = \mathbf{12.5}$. Ratios don't always give whole-number answers, and that is perfectly fine.

Tip: Set up a proportion so matching units line up — keep the same kind of quantity in the same position (top with top, bottom with bottom). Cross-multiply to solve.

Try it: Quick check

In a class of 10 boys and 15 girls, write the ratio of boys to girls in simplest form.

Work it on paper before revealing the answer.

Answer. Answer: 2 : 3

12.5 Using Proportions to Solve Word Problems

The Rule

Many everyday problems are really proportions in disguise: recipes, prices, speeds, and mixtures. To solve one:

1. Write a ratio from the known facts, labeling the units.
2. Set it equal to a second ratio in the *same order*, using a letter for the unknown.
3. Cross-multiply and solve.

Worked Example: a recipe problem

A recipe uses 2 cups of flour to make 12 cookies. How much flour makes 30 cookies?

$$\frac{2 \text{ cups}}{12 \text{ cookies}} = \frac{x \text{ cups}}{30 \text{ cookies}}$$

Cross-multiply: $12x = 2 \times 30 = 60$, so $x = 60 \div 12 = \mathbf{5}$ cups. Notice cups stayed on top and cookies on the bottom on both sides.

Worked Example: a price problem

If 4 tickets cost \$20, how much do 7 tickets cost? Set up $\frac{4}{20} = \frac{7}{x}$ (tickets over dollars). Cross-multiply: $4x = 20 \times 7 = 140$, so $x = 140 \div 4 = \$\mathbf{35}$.

Tip: Always write the units next to your numbers as you set up the proportion. Matching units in matching spots is the surest way to avoid flipping the ratio.

Try it: Quick check

Simplify the ratio $9 : 3$ to its lowest terms.

Work it on paper before revealing the answer.

Answer. Answer: $3 : 1$

12.6 Scale Drawings, Maps & Models

The Rule

A **scale** tells how a drawing, map, or model relates to the real object. It is written as a ratio, such as "1 inch = 20 miles" or $1 : 18$. Every scale problem is a proportion:

$$\frac{\text{scale drawing size}}{\text{real size}} = \frac{\text{drawing measurement}}{\text{actual measurement}}.$$

Set up the scale ratio, then cross-multiply to find the missing distance — either the real distance from a drawing, or the drawing size from a real distance.

Worked Example: reading a map

On a map, 1 inch represents 20 miles. Two cities are 4 inches apart on the map. How far apart are they really?

$$\frac{1 \text{ in}}{20 \text{ mi}} = \frac{4 \text{ in}}{x \text{ mi}} \implies 1 \cdot x = 20 \times 4 = 80.$$

The cities are **80** miles apart.

Worked Example: a scale model

A model car is built at scale $1 : 18$, meaning 1 cm on the model equals 18 cm on the real car. If the model is 25 cm long, the real car is $25 \times 18 = \mathbf{450}$ cm long (that is 4.5 meters).

Tip: Keep units consistent and keep the scale ratio in the same order on both sides (drawing on top, real on bottom). A quick sanity check: real distances should come out much larger than the drawing.

Try it: Quick check

Are the ratios $2:3$ and $6:9$ equivalent? Answer yes or no.

Work it on paper before revealing the answer.

Answer. Answer: Yes

12.7 Similar Figures & Missing Sides

The Rule

Two figures are **similar** when they have the *same shape* but not necessarily the same size — one is a scaled copy of the other. In similar figures, **corresponding sides are proportional**: they all share the same **scale factor**. To find a missing side, set up a proportion of corresponding sides and solve.

$$\frac{\text{side of figure 1}}{\text{matching side of figure 2}} = \frac{\text{another side of figure 1}}{\text{its matching side of figure 2}}.$$

Worked Example: missing side of a similar rectangle

A small rectangle is 3 by 4. A similar large rectangle has its short side equal to 6 (matching the 3). Find the long side x (matching the 4).

$$\frac{3}{6} = \frac{4}{x} \implies 3x = 6 \times 4 = 24 \implies x = \mathbf{8}.$$

The scale factor is 2 (since $6 = 3 \times 2$), and indeed $4 \times 2 = 8$.

Worked Example: shadows and similar triangles

At the same time of day, a 5-ft person casts a 6-ft shadow while a tree casts a 24-ft shadow. The two right triangles are similar, so

$$\frac{5 \text{ ft}}{6 \text{ ft}} = \frac{h}{24 \text{ ft}} \implies 6h = 5 \times 24 = 120 \implies h = \mathbf{20} \text{ ft}.$$

The tree is 20 feet tall.

Tip: Match *corresponding* sides carefully — pair the shortest with the shortest, and so on. Every pair of corresponding sides in similar figures shares the very same scale factor.

Try it: Quick check

Fill in the missing number: $3 : 4 = 6 : __$

Work it on paper before revealing the answer.

Answer. Answer: 8

12.8 Going Deeper: Advanced Ratio & Proportion Ideas

Direct vs. Inverse Variation

When two quantities stay in a **constant ratio**, we say y **varies directly** with x:

$$y = kx \qquad \text{equivalently} \qquad \frac{y}{x} = k \text{ (constant)}.$$

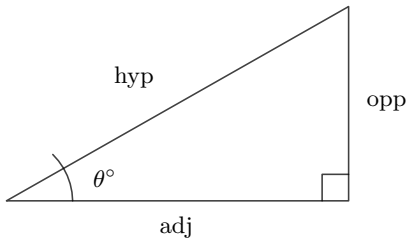
Here k is the **constant of proportionality**. Double x and y doubles too — the graph is a straight line through the origin.

When instead the two quantities have a **constant product**, y **varies inversely** with x:

$$y = \frac{k}{x} \qquad \text{equivalently} \qquad x \cdot y = k \text{ (constant)}.$$

Now doubling x *halves* y. More workers means less time; faster speed means fewer hours for the same trip. The tell-tale sign of inverse variation is that the *product* stays fixed while the *ratio* does not.

Opposite, adjacent and hypotenuse are named from the angle.



Worked Example: spotting and using inverse variation

It takes 6 painters 8 days to paint a mural. Working at the same pace, how long will 16 painters take? More painters means less time, so this is **inverse** variation: painters × days is

constant.

- Find the constant: $k = 6 \times 8 = 48$ (painter-days of work).
- Solve for the new time: $16 \times d = 48$, so $d = 48 \div 16 = \mathbf{3}$ days.

Warning: a beginner might set up a direct proportion $\frac{6}{8} = \frac{16}{d}$ and get the wrong answer. Ask first: as one quantity grows, does the other grow (direct) or shrink (inverse)?

Three-Term Ratios $a : b : c$

A ratio can compare **three or more** quantities at once, such as $2 : 3 : 4$. Think of it as splitting a whole into **parts**: $2 : 3 : 4$ uses $2 + 3 + 4 = 9$ equal parts. To share an amount in a given ratio, divide by the total number of parts to find **one part**, then multiply.

You can also **combine** two separate two-term ratios into one. If $A : B = 2 : 3$ and $B : C = 6 : 5$, first rescale so the shared term B matches. Multiply the first ratio by 2 to get $A : B = 4 : 6$; now B is 6 in both, giving $A : B : C = 4 : 6 : 5$.

Worked Example: sharing money in a ratio

Three friends split \$180 in the ratio $2 : 3 : 4$. Total parts $= 2 + 3 + 4 = 9$, so one part is $\$180 \div 9 = \$20$.

- First friend: $2 \times \$20 = \$\mathbf{40}$.
- Second friend: $3 \times \$20 = \$\mathbf{60}$.
- Third friend: $4 \times \$20 = \$\mathbf{80}$.

Check: $\$40 + \$60 + \$80 = \$180$. ✓

Mean Proportional & Continued Proportions

In the proportion $\frac{a}{b} = \frac{b}{c}$, the repeated middle term b is called the **mean proportional** (or *geometric mean*) of a and c. Cross-multiplying gives

$$b^2 = a \cdot c \qquad \Longrightarrow \qquad b = \sqrt{a \cdot c}.$$

This is different from the ordinary (arithmetic) average $\frac{a+c}{2}$. For example, the mean proportional of 4 and 9 is $\sqrt{4 \cdot 9} = \sqrt{36} = 6$, whereas their arithmetic average is 6.5.

The Golden Ratio

The **golden ratio** appears when a length is split so that *the whole is to the larger part as the*

larger part is to the smaller part. Writing the larger part as a and the smaller as b:

$$\frac{a+b}{a} = \frac{a}{b} = \varphi \approx 1.618.$$

Its exact value is $\varphi = \dfrac{1+\sqrt{5}}{2}$. The golden ratio has a remarkable self-referencing property: $\varphi^2 = \varphi + 1$ and $\dfrac{1}{\varphi} = \varphi - 1$. It shows up in art, architecture, and the spiral arrangement of leaves and seeds in nature.

Weighted Averages & Mixtures

A plain average treats every item equally, but a **weighted average** counts some values more heavily according to a ratio of "weights" (amounts). For two groups,

$$\text{weighted average} = \frac{w_1 v_1 + w_2 v_2}{w_1 + w_2},$$

where each v is a value and each w is how much of it there is. Mixture problems — blending solutions of different concentrations, or averaging test scores of different weights — are exactly this idea. The result always lands *between* the two values, pulled toward whichever group is larger.

Worked Example: mixing two solutions

Mix 2 liters of a 30% salt solution with 3 liters of an 80% salt solution. What is the concentration of the blend?

$$\text{concentration} = \frac{(2)(30\%) + (3)(80\%)}{2+3} = \frac{60+240}{5} = \frac{300}{5} = \mathbf{60}\%.$$

Notice 60% sits between 30% and 80%, closer to 80% because there is more of the stronger solution.

Combined Rates: Work Problems & Average Speed

When two workers (or pipes, or machines) act together, **add their rates**, not their times. If one finishes a job in a hours and the other in b hours, their combined time t satisfies

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{t}.$$

A close cousin appears with **average speed over equal distances**. If you drive one leg at speed v_1 and an equal-length leg at v_2, the average speed is *not* $\frac{v_1+v_2}{2}$ but the **harmonic mean**:

$$\bar{v} = \frac{2 v_1 v_2}{v_1 + v_2}.$$

You spend more time at the slower speed, so the true average is dragged below the plain

average.

Worked Example: a work problem and an average-speed problem

(a) Working together. Pipe A fills a tank in 4 hours, pipe B in 6 hours. Together:

$$\frac{1}{4}+\frac{1}{6}=\frac{3}{12}+\frac{2}{12}=\frac{5}{12}\ \text{tank per hour}\ \Longrightarrow\ t=\frac{12}{5}=\mathbf{2.4}\ \text{hours}.$$

Sensibly, this is faster than either pipe alone.

(b) Average speed. You drive 60 miles at 30 mph, then 60 miles at 60 mph.

$$\bar{v}=\frac{2(30)(60)}{30+60}=\frac{3600}{90}=\mathbf{40}\ \text{mph}.$$

The plain average would be 45 mph, but the correct answer is lower because more time is spent going 30 mph.

Scaling Area & Volume

When a figure is enlarged by a **scale factor** k, its lengths multiply by k — but its **area** multiplies by k^2 and its **volume** by k^3:

$$\text{length}\times k,\qquad \text{area}\times k^2,\qquad \text{volume}\times k^3.$$

This is why doubling a photo's dimensions uses *four* times the ink, and why a scale model at $1:10$ holds only $\frac{1}{1000}$ of the real object's volume. It is the most-missed idea in ratios: side ratios and area (or volume) ratios are *not* the same number.

Big-picture tip: Before setting up any advanced ratio problem, ask three questions. (1) Do the quantities keep a constant *ratio* (direct) or a constant *product* (inverse)? (2) Am I comparing *lengths*, *areas*, or *volumes* — and do I need k, k^2, or k^3? (3) Should I add *rates* rather than times? Answering these first prevents the most common mistakes.

Try it: Quick check

Fill in the missing number: $5:2=__:6$

Work it on paper before revealing the answer.

Answer. Answer: 15

12.9 Why It Works

Cross-multiplication is just clearing denominators: multiplying both sides of $\frac{a}{b}=\frac{c}{d}$ by bd gives $ad=bc$. Since the step is reversible (divide by bd again), $ad=bc$ holds exactly when the ratios are

equal. Proportional reasoning is the same as "$y = kx$": the constant k is the unit rate, and every proportion problem is really finding k from one pair and applying it to another.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

13 Percents

What percent means, converting among fractions/decimals/percents, the three percent problems, percent change, and real-world money math

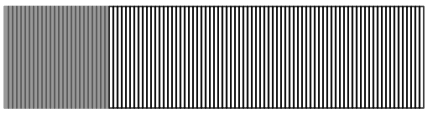
13.1 What Percent Means (Per Hundred)

The Rule

The word **percent** comes from the Latin *per centum*, meaning "**per hundred**." A percent is just a special fraction whose denominator is always 100. So 35% means $\frac{35}{100}$, or "35 out of every 100."

You can picture a percent as part of a 10×10 grid of 100 little squares: 35% shades 35 of the 100 squares. The whole grid is $100\% = \frac{100}{100} = 1$.

Percent means per hundred.



25 of 100 parts shaded

Every percent problem is part = percent × whole with one of the three unknown. Percent increase and decrease are multipliers: +20% is ×1.2, −20% is ×0.8, and successive changes multiply rather than add. Simple interest is a percent applied per year.

Worked Example: reading a percent

A class of 100 students had 60 walk to school. What percent walked? Since "percent" means "per hundred," and there are exactly 100 students, 60 out of 100 is simply **60**%. The other 40 students are 40%.

Where it goes wrong

Believing a 20% rise followed by a 20% fall returns to the start. $1.2 \times 0.8 = 0.96$: you end 4% below where you began.

Worked Example: percent as a fraction of 100

7% means $\frac{7}{100}$, and 150% means $\frac{150}{100}$. A percent can be more than 100%: that just means *more than one whole*. 150% is one whole plus another half.

Tip: Whenever you see the percent sign %, you can mentally replace it with "$\div 100$." So 35% becomes $35 \div 100 = 0.35$.

Try it: Quick check

Out of 100 students, 23 ride the bus. What percent of the students ride the bus?

Work it on paper before revealing the answer.

Answer. Answer: 23%

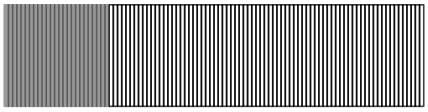
13.2 Converting Among Fractions, Decimals, and Percents

The Rule

These three forms all describe the same amount. To move between them:

- **Percent → decimal:** drop the % and **divide by** 100 (move the point two places *left*).
- **Decimal → percent: multiply by** 100 (move the point two places *right*) and add %.
- **Percent → fraction:** write it over 100, then simplify.
- **Fraction → percent:** divide top by bottom to get a decimal, then multiply by 100; or rewrite the fraction with denominator 100.

Percent means per hundred.



25 of 100 parts shaded

Worked Example: percent ↔ decimal

$45\% = 0.45$ (move the point two places left). $0.6 = 60\%$ (move two places right). $8\% = 0.08$ (you must fill in a zero: 08). $1.25 = 125\%$.

Worked Example: percent ↔ fraction

Convert 40% to a fraction: $\frac{40}{100} = \frac{2}{5}$ (divide top and bottom by 20).

Convert $\frac{3}{4}$ to a percent: $3 \div 4 = 0.75$, and $0.75 = \mathbf{75}\%$. Or notice $\frac{3}{4} = \frac{75}{100} = 75\%$.

Tip: Memorize the "friendly" equivalents: $\frac{1}{2} = 50\%$, $\frac{1}{4} = 25\%$, $\frac{3}{4} = 75\%$, $\frac{1}{5} = 20\%$, $\frac{1}{10} = 10\%$, $\frac{1}{3} \approx 33.3\%$. They make estimating fast.

Try it: Quick check

On a 10×10 grid of 100 squares, 45 squares are shaded. What percent of the grid is shaded?

Work it on paper before revealing the answer.

Answer. Answer: 45%

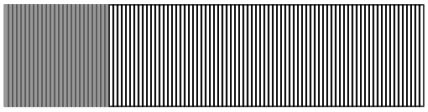
13.3 Finding the Percent of a Number

The Rule

To find a percent **of** a number, change the percent to a decimal (or fraction) and **multiply**. In math, the word "of" almost always means multiply:

$$\text{part} = \text{percent (as a decimal)} \times \text{whole}.$$

Percent means per hundred.



25 of 100 parts shaded

Worked Example: using a decimal

Find 20% of 80. Change 20% to 0.20, then multiply: $0.20 \times 80 = \mathbf{16}$.

Worked Example: using a friendly fraction

Find 25% of 60. Since $25\% = \frac{1}{4}$, just divide by 4: $60 \div 4 = \mathbf{15}$. And 10% of any number is found by moving the point one place left, so 10% of 60 is 6.

Tip: "Of" means **multiply**. To find 10%, move the decimal one place left; to find 1%, move it two places left. Build other percents from these: $30\% = 3 \times (10\%)$.

Try it: Quick check

Write 50% as a decimal.

Work it on paper before revealing the answer.

Answer. Answer: 0.5

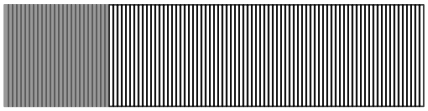
13.4 Finding What Percent One Number Is of Another

The Rule

To find what percent one number is of another, make a fraction $\frac{\text{part}}{\text{whole}}$, then convert it to a percent (divide, then multiply by 100). This uses the **percent proportion**:

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}.$$

Percent means per hundred.



25 of 100 parts shaded

Worked Example: part out of whole

12 is what percent of 48? Write $\frac{12}{48} = \frac{1}{4} = 0.25 = \mathbf{25}\%$.

Worked Example: a test score

A student got 18 out of 20 questions right. What percent is that? $\frac{18}{20} = 0.9 = \mathbf{90}\%$.

Tip: The **whole** is the number that comes after "of" (the total you are comparing to). The **part** is the piece being compared. Put part on top, whole on the bottom.

Try it: Quick check

Write the decimal 0.25 as a percent.

Work it on paper before revealing the answer.

Answer. Answer: 25%

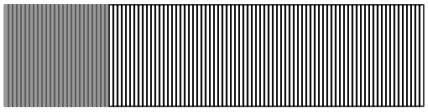
13.5 Finding the Whole (Given a Part and a Percent)

The Rule

Sometimes you know the part and the percent, and you must find the **whole**. Rearrange the percent proportion, or divide the part by the percent (as a decimal):

$$\text{whole} = \frac{\text{part}}{\text{percent (as a decimal)}}.$$

Percent means per hundred.



25 of 100 parts shaded

Worked Example: undo the multiplication

15 is 25% of what number? Since $25\% = 0.25$, divide: $15 \div 0.25 = \mathbf{60}$. Check: $0.25 \times 60 = 15$. ✓

Worked Example: using the proportion

30 is 60% of what number? Set $\frac{30}{w} = \frac{60}{100}$. Cross-multiply: $60w = 3000$, so $w = \mathbf{50}$.

Tip: Finding the whole *undoes* finding the part. Finding the part **multiplies**; finding the whole **divides**. If your answer for the whole is smaller than the part, something went wrong.

Try it: Quick check

Write 10% as a fraction in simplest form.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{1}{10}$

13.6 Percent Increase and Percent Decrease

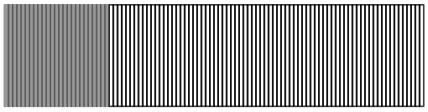
The Rule

Percent change tells how much a quantity grew or shrank, compared to where it started:

$$\text{percent change} = \frac{\text{amount of change}}{\text{original amount}} \times 100\%.$$

If the new value is bigger, it is a percent **increase**; if smaller, a percent **decrease**. Always divide by the *original* amount.

Percent means per hundred.



25 of 100 parts shaded

Worked Example: percent increase

A plant grew from 20 cm to 25 cm. Change $= 25 - 20 = 5$ cm. Percent increase $= \frac{5}{20} \times 100\% = 0.25 \times 100\% = \mathbf{25}\%$.

Worked Example: percent decrease

A price dropped from 80 dollars to 60 dollars. Change $= 80 - 60 = 20$. Percent decrease $= \frac{20}{80} \times 100\% = 0.25 \times 100\% = \mathbf{25}\%$.

Tip: Divide the change by the **original** number, not the new one. A 50% increase followed by a 50% decrease does *not* bring you back to the start, because the two percents are taken of different amounts.

Try it: Quick check

Write the fraction $\frac{3}{4}$ as a percent.

Work it on paper before revealing the answer.

Answer. Answer: 75%

13.7 Real-World Percents: Discount, Tax, Tip, Commission

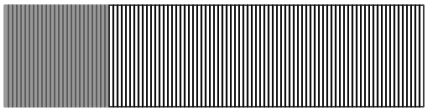
The Rule

Everyday money problems are percent-of-a-number problems in disguise:

- **Discount / sale price:** discount = percent off × original price; sale price = original − discount.
- **Sales tax:** tax = tax rate × price; total = price + tax.

- **Tip:** tip = tip rate × bill.
- **Commission:** commission = commission rate × sales.

Percent means per hundred.



25 of 100 parts shaded

Worked Example: sale price

A \$50 jacket is 20% off. Discount = $0.20 \times \$50 = \$10$. Sale price = $\$50 - \$10 = \$\mathbf{40}$. (Shortcut: paying 80%, so $0.80 \times \$50 = \$40$.)

Worked Example: tax and tip

A dinner bill is \$40. With 10% tax: $0.10 \times \$40 = \$4$, so far \$44. Add a 15% tip on the original \$40 bill: $0.15 \times \$40 = \$6$. Total = $\$44 + \$6 = \$\mathbf{50}$.

Worked Example: commission

A salesperson earns 5% commission on \$3000 of sales. Commission = $0.05 \times \$3000 = \$\mathbf{150}$.

Tip: For a discount, you can either subtract the discount or multiply by (100% − percent off). For tax or tip, you can multiply by (100% + percent) to get the total in one step.

Try it: Quick check

Find 10% of 70.

Work it on paper before revealing the answer.

Answer. Answer: 7

13.8 Simple Interest ($I = Prt$)

The Rule

When money is saved or borrowed, **interest** is a percent paid for the use of that money. **Simple interest** is found with:

$$I = P \times r \times t,$$

where P is the **principal** (starting amount), r is the yearly interest **rate** (as a decimal), and t is the **time** in years. The total amount you end with is $A = P + I$.

Worked Example: interest earned

You deposit $P = \$500$ at a rate $r = 4\% = 0.04$ per year for $t = 3$ years. Interest $= I = 500 \times 0.04 \times 3 = \$\mathbf{60}$. Total in the account: $A = \$500 + \$60 = \$560$.

Worked Example: a shorter time

A \$2000 loan has a rate of 6% for $t = \frac{1}{2}$ year (6 months). $I = 2000 \times 0.06 \times 0.5 = \$\mathbf{60}$. You would repay $\$2000 + \$60 = \$2060$.

Tip: Both r and t must match up: r is *per year*, so t must be in *years*. For months, use $t = \dfrac{\text{months}}{12}$. Always convert the percent rate to a decimal before multiplying.

Try it: Quick check

Find 25% of 40.

Work it on paper before revealing the answer.

Answer. Answer: 10

13.9 Going Deeper: Advanced Percent Ideas

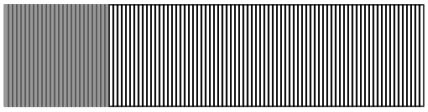
Successive Percent Changes Multiply (They Don't Add)

When one percent change follows another, you **multiply their factors**; you do not add the percents. A change of $p\%$ turns a quantity into a fraction of itself called the **multiplier**:

$$\text{increase of } p\% \;\to\; \text{multiply by } \left(1 + \tfrac{p}{100}\right), \qquad \text{decrease of } p\% \;\to\; \text{multiply by } \left(1 - \tfrac{p}{100}\right).$$

Two changes in a row multiply their multipliers. For example, $+50\%$ then -50% gives $1.50 \times 0.50 = 0.75$, a net *decrease* of 25%, not a return to the start. The order does not matter, since multiplication commutes.

Percent means per hundred.



25 of 100 parts shaded

Worked Example: two changes in a row

A \$200 stock rises 10% one week, then falls 20% the next. Multiply the factors: $\$200 \times 1.10 \times 0.80$. First $\$200 \times 1.10 = \$220$; then $\$220 \times 0.80 = \$\mathbf{176}$. The overall multiplier is $1.10 \times 0.80 = 0.88$, a **12**% net decrease. Notice that $+10\%$ and -20% do *not* combine to -10%.

Percent vs. Percentage Points

A **percentage point** is an absolute gap between two percents; a **percent change** is relative to the starting percent. These are different! If an interest rate rises from 4% to 6%, it went up 2 **percentage points**, but the *relative* increase is $\dfrac{6-4}{4} \times 100\% = 50\%$. Newspapers and ads often blur this, so read carefully: "2 points" and "50% higher" can describe the very same change.

Reverse Percent: Finding the Original After a Change

If a price *already includes* a markup or discount, you cannot just take that percent off the final number to get back. Instead, divide by the multiplier. If a final amount F came from an original X after a $p\%$ increase, then $F = X\left(1 + \frac{p}{100}\right)$, so

$$X = \frac{F}{1 + \frac{p}{100}}.$$

For a discount, divide by $\left(1 - \frac{p}{100}\right)$ instead. The key idea: the percent was taken *of the original*, which is the unknown, so you must undo the multiplication.

Worked Example: reverse percent (undo a tax)

A gadget costs \$54 *after* an 8% sales tax is added. What was the pre-tax price? The final price is the original times 1.08, so

$$X = \frac{\$54}{1.08} = \$\mathbf{50}.$$

Check: $\$50 \times 0.08 = \$4$ tax, and $\$50 + \$4 = \$54$. ✓ A common mistake is to take 8% of \$54 (which is \$4.32) and subtract, giving \$49.68 — wrong, because the tax was 8% of \$50, not of \$54.

Compound Interest: A Preview of $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Simple interest ($I = Prt$) is always figured on the original principal. **Compound** interest instead earns interest *on the interest* already added, so the balance grows by repeated multiplication. If interest compounds n times per year at yearly rate r (a decimal) for t years, the ending amount is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Each period multiplies the balance by $\left(1 + \frac{r}{n}\right)$, and there are nt periods in all. For the same rate and time, compounding always beats simple interest, because later periods earn on a larger balance. (You will study this fully in Algebra; here just notice it is repeated percent increase.)

Worked Example: simple vs. compound

Invest $P = \$1000$ at $r = 10\% = 0.10$ for $t = 2$ years.

Simple: $I = Prt = 1000 \times 0.10 \times 2 = \$200$, so $A = \$1000 + \$200 = \$1200$.

Compounded yearly ($n = 1$): $A = 1000\left(1 + \frac{0.10}{1}\right)^{1\times 2} = 1000(1.10)^2 = 1000 \times 1.21 = \mathbf{\$1210}$.

The extra \$10 is the second year's interest earned on the first year's \$100 of interest — interest on interest.

Percent Error: How Far Off a Measurement Is

In science, **percent error** compares a measured (or estimated) value to the true (accepted) value, relative to that true value:

$$\text{percent error} = \frac{|\text{measured} - \text{actual}|}{|\text{actual}|} \times 100\%.$$

The absolute value bars keep the answer positive — percent error measures *size* of the mistake, not its direction. It is structurally the same as percent change, but the denominator is always the *true* value.

Mixture and Concentration Problems

A **concentration** is a percent: the amount of pure substance divided by the total amount of mixture. The pure amount is

$$\text{pure} = \text{concentration} \times \text{total volume}.$$

When two solutions are combined, the pure amounts **add**, and so do the volumes; the new concentration is the total pure amount over the total volume. Concentrations themselves do *not* simply average unless the two volumes happen to be equal.

Worked Example: mixing two solutions

Mix 2 liters of a 30% salt solution with 3 liters of an 80% salt solution. Find the pure salt in each, then combine:

- First: $0.30 \times 2 = 0.6$ L of salt.
- Second: $0.80 \times 3 = 2.4$ L of salt.

Total salt $= 0.6 + 2.4 = 3.0$ L; total volume $= 2 + 3 = 5$ L. New concentration $= \dfrac{3.0}{5} = 0.60 =$ **60**%. Note this is *not* the plain average of 30% and 80% (which would be 55%), because there was more of the stronger solution.

Big picture: Nearly every advanced percent idea is really about **multipliers**. An increase or decrease of p% is one multiplication by $\left(1 \pm \frac{p}{100}\right)$; chained changes multiply, reversing a change divides, and compounding repeats the multiplication nt times. Keep asking "percent *of what*?" — the denominator (original, actual, or total) is what everything is measured against.

13.10 Why It Works

A p% increase means new $=$ old $+\frac{p}{100}\cdot$old $= (1+\frac{p}{100})\cdot$old, one multiplication. Two changes in a row therefore multiply their factors, and multiplication is commutative — so the order of a rise and a fall does not matter, but they do not cancel unless the factors are reciprocals. This is why percent change is always measured against the original: the multiplier is defined relative to the starting amount.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

14 Rational Numbers

Fractions and decimals, positive and negative, all in one family

14.1 What a Rational Number Is

The Big Idea

A **rational number** is any number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The word "rational" comes from *ratio* — a comparison of two whole numbers.

This one idea gathers up almost every number you have met so far into a single family.

Rational numbers are all the fractions, positive and negative, including the integers and every terminating or repeating decimal. They are closed under the four operations (except division by zero), and the sign rules for integers carry over unchanged. Between any two of them lies another — the number line has no gaps you can point to, though irrationals still fill in more.

14.2 Classifying Numbers

Every one of these is rational, because each can be written as a fraction:

- **Integers:** $7 = \frac{7}{1}, \quad -4 = \frac{-4}{1}, \quad 0 = \frac{0}{1}.$
- **Fractions:** $\frac{3}{5}, \quad -\frac{2}{9}, \quad \frac{11}{4}.$
- **Terminating decimals:** $0.25 = \frac{1}{4}, \quad -1.5 = -\frac{3}{2}.$
- **Repeating decimals:** $0.\overline{3} = \frac{1}{3}.$

Is It Rational?

Show that -2.6 is rational.

Write it as a fraction: $-2.6 = -\frac{26}{10} = -\frac{13}{5}$. Since -13 and 5 are integers and the bottom is not 0, the number is rational.

Where it goes wrong

Treating $-\frac{3}{4}$, $\frac{-3}{4}$ and $\frac{3}{-4}$ as different numbers. They are the same; a fraction has one sign, and it can be written in any of the three places.

Tip: Whole numbers and integers are just rational numbers with a denominator of 1. Every integer is rational, but not every rational number is an integer.

Try it: Quick check

Write the integer -3 as a fraction.

Work it on paper before revealing the answer.

Answer. Answer: $\dfrac{-3}{1}$

14.3 Rational Numbers on the Number Line

Placing and Ordering

Rational numbers fill in the gaps *between* the integers. To place a fraction, split the space between two whole numbers into equal parts. Negative rationals sit to the **left** of 0; positive rationals to the **right**. As always, farther **right** means **greater**.

Comparing Two Fractions

Which is greater, $\dfrac{3}{4}$ or $\dfrac{5}{8}$? Rewrite with a common denominator of 8:

$$\frac{3}{4} = \frac{6}{8}, \qquad \frac{5}{8} = \frac{5}{8}.$$

Since $\dfrac{6}{8} > \dfrac{5}{8}$, we have $\dfrac{3}{4} > \dfrac{5}{8}$.

Ordering with Negatives

Order $-\dfrac{1}{2},\ 0.3,\ -1,\ \dfrac{3}{4}$ from least to greatest.
As decimals: $-0.5,\ 0.3,\ -1,\ 0.75$. Reading the number line left to right:

$$-1 < -\tfrac{1}{2} < 0.3 < \tfrac{3}{4}.$$

Tip: To compare tricky rationals, turn them all into the *same form* — either common-denominator fractions or decimals. With negatives, remember $-\frac{1}{2}$ is *greater* than -1.

Try it: Quick check

Write the decimal 0.7 as a fraction.

Work it on paper before revealing the answer.

Answer. Answer: $\dfrac{7}{10}$

14.4 Absolute Value of Rational Numbers

Distance from Zero

The **absolute value** of a rational number is its distance from 0 on the number line, so it is never negative. We write it with bars:

$$\left|-\tfrac{2}{3}\right| = \tfrac{2}{3}, \qquad |-1.7| = 1.7, \qquad |0.9| = 0.9.$$

Dropping the Sign

$\left|-\dfrac{7}{5}\right| = \dfrac{7}{5}$ because $-\dfrac{7}{5}$ is $\dfrac{7}{5}$ units from 0.

Watch a sign *outside* the bars: $-|-2.4| = -(2.4) = -2.4$. The bars make the inside positive first; the outside sign then stays.

Tip: Absolute value works the same for fractions and decimals as for integers: find the distance from 0, which just means make the inside positive.

Try it: Quick check

Is $\dfrac{5}{2}$ a rational number? Answer yes or no and give one short reason.

Work it on paper before revealing the answer.

Answer. Answer: Yes; it is a ratio of two integers (5 and 2).

14.5 Adding and Subtracting Fractions

Common Denominator First, Then Sign Rules

To add or subtract fractions, first get a **common denominator**. Then combine the numerators using the integer sign rules:

- **Same signs:** add the numerators, keep the sign.
- **Different signs:** subtract, keep the sign of the larger.

Remember: subtracting means *adding the opposite*.

Unlike Denominators with a Negative

Compute $-\frac{1}{3}+\frac{5}{6}$. Common denominator is 6:

$$-\frac{1}{3}+\frac{5}{6}=-\frac{2}{6}+\frac{5}{6}=\frac{-2+5}{6}=\frac{3}{6}=\frac{1}{2}.$$

Subtracting a Negative Fraction

Compute $\frac{1}{4}-\left(-\frac{3}{8}\right)$. Subtracting a negative adds:

$$\frac{1}{4}+\frac{3}{8}=\frac{2}{8}+\frac{3}{8}=\frac{5}{8}.$$

Tip: The denominator only sets the "size" of the pieces — all the sign work happens up in the numerators. Combine numerators using the same rules you learned for integers.

Try it: Quick check

Which is greater, $\frac{1}{2}$ or $\frac{3}{4}$?

Work it on paper before revealing the answer.

Answer. Answer: $\frac{3}{4}$

14.6 Adding and Subtracting Decimals

Line Up the Points, Then Use Sign Rules

To add or subtract decimals, **line up the decimal points** and work as usual. The signs follow the exact same integer rules: same signs add, different signs subtract and keep the larger sign.

Different Signs

Compute $-4.5+1.8$. Different signs, so subtract the absolute values: $4.5-1.8=2.7$. Since $4.5>1.8$, keep the negative sign: -2.7.

Subtracting a Negative Decimal

Compute $2.3 - (-0.9)$. Subtracting a negative adds:

$$2.3 - (-0.9) = 2.3 + 0.9 = 3.2.$$

Tip: Decimals and fractions obey the *same* sign rules as integers. The only extra care with decimals is keeping the decimal points lined up neatly.

Try it: Quick check

Fill in $<$ or $>$: $-\frac{1}{2}$ _____ -1.

Work it on paper before revealing the answer.

Answer. Answer: $-\frac{1}{2} > -1$

14.7 Multiplying and Dividing Rational Numbers

Sign Rule Is the Same; Method Depends on Form

For *both* multiplication and division: **same signs give a positive**, **different signs give a negative**.

Fractions: multiply straight across; to divide, multiply by the reciprocal ("flip the second and multiply").
Decimals: multiply as whole numbers then place the point; to divide, you may shift both points to make the divisor whole.

Multiplying and Dividing Fractions

$\left(-\frac{2}{3}\right)\left(\frac{9}{4}\right)$: different signs $\to$ negative. Multiply across and simplify:

$$-\frac{2 \times 9}{3 \times 4} = -\frac{18}{12} = -\frac{3}{2}.$$

$-\frac{3}{5} \div \left(-\frac{6}{7}\right)$: same signs $\to$ positive. Flip and multiply:

$$\frac{3}{5} \times \frac{7}{6} = \frac{21}{30} = \frac{7}{10}.$$

Multiplying Decimals

$(-1.2)(0.4)$: different signs → negative. Multiply $12 \times 4 = 48$, then place two decimal places: 0.48. Answer: -0.48.

Tip: One motto covers integers, fractions, and decimals: *same signs → positive, different signs → negative.* Decide the sign first, then do the arithmetic.

Try it: Quick check

Order from least to greatest: $\frac{1}{4},\ -\frac{1}{2},\ 0.$

Work it on paper before revealing the answer.

Answer. Answer: $-\frac{1}{2},\ 0,\ \frac{1}{4}$

14.8 Terminating vs. Repeating Decimals

Every Fraction Becomes One or the Other

Convert a fraction to a decimal by **dividing the numerator by the denominator**. The result is always either:

- **Terminating** — the division ends, like $\frac{3}{8} = 0.375$; or
- **Repeating** — a block of digits repeats forever, like $\frac{2}{3} = 0.\overline{6}$.

We write the repeating block with a bar over it: $0.1\overline{6}$ means $0.1666\ldots$

A Terminating Decimal

Convert $\frac{7}{20}$. Divide $7 \div 20 = 0.35$. It terminates.
Shortcut: if the denominator's only prime factors are 2 and 5, the decimal terminates. Here $20 = 2 \times 2 \times 5$.

A Repeating Decimal

Convert $\frac{5}{6}$. Divide $5 \div 6 = 0.8333\ldots = 0.8\overline{3}$. The 3 repeats forever because $6 = 2 \times 3$ contains a prime other than 2 or 5.

Tip: Reduce the fraction first. If the reduced denominator has only 2s and 5s as factors, the decimal terminates; otherwise it repeats.

Try it: Quick check

Evaluate $\left|-\frac{3}{5}\right|$.

Work it on paper before revealing the answer.

Answer. Answer: $\frac{3}{5}$

14.9 Order of Operations with Rational Numbers

PEMDAS, Now with Fractions and Signs

Use the same order every time: **P**arentheses, **E**xponents, **M**ultiply/**D**ivide (left to right), **A**dd/**S**ubtract (left to right). The new challenge is tracking *signs* and *fractions* at each step.

Mixing Fractions and Signs

Evaluate $-\frac{1}{2}+\frac{3}{4}\times(-2)$.

$$\begin{aligned} -\frac{1}{2}+\frac{3}{4}\times(-2) &= -\frac{1}{2}+\left(-\frac{6}{4}\right) && \text{multiply first} \\ &= -\frac{1}{2}-\frac{3}{2} && \text{simplify } \tfrac{6}{4}=\tfrac{3}{2} \\ &= -\frac{4}{2}=-2 && \text{then add} \end{aligned}$$

Decimals and Parentheses

Evaluate $2.5-0.5(3-7)$.

$$\begin{aligned} 2.5-0.5(3-7) &= 2.5-0.5(-4) && \text{parentheses first} \\ &= 2.5+2 && \text{multiply, add the opposite} \\ &= 4.5 \end{aligned}$$

Tip: Do one step at a time and write each line. Decide each sign *before* you compute, and

keep fractions in a common denominator when you add or subtract.

Try it: Quick check

Evaluate $|2.4|$.

Work it on paper before revealing the answer.

Answer. Answer: 2.4

14.10 Word Problems with Rational Numbers

Translate, Then Compute

Real situations mix signs and fractions: temperatures, money owed, elevation, and recipes. Gaining, rising, and depositing are positive; losing, falling, and spending are negative. Decide the sign of each amount, then add, subtract, multiply, or divide.

Temperature with Decimals

At 6 a.m. it was $-2.5°$C. By noon it rose 6.8 degrees. Rising means adding:

$$-2.5 + 6.8 = 4.3.$$

The noon temperature was 4.3°C.

Sharing with Fractions

A recipe needs $\frac{3}{4}$ cup of sugar, but you want to make only half a batch. Multiply:

$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}.$$

You need $\frac{3}{8}$ cup of sugar.

Tip: Read carefully to spot the sign of each quantity. "Below," "owes," "loses," and "drops" all signal negative numbers.

14.11 Going Deeper: Advanced Rational-Number Ideas

The Rationals Are Dense

Between *any* two different rational numbers there is always another rational number — in

fact, infinitely many. This property is called **density**. The integers are *not* dense (there is no integer strictly between 3 and 4), but the rationals are.

The quickest way to find one is the **average** (midpoint): the number halfway between a and b is $\dfrac{a+b}{2}$, and it is always rational when a and b are.

Squeezing a Rational Between Two Others

Find a rational number between $\dfrac{2}{5}$ and $\dfrac{1}{2}$.

Take the average:

$$\frac{1}{2}\left(\frac{2}{5}+\frac{1}{2}\right)=\frac{1}{2}\left(\frac{4}{10}+\frac{5}{10}\right)=\frac{1}{2}\cdot\frac{9}{10}=\frac{9}{20}.$$

Check: $\dfrac{2}{5}=\dfrac{8}{20}$ and $\dfrac{1}{2}=\dfrac{10}{20}$, so indeed $\dfrac{8}{20}<\dfrac{9}{20}<\dfrac{10}{20}$. We could repeat this forever, so there is no "next" rational number.

The Mediant: Another In-Between Number

There is a second trick for landing between two *positive* fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$: add the tops and add the bottoms to form the **mediant** $\dfrac{a+c}{b+d}$. Whenever $\dfrac{a}{b}<\dfrac{c}{d}$, the mediant sits strictly between them:

$$\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}.$$

Warning: this is *not* how you add fractions! It is only a way to build an in-between value.

Using the Mediant

Between $\dfrac{1}{3}$ and $\dfrac{1}{2}$, the mediant is $\dfrac{1+1}{3+2}=\dfrac{2}{5}$.

Check with a common denominator of 30: $\dfrac{1}{3}=\dfrac{10}{30}$, $\dfrac{2}{5}=\dfrac{12}{30}$, $\dfrac{1}{2}=\dfrac{15}{30}$. So $\dfrac{10}{30}<\dfrac{12}{30}<\dfrac{15}{30}$, exactly as promised.

Not Every Number Is Rational: $\sqrt{2}$

Some numbers can *never* be written as a fraction of integers; we call them **irrational**. The famous first example is $\sqrt{2}$. Its decimal $1.41421356\ldots$ never terminates and never repeats. Here is a gentle proof that no fraction equals it.

Why $\sqrt{2}$ Cannot Be a Fraction

Suppose, for contradiction, that $\sqrt{2}=\dfrac{a}{b}$ where $\dfrac{a}{b}$ is **fully reduced** (no common factors).

Squaring both sides:

$$2 = \frac{a^2}{b^2} \quad \Longrightarrow \quad a^2 = 2b^2.$$

Then a^2 is even, which forces a itself to be even, so write $a = 2k$. Substituting:

$$(2k)^2 = 2b^2 \quad \Longrightarrow \quad 4k^2 = 2b^2 \quad \Longrightarrow \quad b^2 = 2k^2.$$

Now b^2 is even, so b is even too. But then a and b are *both* even — they share the factor 2 — contradicting "fully reduced." No such fraction can exist, so $\sqrt{2}$ is irrational.

Closure: Staying Inside the Family

A set is **closed** under an operation if combining two members always gives another member. The rationals are closed under addition, subtraction, and multiplication — add, subtract, or multiply any two fractions and you get a fraction. They are closed under division too, *except* by 0.

The integers, by contrast, are *not* closed under division: $3 \div 4 = \frac{3}{4}$ escapes the integers. That gap is exactly why we needed to build the rationals.

Tip: To see closure at work, note that $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. Since integers are closed under $\times$ and $+$, the top and bottom are integers and $bd \neq 0$, so the sum is again rational.

Absolute Value as Distance Between Two Numbers

You already know $|x|$ measures distance from 0. More powerfully, $|a-b|$ measures the **distance between a and b** on the number line — and the order does not matter, since $|a-b| = |b-a|$.

$$|a - b| = |b - a|.$$

This is why distance is always positive: it does not care which point you start from.

Distance Between Two Rationals

How far apart are $-\frac{3}{4}$ and $\frac{1}{2}$?

$$\left|-\frac{3}{4} - \frac{1}{2}\right| = \left|-\frac{3}{4} - \frac{2}{4}\right| = \left|-\frac{5}{4}\right| = \frac{5}{4}.$$

Reversing the order gives the same distance: $\left|\frac{1}{2} - \left(-\frac{3}{4}\right)\right| = \left|\frac{2}{4} + \frac{3}{4}\right| = \frac{5}{4}$.

A Multi-Step Signed Expression

Evaluate $-\frac{2}{3}-\frac{1}{2}\left(\frac{3}{4}-\frac{5}{2}\right)^2$.

$$
\begin{aligned}
&-\frac{2}{3}-\frac{1}{2}\left(\frac{3}{4}-\frac{5}{2}\right)^2 \\
&=-\frac{2}{3}-\frac{1}{2}\left(\frac{3}{4}-\frac{10}{4}\right)^2 && \text{inside parentheses first} \\
&=-\frac{2}{3}-\frac{1}{2}\left(-\frac{7}{4}\right)^2 && \text{combine to } -\tfrac{7}{4} \\
&=-\frac{2}{3}-\frac{1}{2}\cdot\frac{49}{16} && \text{square: a negative squared is positive} \\
&=-\frac{2}{3}-\frac{49}{32} && \text{multiply} \\
&=-\frac{64}{96}-\frac{147}{96}=-\frac{211}{96} && \text{common denominator 96}
\end{aligned}
$$

The result $-\frac{211}{96}$ is still a rational number — closure at work.

Tip: With exponents and signs together, resolve the parentheses *before* squaring, and remember that squaring a negative gives a positive. Track the leading minus sign in front of a whole term all the way to the last line.

14.12 Why It Works

A repeating decimal is rational, and here is the trick that proves it: let $x = 0.\overline{27}$. Then $100x = 27.\overline{27}$, and subtracting, $99x = 27$, so $x = \frac{27}{99} = \frac{3}{11}$. Multiplying by a power of 10 that matches the period lines the repeating tails up so they cancel — every repeating decimal yields to the same two lines.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

15 Exponents & Roots

Powers, exponent rules, scientific notation, square roots, and order of operations

15.1 Exponent Notation: Base & Exponent

The Rule

An **exponent** is a short way to write repeated multiplication. In the power b^n, the number b is the **base** (the factor being multiplied) and the small raised number n is the **exponent** (how many times to use the base as a factor):

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}.$$

We read 5^2 as "five squared," 5^3 as "five cubed," and 5^4 as "five to the fourth power."

An exponent counts repeated multiplication: $2^5 = 32$. The rules — add exponents when multiplying like bases, multiply them for a power of a power, subtract when dividing — are all consequences of counting factors. A square root undoes squaring; $\sqrt{50} = 5\sqrt{2}$ because a perfect-square factor can be pulled out.

Worked Example: naming the parts

In 7^3, the base is **7** and the exponent is **3**. It means $7 \times 7 \times 7$. *Evaluating* (working it out): $7 \times 7 = 49$, then $49 \times 7 = \mathbf{343}$.

Where it goes wrong

Writing $2^3 \cdot 2^4 = 4^7$ or $(a+b)^2 = a^2 + b^2$. Bases do not multiply when exponents add, and squaring a sum produces a middle term $2ab$.

Worked Example: writing a product as a power

Write $4 \times 4 \times 4 \times 4 \times 4$ as a power. The base is 4 and it appears 5 times, so this is $\mathbf{4^5}$. (Its value is 1024, but writing 4^5 is much shorter.)

Tip: An exponent is *not* multiplication of the base by the exponent. $3^2 = 9$, *not* $3 \times 2 = 6$. Also, any base to the first power is itself: $8^1 = 8$.

Try it: Quick check

In the power 6^4, name the base and the exponent.

Work it on paper before revealing the answer.

Answer. Answer: base 6, exponent 4

15.2 Powers of Whole Numbers & Powers of 10

The Rule

You can raise any whole number to a power by multiplying it by itself the right number of times. **Powers of** 10 are especially friendly: 10^n is simply a 1 followed by n zeros. This is why 10^n matches our place-value system so neatly.

Worked Example: powers of small numbers

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = \mathbf{32}$. $3^4 = 3 \times 3 \times 3 \times 3 = \mathbf{81}$. $6^2 = 6 \times 6 = \mathbf{36}$.

Worked Example: powers of ten

$10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^6 = 1{,}000{,}000$ (one million). Count the zeros: the exponent tells you exactly how many.

Tip: For a power of ten, the exponent equals the number of zeros: 10^4 has 4 zeros, so $10^4 = 10{,}000$. This shortcut is the heart of scientific notation.

Try it: Quick check

Write $3 \times 3 \times 3$ using an exponent.

Work it on paper before revealing the answer.

Answer. Answer: 3^3

15.3 Exponent Rules: Product, Quotient & Power of a Power

The Rule

When the **bases are the same**, three shortcuts save a lot of work:

- **Product rule:** $a^m \cdot a^n = a^{m+n}$ (multiplying → *add* exponents)
- **Quotient rule:** $\dfrac{a^m}{a^n} = a^{m-n}$ (dividing → *subtract* exponents)
- **Power of a power:** $(a^m)^n = a^{m \times n}$ (a power raised to a power → *multiply* exponents)

Each rule works because an exponent is just repeated multiplication, so combining powers lines up more factors of the same base.

Worked Example: product rule

$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = \mathbf{128}$. Check the long way: $2^3 = 8$ and $2^4 = 16$, and $8 \times 16 = 128$. It matches.

Worked Example: quotient and power of a power

Quotient: $\dfrac{5^6}{5^2} = 5^{6-2} = 5^4 = \mathbf{625}$. Power of a power: $(3^2)^3 = 3^{2\times 3} = 3^6 = \mathbf{729}$.

Tip: These shortcuts only work when the *bases match*. You cannot combine $2^3 \cdot 3^2$ this way, because 2 and 3 are different bases. Remember: multiply powers → add, divide → subtract, power of a power → multiply.

Try it: Quick check

Evaluate 2^3.

Work it on paper before revealing the answer.

Answer. Answer: 8

15.4 The Zero Exponent

The Rule

Any nonzero number raised to the power 0 equals 1:

$$a^0 = 1 \quad (\text{for } a \neq 0).$$

Here is why it makes sense. Using the quotient rule, $\dfrac{a^n}{a^n} = a^{n-n} = a^0$. But any number divided by itself is 1. So a^0 must equal 1.

Worked Example: zero exponents

$7^0 = 1$, $100^0 = 1$, $2^0 = 1$. It does not matter how big the base is — as long as it is not zero, a zero exponent gives **1**.

Worked Example: seeing the pattern

Watch the powers of 2 shrink by half as the exponent drops: $2^3 = 8$, $2^2 = 4$, $2^1 = 2$, and the

next step is $2^0 = \mathbf{1}$. The pattern lands exactly on 1.

Tip: $a^0 = 1$ (as long as $a \neq 0$). Do not confuse it with $a^1 = a$. For example, $9^0 = 1$ but $9^1 = 9$.

Try it: Quick check

Evaluate 5^2.

Work it on paper before revealing the answer.

Answer. Answer: 25

15.5 Scientific Notation

The Rule

Scientific notation writes a number as a value from 1 up to (but not including) 10, multiplied by a power of ten:

$$(\text{a number } 1 \leq a < 10) \times 10^n.$$

A **positive** exponent means a *large* number (move the decimal right). A **negative** exponent means a *small* number less than 1 (move the decimal left). The exponent counts how many places the decimal point moves.

Worked Example: large number to scientific notation

Write 52,000 in scientific notation. Put the decimal after the first nonzero digit: 5.2. To get from 5.2 back to 52,000 the decimal moves 4 places to the right, so the exponent is 4: $52{,}000 = \mathbf{5.2 \times 10^4}$.

Worked Example: small number, and converting back

Write 0.00047 in scientific notation. The first nonzero digit gives 4.7; the decimal must move 4 places to the *left* to make the number small, so the exponent is negative: $0.00047 = \mathbf{4.7 \times 10^{-4}}$. Converting back: 3×10^5 means move the decimal 5 places right: **300,000**.

Tip: Big numbers get a *positive* exponent; small numbers (less than 1) get a *negative* exponent. The first factor a must always be at least 1 and less than 10 — so 52×10^3 is *not* yet in correct form.

Try it: Quick check

Evaluate 10^3.

Work it on paper before revealing the answer.

Answer. Answer: 1000

15.6 Square Roots & Perfect Squares

The Rule

A **square root** undoes squaring. The symbol $\sqrt{\ }$ asks: "what nonnegative number, times itself, gives this?" So $\sqrt{25} = 5$ because $5 \times 5 = 25$. A **perfect square** is a whole number times itself, like $1, 4, 9, 16, 25, 36, \ldots$; its square root is a whole number.

Worked Example: exact square roots

$\sqrt{49} = 7$ because $7^2 = 49$. $\sqrt{81} = 9$ because $9^2 = 81$. $\sqrt{144} = 12$ because $12^2 = 144$.

Worked Example: the perfect squares to know

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$, $10^2 = 100$, $11^2 = 121$, $12^2 = 144$. Knowing these by heart makes square roots quick.

Tip: Squaring and taking a square root are opposites, just like adding and subtracting. Because $\sqrt{9} = 3$, we also know $3^2 = 9$.

Try it: Quick check

Evaluate 4^2.

Work it on paper before revealing the answer.

Answer. Answer: 16

15.7 Estimating Square Roots of Non-Perfect Squares

The Rule

Most numbers are *not* perfect squares, so their square roots are not whole numbers. You can still **estimate** one by finding the two nearest perfect squares — one just below and one just above. The square root lands *between* those two whole numbers.

Worked Example: trapping a root between two integers

Estimate $\sqrt{50}$. The nearest perfect squares are 49 and 64, so $\sqrt{49} = 7$ and $\sqrt{64} = 8$. Since 50 is between 49 and 64, we know $7 < \sqrt{50} < 8$. Because 50 is very close to 49, $\sqrt{50}$ is just a bit more than **7**.

Worked Example: which integer is closer

Estimate $\sqrt{30}$. It sits between $\sqrt{25} = 5$ and $\sqrt{36} = 6$, so $5 < \sqrt{30} < 6$. Since 30 is closer to 25 than to 36, $\sqrt{30}$ is closer to **5** (about 5.5).

Tip: To estimate a square root, list perfect squares until you bracket the number. $\sqrt{20}$ is between $\sqrt{16} = 4$ and $\sqrt{25} = 5$, so it is between 4 and 5.

Try it: Quick check

Write $2^2 \cdot 2^3$ as a single power (same base).

Work it on paper before revealing the answer.

Answer. Answer: 2^5

15.8 Order of Operations with Exponents & Roots

The Rule

Exponents and roots fit into the order of operations right after grouping symbols:

1. **P**arentheses (grouping) first.
2. **E**xponents *and roots* next.
3. **M**ultiplication and **D**ivision, left to right.
4. **A**ddition and **S**ubtraction, left to right.

A square-root sign also acts as a grouping symbol: finish the work *under* it before taking the root.

Worked Example: exponent inside PEMDAS

Evaluate $4 + 3 \times 2^3$. Exponent first: $2^3 = 8$, giving $4 + 3 \times 8$. Multiply: $3 \times 8 = 24$. Add: $4 + 24 = \mathbf{28}$.

Worked Example: a root as grouping

Evaluate $\sqrt{9+16}+5^2$. Under the root first: $9+16=25$, and $\sqrt{25}=5$. Then $5^2=25$. Add: $5+25=\mathbf{30}$. (Note $\sqrt{9+16}\neq\sqrt{9}+\sqrt{16}$, which would give $3+4=7$.)

Tip: An exponent applies only to what it *touches*. In 3×2^2, only the 2 is squared: $3\times4=12$. But in $(3\times2)^2$ the parentheses square everything: $6^2=36$.

Try it: Quick check

Write $\dfrac{5^4}{5^2}$ as a single power.

Work it on paper before revealing the answer.

Answer. Answer: 5^2

15.9 Word Problems: Areas & Powers of Ten

The Rule

Exponents appear naturally in the real world. The **area of a square** with side s is s^2 (side times side), which is why we say "squared." **Powers of ten** describe measurements that grow or shrink by factors of ten, such as millimeters, meters, and kilometers.

Worked Example: area of a square

A square garden has sides of 9 meters. Its area is $9^2=9\times9=\mathbf{81}$ square meters. Working backward: if a square tile has an area of 64 square centimeters, each side is $\sqrt{64}=\mathbf{8}$ centimeters.

Worked Example: powers of ten in measurement

A kilometer is $10^3=1000$ meters. So 5 kilometers is $5\times1000=\mathbf{5000}$ meters. A single bacterium about 2×10^{-6} meters wide is tiny — the negative exponent tells us it is far smaller than one meter.

Tip: If a problem gives the *area* of a square and asks for the side, take a square root. If it gives the *side* and asks for area, square it. Squaring and square-rooting undo each other.

15.10 Going Deeper: Advanced Exponent & Root Ideas

Why Negative Exponents Mean "One Over"

The quotient rule not only explains $a^0 = 1$; it also tells us what a **negative** exponent must mean. Keep subtracting exponents past zero:

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}.$$

But writing the same division the long way, $\dfrac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \dfrac{1}{a^3}$. For both answers to agree, we must have

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0).$$

So a negative exponent flips the power into a fraction. It does *not* make the number negative — it makes it *small*.

Worked Example: evaluating negative exponents

Evaluate 2^{-3} and 10^{-2}. Flip each into "one over the positive power":

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \qquad 10^{-2} = \frac{1}{10^2} = \frac{1}{100} = \mathbf{0.01}.$$

Check with the shrinking pattern: $2^2 = 4$, $2^1 = 2$, $2^0 = 1$, $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$, $2^{-3} = \mathbf{\frac{1}{8}}$. Each step still divides by the base, right on through zero into the negatives.

Comparing Powers with Different Bases

When two powers have *different bases and different exponents*, the exponent rules cannot combine them — so you compare by **evaluating** (or estimating) each one. A bigger base does not always win: a smaller base raised to a larger exponent can overtake it. The safe move is to work each power out to a plain number and then compare.

Worked Example: which power is larger?

Compare 4^5 and 5^4. Evaluate each:

$$4^5 = 1024, \qquad 5^4 = 625.$$

So $4^5 > 5^4$, even though 5 is the bigger base — the larger *exponent* did more work here. Another famous near-tie: $2^{10} = 1024$ while $10^3 = 1000$, so 2^{10} is just barely larger. This is why programmers say a "kilobyte" (2^{10} bytes) is *about* a thousand.

Worked Example: last-digit cycles of powers

The ones digit of a power repeats in a short cycle, so you can find it without computing the

whole number. Watch the units digits of the powers of 2:

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 1\underline{6},\ 2^5 = 3\underline{2},\ 2^6 = 6\underline{4}, \ldots$$

The last digits run $2, 4, 8, 6,\ 2, 4, 8, 6, \ldots$ — a cycle of length 4. To find the units digit of 2^{30}, divide the exponent by 4: $30 = 4 \times 7 + 2$, remainder 2. The 2nd entry in the cycle is **4**, so 2^{30} ends in 4. Powers of 3 cycle $3, 9, 7, 1$; powers of 7 cycle $7, 9, 3, 1$.

Arithmetic in Scientific Notation

You can multiply and add numbers written in scientific notation without first expanding them.

- **To multiply:** multiply the front numbers, and *add* the exponents (product rule): $(a \times 10^m)(b \times 10^n) = (a \cdot b) \times 10^{m+n}$.
- **To add or subtract:** first rewrite both numbers with the *same* power of ten, then add the front numbers.

After either step, fix the front number back into the range $1 \le a < 10$ if needed.

Worked Example: multiplying and adding in scientific notation

Product: $(3 \times 10^4) \times (2 \times 10^3)$. Multiply fronts $3 \times 2 = 6$; add exponents $4 + 3 = 7$:

$$(3 \times 10^4)(2 \times 10^3) = 6 \times 10^7 = \mathbf{60{,}000{,}000}.$$

Product needing a fix-up: $(4 \times 10^5)(3 \times 10^{-2}) = 12 \times 10^3$. Since 12 is too big, rewrite $12 = 1.2 \times 10^1$: the answer is $1.2 \times 10^1 \times 10^3 = \mathbf{1.2 \times 10^4}$.
Sum: $3\times10^4+2\times10^3$. Match the powers: $2\times10^3 = 0.2\times10^4$, so $3\times10^4+0.2\times10^4 = \mathbf{3.2 \times 10^4}$.

Estimating Roots More Precisely

Trapping a root between two integers is a first step; you can sharpen the estimate by testing a decimal in the middle. Pick a guess, *square it*, and see whether the result is too big or too small — then nudge the guess up or down. Because squaring grows quickly, a couple of tries pin the root down to a decimal place.

Worked Example: refining $\sqrt{50}$

We already know $7 < \sqrt{50} < 8$, and 50 is just above 49, so try 7.1: $7.1^2 = 50.41$, a little too big. Try 7.07: $7.07^2 = 49.98$, just under 50. So $\sqrt{50} \approx \mathbf{7.07}$. The same "guess, square, adjust" idea works for any root: to estimate $\sqrt{20}$, note $4.4^2 = 19.36$ (low) and $4.5^2 = 20.25$ (high), so $\sqrt{20} \approx 4.47$.

Preview: a Logarithm Undoes an Exponent

A **square root** answers "what number squared gives this?" A **logarithm** answers the matching question for exponents: "what *exponent* do I put on the base to get this number?" Just as squaring and square-rooting are opposites, exponents and logarithms are opposites:

$$\text{if } b^x = y, \quad \text{then } \log_b y = x.$$

For example, $10^3 = 1000$, so $\log_{10} 1000 = 3$ — the logarithm simply counts the zeros. And $2^5 = 32$, so $\log_2 32 = 5$. You will meet logarithms fully in later courses, but you already understand the idea: a logarithm is the "hidden exponent."

Tip — growth vs. decay: Repeated multiplying by a number *bigger* than 1 is **growth** (it gets large fast): doubling gives 2^n, so after 10 doublings you have $2^{10} = 1024$ times as much. Repeated multiplying by a number *between* 0 and 1 is **decay** (it shrinks toward zero): halving gives $\left(\frac{1}{2}\right)^n = 2^{-n}$, so after 10 halvings only $\frac{1}{1024}$ is left. Positive exponents grow; negative exponents shrink.

15.11 Why It Works

$a^m \cdot a^n = a^{m+n}$ because m copies of a followed by n copies is $m + n$ copies. Dividing removes copies: $a^m / a^n = a^{m-n}$, and applying that with $m = n$ forces $a^0 = 1$, while $m < n$ forces negative exponents to mean reciprocals. Fractional exponents follow from $(a^{1/2})^2 = a^1$, so $a^{1/2}$ must be the square root. None of these are extra rules; they are the single counting rule pushed into new territory.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

16 Course Review

All 14 topics: definitions, rules, and formulas for whole numbers through statistics and probability

16.1 Whole Numbers & Operations

Place Value & Rounding

Each place is worth **ten times** the one to its right: ones, tens, hundreds, thousands, ten-thousands, … Digits are grouped into **periods** of three (ones, thousands, millions) by commas.

To round to a place: look at the digit *just to the right*. If it is 5 or more, round **up**; if 4 or

less, round **down**. Replace all digits to the right with zeros.

Order of Operations (PEMDAS)

1. **P**arentheses / grouping symbols first.
2. **E**xponents.
3. **M**ultiplication and **D**ivision, left to right (equal rank).
4. **A**ddition and **S**ubtraction, left to right (equal rank).

Exponents & Perfect Squares

In b^n, b is the **base** and n the **exponent**: $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$. Any base to the first power is itself ($8^1 = 8$).

Perfect squares: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144$ (that is 1^2 through 12^2).

Worked Example: PEMDAS

$5 + 3 \times (8 - 6)^2$. Parentheses: $8 - 6 = 2$. Exponent: $2^2 = 4$. Multiply: $3 \times 4 = 12$. Add: $5 + 12 = \mathbf{17}$.

Tip: PEMDAS does *not* mean all multiplication before all division. M/D tie and A/S tie; break ties left to right. Also $3^2 = 9$, *not* 3×2.

16.2 Integers

Number Line & Absolute Value

Integers are the whole numbers and their opposites: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ On a number line, numbers increase to the right. So $-5 < -2$ (farther left is smaller).

Absolute value $|a|$ is the distance from 0, always ≥ 0: $|-7| = 7$ and $|7| = 7$.

Sign Rules

Adding: same signs $\rightarrow$ add and keep the sign; different signs $\rightarrow$ subtract and keep the sign of the larger absolute value.

Subtracting: add the opposite: $a - b = a + (-b)$.

Multiplying / Dividing: same signs $\rightarrow$ **positive**; different signs $\rightarrow$ **negative**.

$$(+)(+) = + \quad (-)(-) = + \quad (+)(-) = - \quad (-)(+) = -$$

Worked Examples

$-8+3 = -5$ (different signs, $8-3=5$, keep $-$). $-4-(-9) = -4+9 = 5$. $(-6)(-7) = 42$. $-20 \div 4 = -5$.

Tip: Multiplying/dividing an *even* number of negatives gives a positive; an *odd* number gives a negative. Subtracting a negative makes a number bigger.

16.3 Factors & Multiples

Primes, Factors & Prime Factorization

A **factor** divides a number evenly; a **multiple** is what you get counting by that number. A **prime** has exactly two factors (1 and itself): $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$ A **composite** has more than two factors. (1 is neither.)

Prime factorization writes a number as a product of primes (use a factor tree): $60 = 2^2 \times 3 \times 5$.

GCF & LCM

GCF (greatest common factor): the largest factor two numbers share — multiply the primes they have *in common*. **LCM** (least common multiple): the smallest multiple they share — take each prime to its *highest* power.

Useful identity: $\text{GCF}(a,b) \times \text{LCM}(a,b) = a \times b$.

Divisibility Rules

- **2:** last digit even.
- **3:** digit sum divisible by 3.
- **4:** last two digits divisible by 4.
- **5:** ends in 0 or 5.
- **6:** divisible by 2 *and* 3.
- **9:** digit sum divisible by 9.
- **10:** ends in 0.

Worked Example: GCF and LCM of 12 and 18

$12 = 2^2 \times 3$, $18 = 2 \times 3^2$. GCF $= 2 \times 3 =$ **6** (lowest power of shared primes). LCM $= 2^2 \times 3^2 =$ **36** (highest power of each prime). Check: $6 \times 36 = 216 = 12 \times 18$.

Tip: GCF is used to *simplify* fractions; LCM gives the *least common denominator* for adding fractions.

16.4 Fractions

Equivalent, Simplify & Compare

Multiply or divide top *and* bottom by the same number for an **equivalent** fraction. **Simplify** by dividing both by their GCF. To **compare**, use a common denominator or cross-multiply: $\frac{a}{b}$ vs $\frac{c}{d}$ compare $a \cdot d$ vs $b \cdot c$.

The Four Operations

- **Add / Subtract:** need a *common denominator*, then add/subtract numerators: $\dfrac{a}{c} \pm \dfrac{b}{c} = \dfrac{a \pm b}{c}$.
- **Multiply:** straight across: $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$.
- **Divide:** multiply by the reciprocal: $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c}$.

Mixed Numbers

Mixed $\to$ improper: $a\frac{b}{c} = \dfrac{a \cdot c + b}{c}$. Improper $\to$ mixed: divide; quotient is the whole part, remainder over the divisor.

Worked Example

$\dfrac{2}{3} + \dfrac{1}{4}$. LCD $= 12$: $\dfrac{8}{12} + \dfrac{3}{12} = \dfrac{11}{12}$. $\dfrac{3}{4} \div \dfrac{2}{5} = \dfrac{3}{4} \times \dfrac{5}{2} = \dfrac{15}{8} = 1\frac{7}{8}$.

Tip: You need a common denominator to add or subtract, but *not* to multiply or divide. Always simplify your final answer.

16.5 Decimals

Place Value & The Four Operations

Places after the point: tenths, hundredths, thousandths, …

- **Add / Subtract:** line up the decimal points, then compute.

- **Multiply:** ignore points, multiply, then place the point so the product has as many decimal places as both factors *combined*.
- **Divide:** move the divisor's point to make it whole; move the dividend's point the same number of places; then divide.

Fraction ↔ Decimal

Fraction → decimal: divide numerator by denominator. **Decimal → fraction:** write over its place value, then simplify ($0.75 = \frac{75}{100} = \frac{3}{4}$).

A decimal **terminates** if the simplified denominator has only factors of 2 and/or 5; otherwise it **repeats**. Repeating: $0.\overline{3} = \frac{1}{3}$, shown with a bar over the repeating block.

Worked Example

1.2×0.05: $12 \times 5 = 60$; $1 + 2 = 3$ decimal places → **0.060** $= 0.06$. $\frac{5}{8} = 5 \div 8 =$ **0.625** (terminates, since $8 = 2^3$).

Tip: When *multiplying* decimals you do *not* line up the points — you count decimal places at the end. Only for *adding/subtracting* do you line up the points.

16.6 Ratios, Rates & Proportions

Ratios, Rates & Unit Rates

A **ratio** compares two quantities: a to b, a:b, or $\frac{a}{b}$. A **rate** compares different units (miles per hour). A **unit rate** has a denominator of 1 — divide to find "per one" ($3 for 2 lb = $1.50 per lb).

Proportions & Cross-Multiplication

A **proportion** states two ratios are equal: $\dfrac{a}{b} = \dfrac{c}{d}$. Solve by **cross-multiplying**: $a \cdot d = b \cdot c$, then isolate the unknown.

Scale & similar figures: similar figures have equal corresponding angles and proportional sides. Set up matching sides as a proportion to find a missing length. A **scale** (e.g. 1 in : 50 mi) is a ratio between a drawing and reality.

Worked Example

If 3 pens cost $4.50, find the cost of 7. $\dfrac{4.50}{3} = \dfrac{x}{7}$. Cross-multiply: $3x = 31.50$, so $x =$ **$10.50**.

(Unit rate is \$1.50/pen $\times 7$.)

Tip: Keep units in the same position on both sides of a proportion (miles over hours = miles over hours). Mixing them up gives a wrong answer.

16.7 Percents

Converting

Percent means "per 100." **Percent → decimal:** divide by 100 (move point 2 left): $45\% = 0.45$. **Decimal → percent:** multiply by 100. **Percent → fraction:** write over 100 and simplify: $45\% = \frac{45}{100} = \frac{9}{20}$.

Percent Of, Percent Change & Money

- **Percent of a number:** part = percent(as decimal) × whole.
- **Percent equation:** $\dfrac{\text{part}}{\text{whole}} = \dfrac{\%}{100}$.
- **Percent change:** $\dfrac{\text{new} - \text{old}}{\text{old}} \times 100\%$ (increase if +, decrease if −).
- **Tax / tip:** add % of price. **Discount:** subtract % of price.
- **Simple interest:** $I = Prt$ (P = principal, r = rate as decimal, t = years).

Worked Example

A \$40 shirt is 25% off. Discount $= 0.25 \times 40 = \$10$, so sale price = \$30. Interest on \$500 at 4% for 3 years: $I = 500 \times 0.04 \times 3 = \$\mathbf{60}$.

Tip: Percent change is always divided by the *original* amount, not the new one. Convert every percent to a decimal before multiplying.

16.8 Rational Numbers

What They Are

A **rational number** can be written as a fraction $\dfrac{a}{b}$ of integers with $b \neq 0$. This includes integers, fractions, terminating decimals, and repeating decimals. Operations follow the fraction/decimal rules *plus* the integer sign rules.

Terminating vs. Repeating

Write the fraction in lowest terms and look at the denominator:

- Only factors 2 and/or 5 ⇒ **terminates** (e.g. $\frac{7}{20} = 0.35$).
- Any other prime factor ⇒ **repeats** (e.g. $\frac{5}{6} = 0.8\overline{3}$).

Worked Example: signed fractions

$-\frac{2}{3} \times \frac{9}{4} = -\frac{18}{12} = -\frac{3}{2}$ (different signs → negative). $-0.6 + 1.25 = 0.65$.

Tip: Determine the sign first, then compute with absolute values. A negative fraction can carry its sign on top, bottom, or in front — they all mean the same thing.

16.9 Exponents & Roots

Exponent Rules

- Product: $a^m \cdot a^n = a^{m+n}$
- Quotient: $\frac{a^m}{a^n} = a^{m-n}$
- Power of a power: $(a^m)^n = a^{mn}$
- Power of a product: $(ab)^n = a^n b^n$
- Zero: $a^0 = 1$ $(a \neq 0)$
- Negative: $a^{-n} = \frac{1}{a^n}$

Scientific Notation

Write as $a \times 10^n$ where $1 \leq a < 10$ and n is an integer. Large numbers use *positive* n; small numbers (less than 1) use *negative* n.

$$4.2 \times 10^5 = 420{,}000 \qquad 3 \times 10^{-4} = 0.0003$$

Square Roots

$\sqrt{a}$ asks "what number squared gives a?" Squaring and square-rooting undo each other: $\sqrt{49} = 7$ since $7^2 = 49$. Perfect-square roots are whole numbers; others are irrational (e.g. $\sqrt{2} \approx 1.41$) and can be estimated between the two nearest perfect squares.

Worked Example

$x^3 \cdot x^4 = x^7$. $(2^3)^2 = 2^6 = 64$. $5^{-2} = \frac{1}{25}$. $\sqrt{81} = 9$. $\sqrt{50}$ is between $\sqrt{49} = 7$ and

$\sqrt{64} = 8$, so about 7.1.

Tip: Add exponents only when the *bases match and you are multiplying.* A negative exponent means reciprocal, *not* a negative number: $2^{-3} = \frac{1}{8}$, not -8.

16.10 Introduction to Algebra

Variables, Terms & Evaluating

A **variable** is a letter standing for an unknown number. A **term** is a number, a variable, or their product (e.g. $3x$); the number is the **coefficient**. **Evaluate** an expression by substituting values for the variables and using PEMDAS.

Distributive Property & Like Terms

Distributive: $a(b + c) = ab + ac$ (multiply the outside by each inside term).

Like terms have the same variable(s) to the same power; combine them by adding coefficients: $5x + 3x = 8x$, but $5x + 3x^2$ cannot combine.

Translating Phrases

- sum / more than $\rightarrow +$
- difference / less than $\rightarrow -$
- product / of / times $\rightarrow \times$
- quotient / per $\rightarrow \div$
- is / equals $\rightarrow =$

"7 less than x" is $x - 7$ (order flips).

Worked Example

Evaluate $2a + b^2$ at $a = 3, b = 4$: $2(3) + 4^2 = 6 + 16 = \mathbf{22}$. Simplify $3(x + 2) + 4x = 3x + 6 + 4x = \mathbf{7x + 6}$.

Tip: "7 less than x" and "x less than 7" are different: $x - 7$ vs. $7 - x$. Only combine like terms; never add x terms to constant terms.

16.11 Equations & Inequalities

Solving Equations

An equation is a balance. Use **inverse operations** to isolate the variable, doing the *same thing to both sides*. Undo in reverse PEMDAS order: addition/subtraction first, then multiplication/division.

- One-step: $x + 5 = 12 \Rightarrow x = 7$.
- Two-step: $2x + 3 = 11 \Rightarrow 2x = 8 \Rightarrow x = 4$.

Inequalities

Solve like equations, but the sign shows a range of solutions ($<, \leq, >, \geq$). **Key rule:** when you **multiply or divide both sides by a negative number, flip the inequality sign.**

$$-2x < 6 \;\Rightarrow\; x > -3$$

Graph on a number line: open circle for $<$ or $>$, closed circle for $\leq$ or $\geq$.

Worked Example

Solve $5x - 4 = 21$: add $4 \to 5x = 25$; divide by $5 \to x = \mathbf{5}$. Check: $5(5) - 4 = 21$. ✓

Solve $-3x \geq 12$: divide by -3 and flip: $x \leq \mathbf{-4}$.

Tip: Always check by substituting your answer back in. The *only* time the inequality sign flips is multiplying or dividing by a negative — not for adding or subtracting a negative.

16.12 The Coordinate Plane

Points, Axes & Quadrants

The plane is formed by a horizontal **x-axis** and vertical **y-axis** meeting at the **origin** $(0, 0)$. An **ordered pair** (x, y) gives horizontal then vertical position (over, then up/down).

- **QI** $(+, +)$ **QII** $(-, +)$
- **QIII** $(-, -)$ **QIV** $(+, -)$

Points on an axis are in *no* quadrant.

Tables & Rules

A rule like $y = 2x + 1$ pairs each x with a y. Make a table, plot the (x, y) pairs, and connect them — a straight-line rule graphs as a line.

Worked Example

Plot $(-3, 2)$: start at origin, go *left* 3, then *up* 2 — that lands in Quadrant II. The point $(4, 0)$ sits on the x-axis (not in any quadrant).

Tip: (x, y) order matters: $(3, 5) \neq (5, 3)$. Read x first (left/right), then y (up/down). Remember: "over before up."

16.13 Geometry Basics

Angles

Measured in degrees. **Acute** $< 90°$; **right** $= 90°$; **obtuse** between $90°$ and $180°$; **straight** $= 180°$. **Complementary** angles sum to $90°$; **supplementary** sum to $180°$. Angles in a triangle sum to $180°$.

Perimeter & Area

- Rectangle: $P = 2l + 2w$, $A = lw$
- Square: $P = 4s$, $A = s^2$
- Triangle: $A = \frac{1}{2}bh$
- Parallelogram: $A = bh$
- Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$
- Circle: $C = 2\pi r = \pi d$, $A = \pi r^2$

Pythagorean Theorem

For a *right* triangle with legs a, b and hypotenuse c (the side opposite the right angle):

$$a^2 + b^2 = c^2$$

Volume & Surface Area

- Rect. prism: $V = lwh$
- Cube: $V = s^3$
- Prism: $V =$ (base area) $\times\, h$
- Cylinder: $V = \pi r^2 h$
- Rect. prism surface: $SA = 2(lw + lh + wh)$
- Cylinder surface: $SA = 2\pi r^2 + 2\pi rh$

Worked Example

A right triangle has legs 6 and 8: $c^2 = 6^2 + 8^2 = 36 + 64 = 100$, so $c = \sqrt{100} = \mathbf{10}$. Circle

with $r = 3$: $A = \pi(3)^2 = 9\pi \approx \mathbf{28.3}$.

Tip: Perimeter and circumference are *lengths* (units); area is in *square* units; volume in *cubic* units. Use $\pi \approx 3.14$ (or $\frac{22}{7}$) when a decimal is required.

16.14 Data, Statistics & Probability

Measures of Center & Spread

- **Mean** (average): $\dfrac{\text{sum of values}}{\text{number of values}}$.
- **Median:** middle value when ordered (average the two middle values if there is an even count).
- **Mode:** value that appears most often (can be none or several).
- **Range:** max − min.

Probability & Counting

Probability of an event: $P(E) = \dfrac{\text{favorable outcomes}}{\text{total outcomes}}$, a value from 0 (impossible) to 1 (certain). Complement: $P(\text{not } E) = 1 - P(E)$.

Counting principle: if one choice has m options and another has n, together there are $m \times n$ outcomes. Independent events: $P(A \text{ and } B) = P(A) \times P(B)$.

Worked Example

Data $\{4, 7, 7, 10, 12\}$: mean $= \frac{40}{5} = 8$; median $= 7$; mode $= 7$; range $= 12 - 4 = 8$. Rolling a 6-sided die: $P(\text{even}) = \frac{3}{6} = \mathbf{\frac{1}{2}}$.

Tip: Always *order* the data before finding the median. The mean is pulled toward unusually large or small values (outliers); the median is not.