MathCastle

Grade 5

Complete course booklet • 7 topics

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1 Decimal Operations

Multiplying and dividing decimals

1.1 Counting Decimal Places

The Big Idea

Multiply as if the points were not there, then put the point back. The answer has as many decimal places as the two factors have *together*.

Adding and subtracting decimals is exactly like whole numbers as long as the decimal points are lined up, because then tenths sit under tenths and hundredths under hundredths. Multiplying is done as whole numbers first; the decimal point is placed afterwards by counting decimal places.

Finding 2.4×3

Ignore the point: $24 \times 3 = 72$. There was one decimal place, so the answer is 7.2.

Where it goes wrong

Lining decimals up by their right-hand ends instead of by the decimal point: $3.5 + 1.25$ is 4.75, not 1.60. Pad with zeros (3.50) so every number has the same number of decimal places.

Sanity check

2.4 is close to 2, so the answer should be near 6 — and 7.2 is. A misplaced point would give 0.72 or 72, both clearly wrong.

1.2 In Plain Terms & More Examples

In plain terms

Multiply as if the points were not there, then count how many decimal places the two numbers had together and put the point back that far from the right.

Finding 1.5×4

Ignore the point: $15 \times 4 = 60$. One decimal place, so the answer is 6.0.

Finding 0.3×0.2

Ignore the points: $3 \times 2 = 6$. Two decimal places, so the answer is 0.06.

A multi-step problem

You buy 3 notebooks at \$2.40 each and pay with a \$10 bill. Step 1: $3 \times 2.40 = 7.20$. Step 2: the change is $10 - 7.20 = \$2.80$.

1.3 Dividing Decimals

Make the divisor whole

To divide by a decimal, move the point in BOTH numbers the same number of places until the divisor is whole. $4.5 \div 0.5$ becomes $45 \div 5 = 9$. The answer does not change because you multiplied both by the same power of ten.

Finding $7.2 \div 0.06$

Move two places: $720 \div 6 = 120$.

Try it: $3.36 \div 1.2$

Shift the point, divide, and check by multiplying back.

Answer. Shift one place: $33.6 \div 12 = \mathbf{2.8}$. Check: $1.2 \times 2.8 = 3.36$.

1.4 Word Problems

Price per kilogram

Apples cost \$5.40 for 3 kg. Per kilogram: $5.40 \div 3 = \$1.80$.

Try it: the trip

A car uses 0.08 liters of fuel per kilometer. How much fuel for a 125 km trip?

Answer. $125 \times 0.08 = 10.00$: **10** liters. (Two decimal places in the factors, two in the product: 10.00.)

1.5 Why It Works

A decimal with two places is a whole number of hundredths: $1.25 = 125$ hundredths. So $3.50 + 1.25 = 350 + 125 = 475$ hundredths $= 4.75$ — which is why lining up the points (lining up the hundredths) makes ordinary addition work. For multiplication, $0.3 \times 0.2 = \frac{3}{10} \times \frac{2}{10} = \frac{6}{100} = 0.06$: the denominators multiply, so the decimal places ADD ($1 + 1 = 2$ places).

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

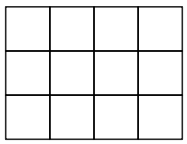
2 Multiplying Fractions

Straight across, and why

2.1 Why Straight Across Works

The Big Idea

Multiplying fractions means taking a **part of a part**. Cutting a rectangle both ways shows why you multiply tops and bottoms separately.



a part of a part

Multiplying by a fraction means taking a part of something: $\frac{1}{2} \times \frac{1}{3}$ is half of a third. Multiply the tops and multiply the bottoms — the picture is a rectangle cut both ways, and the doubly-shaded piece is the answer. Multiplying by a fraction less than 1 makes things smaller, which surprises people who expect multiplication to grow.

Finding $\frac{2}{3} \times \frac{3}{4}$

Straight across: $\dfrac{2 \times 3}{3 \times 4} = \dfrac{6}{12} = \dfrac{1}{2}$.

Where it goes wrong

Finding a common denominator before multiplying. That is the rule for adding; for multiplying you go straight across.

It gets smaller

Multiplying by a fraction less than 1 makes the result *smaller* — surprising at first, but taking part of something gives less than you started with.

2.2 In Plain Terms & More Examples

In plain terms

Multiply the top numbers together and the bottom numbers together, then simplify. Taking a part of a part gives a smaller result.

Finding $\frac{3}{5} \times \frac{5}{6}$

Straight across: $\dfrac{3 \times 5}{5 \times 6} = \dfrac{15}{30} = \dfrac{1}{2}$.

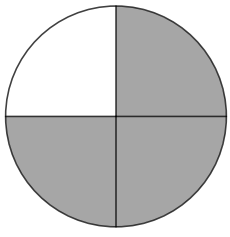
Finding $\frac{1}{2} \times \frac{2}{3}$

Straight across: $\dfrac{1 \times 2}{2 \times 3} = \dfrac{2}{6} = \dfrac{1}{3}$.

2.3 Mixed Numbers and Whole Numbers

Convert first

Turn mixed numbers into improper fractions before multiplying: $1\frac{1}{2} = \frac{3}{2}$. A whole number is a fraction over 1.



3/4 shaded

Finding $2\frac{1}{3} \times \frac{3}{5}$

$\frac{7}{3} \times \frac{3}{5} = \frac{21}{15} = \frac{7}{5} = 1\frac{2}{5}$.

Try it: cancel before you multiply

Compute $\frac{4}{9} \times \frac{3}{8}$ by cancelling common factors first.

Answer. Cancel 4 with 8 and 3 with 9: $\frac{1}{3} \times \frac{1}{2} = \mathbf{\frac{1}{6}}$.

2.4 Word Problems

Part of a part

Two thirds of a class are girls and $\frac{3}{4}$ of the girls play soccer. The fraction of the class that are soccer-playing girls: $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$.

Try it: the recipe

A recipe needs $\frac{3}{4}$ cup of sugar. You make $2\frac{1}{2}$ batches. How much sugar?

Answer. $\frac{5}{2} \times \frac{3}{4} = \frac{15}{8} = \mathbf{1\frac{7}{8}}$ cups.

2.5 Why It Works

Draw a unit square. Shade $\frac{1}{3}$ of it with vertical stripes and $\frac{1}{2}$ of it with horizontal stripes. The doubly shaded part is 1 of the 6 equal rectangles the two cuts create, so $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. In general the cuts make $b \times d$ equal pieces and the doubly shaded region has $a \times c$ of them: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Reading a proof

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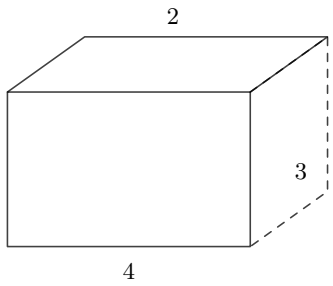
3 Volume

Filling a box with unit cubes

3.1 Counting Cubes

The Big Idea

Volume counts the **unit cubes** that fill a solid. One layer holds length × width cubes, and there are height layers.



Volume counts how many unit cubes fill a solid. For a box, a layer of cubes on the bottom has length × width cubes, and there are height-many layers, so $V = \ell \times w \times h$. Units are cubed — cubic centimeters, cubic feet — because three lengths were multiplied.

The formula

For a rectangular box, $V = l \times w \times h$, measured in **cubic** units.

A 4 by 3 by 2 box

One layer is $4 \times 3 = 12$ cubes, and there are 2 layers: 24 cubic units.

Where it goes wrong

Mixing units, like a length in centimeters and a height in meters, or reporting a volume in square units. Convert everything to one unit first, then cube it.

3.2 In Plain Terms & More Examples

In plain terms

Volume counts the unit cubes that fill a box: multiply length by width by height. The answer is in cubic units.

A 5 by 2 by 3 box

$V = 5 \times 2 \times 3 = 30$ cubic units.

A 6 by 4 by 2 box

$V = 6 \times 4 \times 2 = 48$ cubic units.

3.3 Solids Made of Two Boxes

Split, then add

An L-shaped solid is two boxes glued together. Find each box's volume and add — the same idea as the area model for multiplication.

An L-shape

A $6 \times 2 \times 3$ box next to a $2 \times 4 \times 3$ box: $36 + 24 = 60$ cubic units.

Try it: the missing side

A box has volume 180 cm^3, length 9 cm and width 4 cm. What is its height?

Answer. Base area $9 \times 4 = 36$; height $= 180 \div 36 =$ **5** cm.

3.4 Word Problems

A fish tank

A tank is 50 cm long, 20 cm wide and 30 cm tall: $50 \times 20 \times 30 = 30{,}000$ cm^3, which is 30 liters (since 1000 $\text{cm}^3 = 1$ L).

Try it: sandbox

A sandbox is 2 m by 1.5 m and is filled to a depth of 0.2 m. How many cubic meters of sand is that?

Answer. $2 \times 1.5 \times 0.2 =$ **0.6** m^3.

3.5 Why It Works

The bottom layer of a $4 \times 3 \times 2$ box is a 4×3 rectangle of cubes — 12 cubes, by the same reasoning as area. Stacking 2 identical layers gives $12 \times 2 = 24$ cubes. That is the whole argument: volume of a box = (area of the base) $\times$ (height). The same formula works for any prism or cylinder, because they are stacks of identical layers too — which is why a cylinder's volume is $\pi r^2 h$.

Reading a proof

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forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

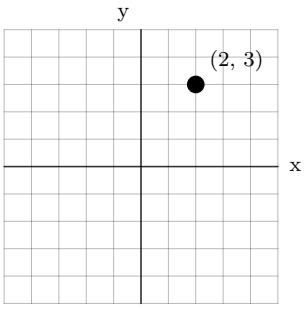
4 The Coordinate Plane

Plotting ordered pairs

4.1 Across Then Up

The Big Idea

An ordered pair (x, y) is a set of directions from the origin: go **across** x, then **up** y. The order matters — $(2, 5)$ and $(5, 2)$ are different places.



Two number lines crossed at right angles let you name any point with two numbers: how far right (the x-coordinate), then how far up (the y-coordinate). The order matters — $(3, 5)$ and $(5, 3)$ are different points. Reading a graph is just reading these two numbers off.

Plotting (2, 3)

From the origin, move 2 right and then 3 up.

Where it goes wrong

Reading the pair in the wrong order. "Along the hall, then up the stairs": x first, y second.

4.2 In Plain Terms & More Examples

In plain terms

An ordered pair (x, y) means go across x first, then up y. Swapping the numbers lands you somewhere else.

Plotting $(4, 1)$

From the origin, move 4 right and 1 up.

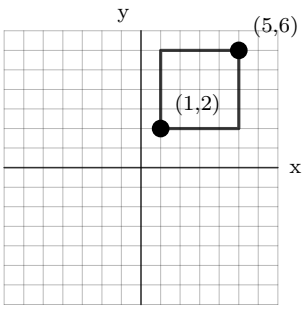
Plotting $(0, 3)$

Move 0 across (stay on the vertical axis) and 3 up.

4.3 Distances and Shapes

Same x or same y

Two points with the same y lie on a horizontal line; the distance between them is the difference of the x-coordinates. Same x: vertical, subtract the y's.



A rectangle

Corners $(1, 2), (5, 2), (5, 6), (1, 6)$: width $5 - 1 = 4$, height $6 - 2 = 4$. It is a square of area 16.

Try it: the fourth corner

Three corners of a rectangle are $(2, 1), (7, 1)$ and $(7, 4)$. Find the fourth corner and the perimeter.

Answer. Fourth corner $(\mathbf{2}, \mathbf{4})$. Width 5, height 3, perimeter $2(5 + 3) = \mathbf{16}$.

4.4 Word Problems

A treasure map

Start at $(3, 2)$, go 4 east and 5 north: you land at $(3 + 4, 2 + 5) = (7, 7)$.

Try it: the walk home

From school at $(6, 8)$, Ravi walks 6 west and then 8 south. Where does he end up, and what point is that?

Answer. $(6 - 6, 8 - 8) = (\mathbf{0}, \mathbf{0})$, the origin.

4.5 Why It Works

The coordinate plane turns geometry into arithmetic. A horizontal move from $(2, 5)$ to $(7, 5)$ is $7 - 2 = 5$ units, because only the x-coordinate changed. A vertical move is a difference of y-coordinates. Every fact about distances, shapes and lines on the plane can be computed from coordinates — which is what makes graphing, mapping, and every computer screen work.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

5 Order of Operations

Which operation goes first

5.1 The Agreed Order

The Big Idea

Without a rule, $2 + 3 \times 4$ could be 20 or 14. Everyone agrees to do **multiplication and division before addition and subtraction**, with brackets first of all.

An expression like $2 + 3 \times 4$ needs a rule for which operation goes first, or two people would get different answers. The agreement: grouping symbols first, then powers, then multiplication and division from left to right, then addition and subtraction from left to right. It is a convention — but one the whole world shares.

Evaluating $2 + 3 \times 4$

Multiply first: $3 \times 4 = 12$. Then add: $2 + 12 = 14$.

Where it goes wrong

Treating multiplication as always before division, or addition as before subtraction. They are ties, done left to right: $8 \div 2 \times 4 = 4 \times 4 = 16$.

5.2 In Plain Terms & More Examples

In plain terms

Do brackets first, then multiplication and division, and only then addition and subtraction — working left to right within each step. This agreed rule is the **order of operations** (often remembered as PEMDAS).

Evaluating $10 - 2 \times 3$

Multiply first: $2 \times 3 = 6$. Then subtract: $10 - 6 = 4$.

Evaluating $(6 + 2) \times 3$

Brackets first: $6 + 2 = 8$. Then multiply: $8 \times 3 = 24$.

5.3 Nested Grouping

Inside out

With brackets inside parentheses, work from the innermost group outward. Each layer is finished before the next.

Finding $2 \times [3 + (8 - 5)^2]$

Inner: $8 - 5 = 3$. Square: 9. Bracket: $3 + 9 = 12$. Multiply: 24.

Try it: evaluate

Compute $36 \div (2 + 4) \times 3 - 1$.

Answer. Parentheses: $36 \div 6 \times 3 - 1$. Left to right: $6 \times 3 = 18$, then $18 - 1 = \mathbf{17}$.

5.4 Writing Expressions

From words

"Add 5 and 7, then double the result" is $2 \times (5+7) = 24$. Without the parentheses, $2 \times 5 + 7 = 17$ says something different.

Try it: place the parentheses

Insert parentheses so that $4 + 2 \times 3 - 1$ equals 17.

Answer. $(4+2)\times 3-1=6\times 3-1=\mathbf{17}$.

5.5 Why It Works

The rule exists because multiplication is repeated addition: $2+3\times 4$ means $2+4+4+4$, and the additions hidden inside the multiplication must be finished before the 2 is added. Parentheses override this when a problem needs it, which is exactly why they exist: $(2+3)\times 4$ means "four copies of the sum". Powers come before multiplication for the same reason one level up — a power is repeated multiplication.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

6 Converting Units

Metric prefixes and time

6.1 Multiply or Divide?

The Big Idea

Going to a **smaller** unit means you need *more* of them, so multiply. Going to a bigger unit means fewer, so divide.

Converting units is multiplying by 1 in disguise: since 1 foot = 12 inches, the fraction $\frac{12\text{ in}}{1\text{ ft}}$ equals 1. Multiply 3 feet by it and the feet cancel, leaving 36 inches. Write the conversion so the unit you want to remove is on the opposite side of the fraction.

Metres to centimetres

One metre is 100 centimetres, so 3 metres is $3\times 100=300$ centimetres — more of a smaller unit.

Where it goes wrong

Multiplying when you should divide (or the reverse). Let the units decide: if the units do not cancel, the fraction is upside down.

6.2 In Plain Terms & More Examples

In plain terms

Changing to a smaller unit needs more of them, so multiply; changing to a bigger unit needs fewer, so divide.

Kilometres to metres

One kilometre is 1000 metres, so $5 \text{ km} = 5 \times 1000 = 5000 \text{ m}$.

Minutes to hours

One hour is 60 minutes, so $240 \text{ min} = 240 \div 60 = 4$ hours.

6.3 Two-Step Conversions

Chain the steps

To go from hours to seconds, go through minutes: $1 \text{ h} = 60 \text{ min} = 60 \times 60 = 3600 \text{ s}$. Each step multiplies (bigger unit to smaller) or divides (smaller to bigger).

Kilometers to centimeters

$2.5 \text{ km} = 2500 \text{ m} = 250{,}000 \text{ cm}$.

Try it: convert

Convert 3 hours 20 minutes to minutes, and 4500 g to kilograms.

Answer. $3 \times 60 + 20 = \mathbf{200}$ minutes. $4500 \div 1000 = \mathbf{4.5}$ kg.

6.4 Word Problems

Juice boxes

Each juice box holds 250 mL. Six boxes hold $1500 \text{ mL} = 1.5 \text{ L}$.

Try it: the race

A race is 5 km. Lena has run 3250 m. How many kilometers are left?

Answer. $5000 - 3250 = 1750 \text{ m} = \mathbf{1.75}$ km.

6.5 Why It Works

Because 12 inches and 1 foot are the same length, the ratio $\frac{12\text{ in}}{1\text{ ft}}$ is exactly 1, and multiplying any quantity by 1 leaves it unchanged — only its name changes. Chaining several such fractions converts through intermediate units (km → m → cm) without ever deciding whether to multiply or divide: the cancellation of units does the deciding.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

7 Powers of Ten

What happens to the digits

7.1 Shifting Places

The Big Idea

Multiplying by 10^n moves every digit n places to the **left**. The digits never change — only what each place is worth.



Multiplying by 10 shifts every digit one place to the left; dividing by 10 shifts it right. So $10^3 = 1000$ moves digits three places, and $3.7 \times 10^3 = 3700$. The exponent counts the shifts, which is why scientific notation can write huge and tiny numbers with a small exponent.

Finding 43×10^2

Two places left: 43 becomes 4300.

Where it goes wrong

Thinking 10^3 has three zeros AND a leading 1 that counts as a digit shift — it does not: $10^3 = 1000$

is a shift of exactly three places. Also, $10^0 = 1$, not 0.

7.2 In Plain Terms & More Examples

In plain terms

Multiplying by 10^n slides every digit n places to the left; the digits stay the same, only their value grows.

Finding 6×10^3

Three places left: 6 becomes 6000.

Finding 25×10^2

Two places left: 25 becomes 2500.

7.3 Dividing by Powers of Ten

Shift right

Dividing by 10^n moves every digit n places to the right. Past the ones place you get decimals: $42 \div 10^3 = 0.042$.

Finding $6.5 \div 10^2$

Two places right: 0.065.

Try it: both directions

Compute 0.37×10^4 and $815 \div 10^2$.

Answer. $0.37 \times 10^4 = \mathbf{3700}$. $815 \div 10^2 = \mathbf{8.15}$.

7.4 Word Problems

Distance to the Moon

The Moon is about 3.84×10^5 km away — that is 384,000 km. Writing 3.84×10^5 keeps the size clear without counting zeros.

Try it: cells

A drop of blood holds about 5×10^6 red blood cells. Write that as an ordinary number, and say how many cells are in 10 drops.

Answer. $5 \times 10^6 = \mathbf{5{,}000{,}000}$. Ten drops: $5 \times 10^7 = \mathbf{50{,}000{,}000}$.

7.5 Why It Works

Each place in our number system is worth ten times the place to its right. Multiplying a number by 10 therefore makes every digit worth ten times as much, which is the same as moving it one place left. Repeating this n times is multiplying by 10^n. Division by 10 is the reverse shift, and moving past the ones place produces decimals — the same rule, continued.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.