MathCastle

Grade 6

Complete course booklet • 11 topics

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1 Ratios and Rates

Comparing two quantities

1.1 Rates as Ratios

The Big Idea

A **ratio** compares two amounts. A **unit rate** is a ratio with 1 in the second slot, which makes scaling up easy: just multiply.

A ratio compares two quantities by division: 3 red to 2 blue, written $3 : 2$. A rate is a ratio of two different units, like 60 miles per hour, and a unit rate makes the second quantity 1. Equivalent ratios scale both numbers by the same factor — a ratio table is just multiplication in rows.

Scaling a recipe

If 12 grams of sugar go with one cup of flour, then 5 cups need $12 \times 5 = 60$ grams.

Where it goes wrong

Comparing ratios by subtracting instead of dividing. $2 : 4$ and $3 : 6$ are the same ratio even though the differences 2 and 3 differ.

1.2 In Plain Terms & More Examples

In plain terms

A ratio compares two amounts. A unit rate rewrites it with 1 in the second slot, which makes scaling as simple as multiplying.

A pay rate

If \$60 is earned in 5 hours, the unit rate is $60 \div 5 = \$12$ per hour.

Scaling a ratio

The ratio $2 : 3$ scaled up by 4 becomes $8 : 12$.

1.3 Ratio Tables and Unit Rates

Scale the pair

A ratio table multiplies both numbers by the same factor row by row. Dividing both by the second number gives the unit rate — "per one".

Unit price

5 pens cost \$3.75. Per pen: $3.75 \div 5 = \$0.75$. So 12 pens cost $12 \times 0.75 = \$9$.

Try it: the mix

Paint is mixed 3 parts blue to 5 parts yellow. How much blue for 40 liters of paint?

Answer. Each batch of 8 liters has 3 blue. $40 \div 8 = 5$ batches, so $5 \times 3 = \mathbf{15}$ liters of blue.

1.4 Word Problems

Speed

A train goes 180 km in 1.5 hours: $180 \div 1.5 = 120$ km/h. In 4 hours it goes 480 km.

Try it: better buy

Store A sells 6 bagels for \$4.50; Store B sells 8 for \$5.60. Which is cheaper per bagel?

Answer. A: $4.50 \div 6 = \$0.75$. B: $5.60 \div 8 = \$0.70$. **Store B** is cheaper.

1.5 Why It Works

Two ratios $a : b$ and $c : d$ are equivalent exactly when $\frac{a}{b} = \frac{c}{d}$, i.e. when $ad = bc$ (multiply both sides by bd). Scaling a ratio by k keeps the fraction the same because $\frac{ka}{kb} = \frac{a}{b}$ — this is equivalent fractions again. A unit rate is the fraction $\frac{a}{b}$ itself: it answers "how much of the first for one of the second".

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

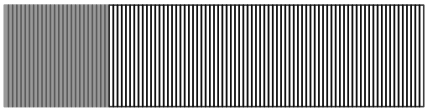
2 Percent

Per hundred

2.1 Percent Means Out of 100

The Big Idea

"Percent" literally means **per hundred**, so $p\%$ is the fraction $\frac{p}{100}$. Finding a percent of a number is just multiplying by that fraction.



25 of 100 parts shaded

Percent means "per hundred": 35% is $\frac{35}{100}$, or 0.35. Finding a percent of a number is multiplying by that decimal; finding what percent one number is of another is dividing and then multiplying by 100. Every percent problem is one of three questions: find the part, find the whole, or find the percent.

Finding 25% of 80

$$\frac{25}{100} \times 80 = \frac{1}{4} \times 80 = 20.$$

Where it goes wrong

Percent change divided by the wrong base. A rise from 40 to 50 is $\frac{10}{40} = 25\%$, not $\frac{10}{50} = 20\%$. Always divide by the ORIGINAL amount.

2.2 In Plain Terms & More Examples

In plain terms

A percent is a fraction out of 100. To find a percent of a number, turn it into that fraction and multiply.

Finding 10% of 50

$$\frac{10}{100} \times 50 = \frac{1}{10} \times 50 = 5.$$

Finding 75% of 40

$$\frac{75}{100} \times 40 = \frac{3}{4} \times 40 = 30.$$

A multi-step problem

A \$24 shirt is 25% off, and you also have a \$2 coupon. Step 1: 25% of 24 is 6, so the sale price is $24 - 6 = 18$. Step 2: the coupon brings it to $18 - 2 = \$16$.

2.3 Finding the Whole and the Percent

Three questions, one equation

part = percent × whole. Cover the unknown: whole = part ÷ percent; percent = part ÷ whole (then ×100).

What is the whole?

30% of a number is 12. Whole $= 12 \div 0.30 = 40$.

Try it: what percent?

Sam got 34 out of 40 on a quiz. What percent is that? And 15% of what number is 9?

Answer. $34 \div 40 = 0.85 =$ **85**%. $9 \div 0.15 =$ **60**.

2.4 Word Problems

A discount

A \$48 jacket is 25% off. Discount $0.25 \times 48 = \$12$, sale price \$36 — or directly $0.75 \times 48 = 36$.

Try it: tax

A meal costs \$32 and tax is 8%. What is the total?

Answer. $1.08 \times 32 =$ **\$34.56**.

2.5 Why It Works

Percents are a common denominator. Comparing $\frac{7}{20}$ with $\frac{9}{25}$ is awkward; comparing 35% with 36% is instant, because both have been rewritten with denominator 100. So "p% of N" $= \frac{p}{100} \times N$ is nothing more than multiplying by a fraction, and every percent rule is a fraction rule with 100 on the bottom.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

3 Dividing Fractions

Multiply by the reciprocal

3.1 Why Flip and Multiply

The Big Idea

Dividing asks "how many of these fit?" Dividing by $\frac{c}{d}$ is the same as multiplying by $\frac{d}{c}$, because that undoes the division.



1 of 2 parts shaded

Dividing asks "how many of these fit into that?" $6 \div \frac{1}{2} = 12$ because twelve halves make six. The shortcut — multiply by the reciprocal, flipping the second fraction — always gives the same answer as this counting, which is why it works even when the numbers are ugly.

Finding $\frac{1}{2} \div \frac{1}{4}$

Flip and multiply: $\frac{1}{2} \times \frac{4}{1} = 2$. And indeed two quarters fit inside a half.

Where it goes wrong

Flipping the first fraction instead of the second, or flipping both. Only the divisor (the number you divide BY) is inverted.

3.2 In Plain Terms & More Examples

In plain terms

To divide by a fraction, flip it (its reciprocal) and multiply. Dividing asks how many of the second fraction fit inside the first.

Finding $\frac{3}{4} \div \frac{1}{2}$

Flip and multiply: $\frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = \frac{3}{2}$.

Finding $\frac{2}{3} \div \frac{1}{6}$

Flip and multiply: $\frac{2}{3} \times \frac{6}{1} = \frac{12}{3} = 4$.

3.3 Mixed Numbers and Whole Numbers

Convert, flip the second, multiply

Change mixed numbers to improper fractions, keep the first, flip the divisor, multiply. A whole number divisor 4 is $\frac{4}{1}$, so dividing by 4 is multiplying by $\frac{1}{4}$.

Finding $2\frac{1}{2} \div \frac{3}{4}$

$\frac{5}{2} \times \frac{4}{3} = \frac{20}{6} = \frac{10}{3} = 3\frac{1}{3}$. Sense check: $\frac{3}{4}$ fits into $2\frac{1}{2}$ a bit more than 3 times. ✓

Try it: two ways

Compute $\frac{5}{6} \div 5$ and $6 \div \frac{2}{3}$.

Answer. $\frac{5}{6} \times \frac{1}{5} = \mathbf{\frac{1}{6}}$. $6 \times \frac{3}{2} = \mathbf{9}$ (nine two-thirds make six).

3.4 Word Problems

Servings

A $3\frac{1}{2}$-cup bag of rice makes servings of $\frac{1}{2}$ cup: $\frac{7}{2} \div \frac{1}{2} = 7$ servings.

Try it: the board

A board $4\frac{1}{2}$ feet long is cut into pieces $\frac{3}{4}$ foot long. How many pieces?

Answer. $\frac{9}{2} \times \frac{4}{3} = \frac{36}{6} = \mathbf{6}$ pieces.

3.5 Why It Works

Dividing by $\frac{a}{b}$ must undo multiplying by $\frac{a}{b}$. Multiplying by $\frac{b}{a}$ does exactly that, since $\frac{a}{b} \times \frac{b}{a} = 1$. So $x \div \frac{a}{b} = x \times \frac{b}{a}$ — the reciprocal rule is just "division is the inverse of multiplication" written for fractions. Check with $6 \div \frac{1}{2} = 6 \times \frac{2}{1} = 12$ ✓.

Reading a proof

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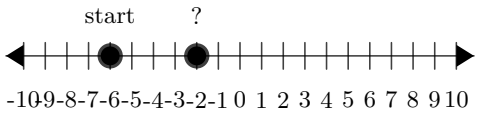
4 Negative Numbers

Numbers below zero

4.1 Both Sides of Zero

The Big Idea

Negative numbers continue the number line to the **left** of zero. Adding still means moving right, and subtracting means moving left — the rule never changes.



Negative numbers extend the number line to the left of zero. Adding a negative moves left; subtracting a negative moves right, which is why $5 - (-3) = 8$. Multiplying two negatives gives a positive — not by decree, but because the rules of arithmetic force it.

Finding $-6 + 4$

Start at -6 and move 4 to the right, landing on -2.

Where it goes wrong

Believing "two negatives make a positive" applies to addition. $-3 + (-5) = -8$: adding two debts makes a bigger debt. The sign rule for two negatives is about multiplication and division only.

4.2 In Plain Terms & More Examples

In plain terms

Negatives are the numbers left of zero. Adding still moves right on the number line; subtracting still moves left.

Finding $-3 + 7$

Start at -3 and move 7 right, landing on 4.

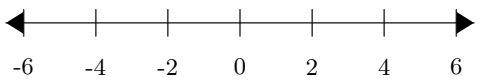
Finding $2 - 5$

Start at 2 and move 5 left, landing on -3.

4.3 Multiplying and Dividing

The sign rule

Same signs give a positive; different signs give a negative. $(-3)(-4) = 12$, $(-3)(4) = -12$, $-12 \div (-3) = 4$.



Three factors

$(-2)(-3)(-1)$: the first two give 6, then $6 \times (-1) = -6$. An odd number of negative factors

gives a negative.

Try it: evaluate

Compute $-8-(-5)$, $(-6)(7)$, and $-45 \div (-9)$.

Answer. $-8+5=\mathbf{-3}$; $\mathbf{-42}$; $\mathbf{5}$.

4.4 Word Problems

Temperature drop

The temperature was $4°$ and fell $9°$: $4-9=-5°$.

Try it: the elevator

An elevator starts at floor -2 (a basement level), goes up 7 floors, then down 10. Where is it?

Answer. $-2+7-10=\mathbf{-5}$, five levels below ground.

4.5 Why It Works

Why is $(-1)\times(-1)=1$? Start from $(-1)\times 0=0$ and write $0=1+(-1)$: then $(-1)\times 1+(-1)\times(-1)=0$, so $-1+(-1)(-1)=0$, hence $(-1)(-1)=1$. Nothing was decreed; keeping the distributive property true for negatives leaves no other choice. Subtracting a negative is adding its opposite for the same reason — the opposite of -3 is 3.

Reading a proof

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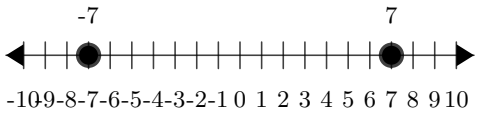
5 Absolute Value

Distance from zero

5.1 Distance Is Never Negative

The Big Idea

$|x|$ is how **far** x is from zero, ignoring direction. Distance cannot be negative, so the answer is never negative.



Absolute value is distance from zero, and distance is never negative: $|-7| = 7$ and $|7| = 7$. It ignores direction and keeps size. The distance between two numbers on the line is the absolute value of their difference, $|a - b|$, in either order.

Why $|-7| = 7$

-7 sits seven steps from zero, so its absolute value is 7 — the same distance as $+7$.

Where it goes wrong

Thinking $|x|$ "makes the sign positive" and so writing $|x| = x$ always. For $x = -7$ that is false; $|x| = -x$ when x is negative, because $-(-7) = 7$.

5.2 In Plain Terms & More Examples

In plain terms

Absolute value is distance from zero, so it is never negative — just drop the sign.

Finding $|-12|$

It is 12 steps from zero, so $|-12| = 12$.

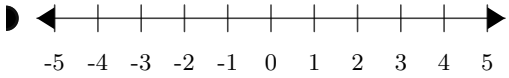
Finding $|8|$

It is already positive: $|8| = 8$.

5.3 Distance Between Numbers

Subtract, then take the size

The distance between a and b on the number line is $|a-b|$. Order does not matter: $|3-(-4)| = |-4-3| = 7$.



Ordering

Sort $|-6|, -3, |2|, 0$: the values are $6, -3, 2, 0$, so in order: $-3, 0, |2|, |-6|$.

Try it: distance

How far apart are -7 and 5? And which is greater, $-|-8|$ or -3?

Answer. $|-7-5| = \mathbf{12}$. $-|-8| = -8$, which is less than -3, so $\mathbf{-3}$ is greater.

5.4 Word Problems

Elevation

Death Valley is at -86 m and a nearby peak at 3366 m. Their elevation difference is $|3366 - (-86)| = 3452$ m.

Try it: debt

Kai's account is at $-\$45$. How much must he deposit to reach \$0? Express it with absolute value.

Answer. $|-45| = \mathbf{\$45}$.

5.5 Why It Works

By definition $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$. Both branches are non-negative, so $|x| \geq 0$ always, and $|x| = 0$ only for $x = 0$. The distance between a and b is $|a - b|$: if $a \geq b$ the distance is $a - b$; if $a < b$ it is $b - a = -(a - b)$ — exactly the two branches of the definition.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line

forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

6 Expressions

Letters standing for numbers

6.1 Substituting a Value

The Big Idea

A letter holds a number we have not chosen yet. To **evaluate**, replace the letter with its value and follow the order of operations.

A variable is a letter standing for a number we do not know yet, or that can change. An expression like $3x + 2$ is a recipe: triple the number, then add two. Evaluating means substituting a value; simplifying means combining like terms — terms with the same variable part — because $3x + 2x$ is five of the same thing.

Evaluating $3x + 5$ at $x = 4$

Substitute: $3(4) + 5 = 12 + 5 = 17$.

Where it goes wrong

Combining unlike terms: $3x + 2$ is NOT $5x$. Only terms with identical variable parts can be added, the way 3 apples and 2 oranges stay separate.

6.2 In Plain Terms & More Examples

In plain terms

A letter stands for a number. To evaluate, swap in the value and follow the order of operations.

Evaluating $2x + 7$ at $x = 5$

Substitute: $2(5) + 7 = 10 + 7 = 17$.

Evaluating $5a - 3$ at $a = 4$

Substitute: $5(4) - 3 = 20 - 3 = 17$.

6.3 Distributing and Factoring

Expand or pull out

$3(x+4) = 3x+12$ by distributing. Backwards, $6x+9 = 3(2x+3)$ by pulling out the common factor 3.

Simplify $2(x+5)+3x-4$

Distribute: $2x+10+3x-4$. Combine: $5x+6$.

Try it: simplify and evaluate

Simplify $4(2y-1)+y$, then evaluate it at $y=3$.

Answer. $8y-4+y = \mathbf{9y-4}$. At $y=3$: $27-4 = \mathbf{23}$.

6.4 Writing Expressions

Perimeter

A rectangle has width w and length $2w+3$. Its perimeter is $2w+2(2w+3) = 6w+6$.

Try it: the pattern

Each table seats 4 people, but joining n tables in a row loses 2 seats per join. Write the seats for n tables in a row, and evaluate for $n=5$.

Answer. There are $n-1$ joins: $4n-2(n-1) = \mathbf{2n+2}$ seats. For $n=5$: **12**.

6.5 Why It Works

Like terms combine because of the distributive property run backwards: $3x+2x = (3+2)x = 5x$. Unlike terms have no common factor to pull out, so they stay apart. Every simplification rule you will learn — collecting terms, expanding brackets, factoring — is the distributive property $a(b+c) = ab+ac$ read in one direction or the other.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

7 One-Step Equations

Undoing what was done to x

7.1 Keep the Balance

The Big Idea

An equation is a balance. Whatever you do to one side you must do to the other, and you undo an operation with its **opposite**.

An equation is a balance: both sides weigh the same. To find the unknown, undo whatever was done to it, and do the same thing to both sides so the balance stays level. If $x + 7 = 12$, subtract 7 from both sides; if $4x = 20$, divide both sides by 4.

Solving $4x = 28$

x was multiplied by 4, so divide both sides by 4: $x = 7$. Check: $4 \times 7 = 28$.

Where it goes wrong

Doing the operation to one side only, or undoing with the same operation instead of the inverse ($x+7 = 12$ needs subtraction, not more addition).

7.2 In Plain Terms & More Examples

In plain terms

Undo whatever was done to the letter using the opposite operation — an **inverse operation** — doing the same thing to both sides so the balance holds (the **properties of equality**).

Solving $x + 9 = 15$

Subtract 9 from both sides: $x = 6$. Check: $6 + 9 = 15$.

Solving $\frac{x}{3} = 4$

Multiply both sides by 3: $x = 12$. Check: $12 \div 3 = 4$.

7.3 Equations with Fractions and Decimals

Same moves, uglier numbers

To undo multiplication by a fraction, multiply by its reciprocal. To undo a decimal, divide by it — or multiply everything by 10 first.

Solve $\frac{2}{3}x = 8$

Multiply both sides by $\frac{3}{2}$: $x = 8 \times \frac{3}{2} = 12$. Check: $\frac{2}{3} \times 12 = 8$. ✓

Try it: solve

Solve $x - 3.5 = 7.2$ and $0.4x = 6$.

Answer. $x = 7.2 + 3.5 = \mathbf{10.7}$. $x = 6 \div 0.4 = \mathbf{15}$.

7.4 Word Problems

Setting up

"After spending \$18, Ana has \$27." Let m be what she started with: $m - 18 = 27$, so $m = 45$.

Try it: write and solve

Six identical books weigh 9 kg. Write an equation and find the weight of one book.

Answer. $6b = 9$, so $b = \mathbf{1.5}$ kg.

7.5 Why It Works

If two quantities are equal and you change both in the same way, they stay equal — that is what equality means. Subtracting 7 from both sides of $x + 7 = 12$ gives $x + 7 - 7 = 12 - 7$, and $7 - 7 = 0$ leaves x alone: $x = 5$. Checking by substitution ($5 + 7 = 12$ ✓) is not optional; it is the proof that the balance was kept at every step.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

8 Area of Triangles

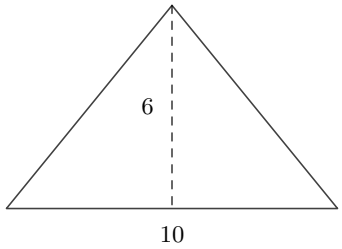
Half of a rectangle

8.1 Where the Half Comes From

The Big Idea

Any triangle is exactly **half** of a rectangle with the same base and height — that is where the

$\frac{1}{2}$ in the formula comes from.



A triangle is half of a parallelogram: two copies of any triangle fit together into one. So its area is half of base × height, where the height is measured straight up from the base, not along a slanted side. Any side can be the base — the answer comes out the same.

The formula

Area $= \frac{1}{2}bh$, where h is measured **perpendicular** to the base, not along a slanted side.

Base 10, height 6

Area $= \frac{1}{2}(10)(6) = 30$ square units.

Where it goes wrong

Using a slanted side as the height. The height must meet the base at a right angle; in an obtuse triangle it can even fall outside the triangle.

8.2 In Plain Terms & More Examples

In plain terms

A triangle is half a rectangle with the same base and height, so its area is $\frac{1}{2}$ × base × height.

Base 8, height 5

Area $= \frac{1}{2}(8)(5) = 20$ square units.

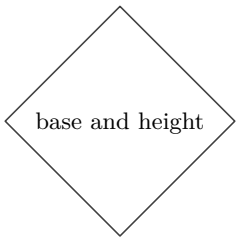
Base 12, height 3

Area $= \dfrac{1}{2}(12)(3) = 18$ square units.

8.3 Parallelograms and Trapezoids

Cut and slide

A parallelogram is a rectangle with a triangle moved from one end to the other: area $= bh$. A trapezoid is two triangles sharing the height: area $= \frac{1}{2}(b_1 + b_2)h$.



A trapezoid

Bases 6 and 10, height 4: $\frac{1}{2}(6 + 10) \times 4 = 32$.

Try it: composite shape

A shape is a 6×4 rectangle with a triangle of base 6 and height 3 on top. Find the total area.

Answer. $24 + \frac{1}{2} \cdot 6 \cdot 3 = 24 + 9 = \mathbf{33}$.

8.4 Word Problems

A sail

A triangular sail has base 3 m and height 5 m: area $\frac{1}{2} \cdot 3 \cdot 5 = 7.5$ m^2.

Try it: the missing height

A triangle has area 42 cm^2 and base 12 cm. What is its height?

Answer. $\frac{1}{2} \cdot 12 \cdot h = 42$, so $6h = 42$ and $h = \mathbf{7}$ cm.

8.5 Why It Works

Take a triangle with base b and height h and a second copy rotated 180°. They fit together along the matching side into a parallelogram of base b and height h, and a parallelogram's area is bh (slice off the triangle on one end and slide it to the other to make a $b \times h$ rectangle). Each copy is half of that: $A = \frac{1}{2}bh$. The argument never used which side was called the base, which is why all three choices agree.

Reading a proof

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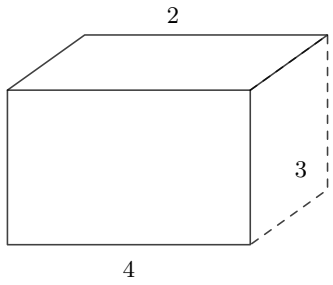
9 Surface Area and Volume

Wrapping versus filling

9.1 Two Different Measures

The Big Idea

Volume fills the inside (cubic units). **Surface area** wraps the outside (square units) — add up the six faces, which come in matching pairs.



Surface area is the total area of all the faces — what you would paint; volume is the space inside — what you would fill. A net (the solid unfolded flat) shows every face once, so adding the areas of the net's pieces gives the surface area. Volume of a box is length × width × height.

A 4 by 3 by 2 box

Volume $= 4 \cdot 3 \cdot 2 = 24$. Surface area $= 2(12 + 6 + 8) = 52$.

Where it goes wrong

Counting only the visible faces. A box has six faces, in three matching pairs; a net shows all six, which is why drawing one prevents the error.

9.2 In Plain Terms & More Examples

In plain terms

Volume fills the inside (cubic units); surface area covers the outside (square units) — add the areas of all six faces.

A cube of side 3

Volume $= 3^3 = 27$. Surface area $= 6 \times 3^2 = 54$.

A 5 by 2 by 2 box

Volume $= 5 \cdot 2 \cdot 2 = 20$. Surface area $= 2(10 + 4 + 10) = 48$.

9.3 Triangular Prisms and Fractional Edges

Base area times height

Any prism's volume is (area of the base shape) $\times$ (length). For a triangular prism the base is a triangle. Fractional edges use the same formulas.

A wedge

Triangle base 4, height 3, prism length 10: base area 6, volume 60.

Try it: fractional box

A box is $2\frac{1}{2}$ by 4 by $1\frac{1}{2}$ units. Find its volume and surface area.

Answer. Volume $\frac{5}{2} \cdot 4 \cdot \frac{3}{2} = \mathbf{15}$. Faces: $2(10 + 3.75 + 6) = \mathbf{39.5}$.

9.4 Word Problems

Wrapping a gift

A box $20 \times 15 \times 10$ cm needs paper for all six faces: $2(300 + 200 + 150) = 1300$ cm^2.

Try it: painting a cube

A cube has edge 6 cm. What is its surface area, and how many 1-cm cubes fit inside?

Answer. Surface: $6 \times 36 = \mathbf{216}$ cm^2. Volume: $6^3 = \mathbf{216}$ cubes — the same number, a coincidence special to edge 6.

9.5 Why It Works

A rectangular box $\ell \times w \times h$ has two faces of each size: ℓw, ℓh, and wh. So $S = 2\ell w + 2\ell h + 2wh = 2(\ell w + \ell h + wh)$. Its volume counts unit cubes: ℓw cubes per layer, h layers, so $V = \ell wh$. Surface area has square units and volume cubic units because two lengths were multiplied in one case and three in the other — the units are a built-in check.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

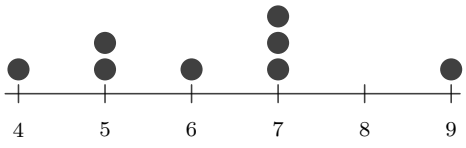
10 Statistics

Mean, median and spread

10.1 Two Kinds of Middle

The Big Idea

The **mean** shares the total out equally. The **median** is the value in the middle once the data is in order. One outlier drags the mean but barely moves the median.



The mean is the fair-share average: total divided by count. The median is the middle value when the data are sorted, and the mode is the most common value. The mean is pulled by extreme values; the median is not, which is why house prices are reported as medians.

Even-sized sets

With an even count there is no single middle value, so the median is the **average of the two middle** numbers.

Where it goes wrong

Forgetting to sort before finding the median, or taking the mean of the two middle values only when there is an even number of data points — with an odd count there is a single middle value.

10.2 In Plain Terms & More Examples

In plain terms

The mean shares the total out equally; the median is the middle value once the numbers are in order.

Mean of $4, 6, 8, 10$

Total $= 28$, and $28 \div 4 = 7$, so the mean is 7.

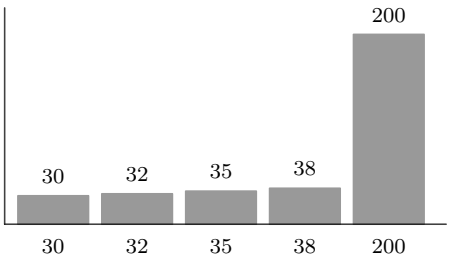
Median of $3, 5, 9$

In order, the middle value is 5.

10.3 Which Average? And the Range

Outliers move the mean

One extreme value drags the mean but leaves the median alone. The range (largest minus smallest) measures spread but is also hit by outliers.



Salaries

Five workers earn $30, 32, 35, 38$ and 200 (thousands). Mean $= 335 \div 5 = 67$; median $= 35$. The median describes a typical worker better.

Try it: compute all four

Data: $8, 3, 5, 8, 6$. Find the mean, median, mode and range.

Answer. Sorted: $3, 5, 6, 8, 8$. Mean $30 \div 5 = \mathbf{6}$; median $\mathbf{6}$; mode $\mathbf{8}$; range $8 - 3 = \mathbf{5}$.

10.4 Working Backwards

The missing score

Four tests average 85, so the total is 340. Three scores are $80, 90, 88$ — that is 258 — so the fourth is $340 - 258 = 82$.

Try it: what do I need?

Priya has scores $72, 85, 91$. What must she score on a fourth test to average exactly 85?

Answer. Needed total $4 \times 85 = 340$. Current total 248. She needs $340 - 248 = \mathbf{92}$.

10.5 Why It Works

The mean is the value every data point would have if the total were shared equally: n copies of the mean add to the same total as the data. That is why the deviations from the mean always sum to zero — the amounts above balance the amounts below. The median splits the sorted list into two equal halves by count, so it does not care how far away the largest value is; that is its robustness.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

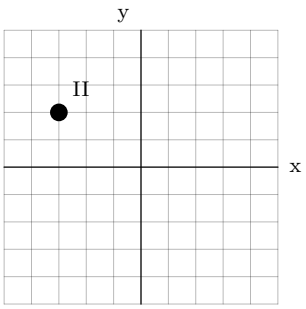
11 Quadrants

The four regions of the plane

11.1 Reading the Signs

The Big Idea

The axes cut the plane into four quadrants, numbered anticlockwise from the top right. The **signs** of x and y tell you which one you are in.



With negative numbers on both axes, the plane has four quadrants, numbered counterclockwise starting from the upper right. The signs of the coordinates tell the quadrant: $(+,+)$ is I, $(-,+)$ is II, $(-,-)$ is III, $(+,-)$ is IV. Points on an axis have a zero coordinate and belong to no quadrant.

The pattern

I is $(+,+)$, II is $(-,+)$, III is $(-,-)$, IV is $(+,-)$.

Where it goes wrong

Reflections: reflecting across the x-axis changes the sign of y (the point moves up or down), not x. Sketch it — the axis you reflect across stays fixed.

11.2 In Plain Terms & More Examples

In plain terms

The signs of x and y tell you the quadrant: quadrants are numbered anticlockwise from the top right.

Where is $(-4,-1)$?

Both are negative, so it is in quadrant III.

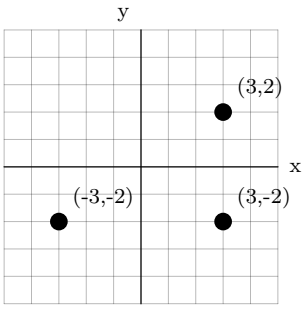
Where is $(5,-2)$?

x is positive and y is negative, so it is in quadrant IV.

11.3 Reflections and Distances

Flip a sign

Reflecting across the x-axis changes the sign of y; across the y-axis changes the sign of x. Distance along a horizontal or vertical line is the absolute difference of the coordinates that change.



Reflect $(3, -2)$

Across the x-axis: $(3, 2)$. Across the y-axis: $(-3, -2)$. Across both: $(-3, 2)$.

Try it: distance and quadrant

How far is $(-4, 7)$ from $(-4, -3)$? In which quadrant is $(-5, -1)$?

Answer. Same x, so $|7 - (-3)| = \mathbf{10}$. Both coordinates negative: quadrant **III**.

11.4 Word Problems

A polygon on the grid

Vertices $(-2, 3), (4, 3), (4, -1), (-2, -1)$ form a rectangle 6 wide and 4 tall: perimeter 20, area 24.

Try it: the park

A square park has corners at $(-3, -3)$ and $(3, -3)$ on its south side. Where are the north corners, and what is the park's area?

Answer. Side length 6, so the north corners are $(\mathbf{-3}, \mathbf{3})$ and $(\mathbf{3}, \mathbf{3})$; area **36**.

11.5 Why It Works

Distance along a horizontal line is $|x_2 - x_1|$ and along a vertical line is $|y_2 - y_1|$ — absolute values, because a distance is never negative. So $(-3, 2)$ and $(5, 2)$ are $|5 - (-3)| = 8$ apart. Reflecting

across the y-axis sends (x, y) to $(-x, y)$ because the mirror keeps the height and flips the side; this is the first appearance of a transformation rule written in coordinates.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.