MathCastle

Geometry

Complete course booklet • 12 topics

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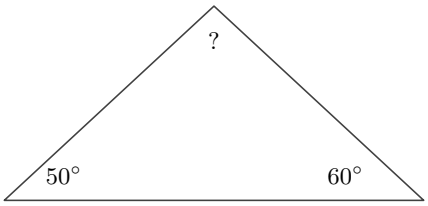
1 Core Ideas in Plain Terms

Shapes and reasoning, in everyday language

1.1 Angles and Lines

Angles in a shape add to a fixed total

The **Triangle Angle-Sum Theorem** says the three angles of any triangle add to 180°; a quadrilateral adds to 360° — both come from the polygon angle-sum formula $(n-2)\cdot 180°$. So a missing angle is just the total minus the ones you know. When a line crosses two parallel lines, it makes matching (equal) angles — the key to "angle chasing."



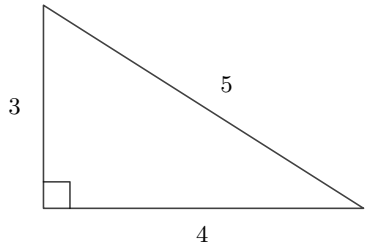
Find the third angle

A triangle has angles 50° and 60°; by the Triangle Angle-Sum Theorem the third is $180° - 110° = 70°$.

1.2 Triangles and the Pythagorean Theorem

Right triangles: a + b = c

In a right triangle, the two short sides (legs) and the longest side (hypotenuse) satisfy the **Pythagorean theorem**: $a^2 + b^2 = c^2$. A few side-triples come up constantly — 3-4-5, 5-12-13 — so recognizing them saves time.



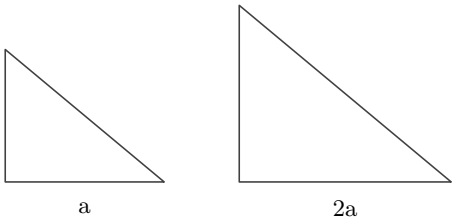
Find the hypotenuse

Legs 3 and 4: by the Pythagorean theorem, $c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

1.3 Similarity, Area, and Volume

Same shape, scaled size

Two figures are **similar** if one is a scaled copy of the other — all angles equal, all sides in the same ratio. If you scale lengths by k, areas scale by k^2 and volumes by k^3. That is why doubling a shape's size quadruples its area.



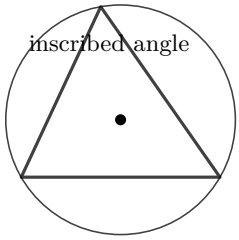
Area of a triangle

Using the triangle-area formula $A = \frac{1}{2}bh$: base 6, height 4 gives $A = \frac{1}{2} \cdot 6 \cdot 4 = 12$.

1.4 Going Deeper: Circles and Coordinates

Inscribed angles are half the central angle

An angle drawn from the circle's edge sees an arc at half the angle drawn from the center. Two consequences worth knowing cold: every inscribed angle in a semicircle is 90°, and inscribed angles on the same arc are equal.



Use it

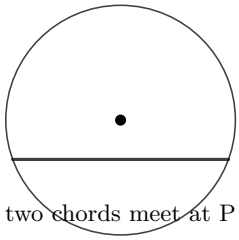
If arc AB measures $80°$, an inscribed angle standing on AB measures $40°$ — no matter where on the far side of the circle its vertex sits.

Distance and midpoint are Pythagoras in disguise

The distance between two points is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ — the hypotenuse of the right triangle their coordinates make. From $(0, 0)$ to $(3, 4)$: $\sqrt{9 + 16} = 5$. The midpoint just averages the coordinates.

Power of a Point (multi-step)

Chords AB and CD meet at P inside a circle, with $AP = 4$, $PB = 6$, and $CP = 3$. The **Power of a Point** theorem says $AP \cdot PB = CP \cdot PD$. Step 1: $AP \cdot PB = 4 \cdot 6 = 24$. Step 2: $PD = 24 \div 3 = 8$.



Ptolemy's theorem (multi-step)

Quadrilateral $ABCD$ is inscribed in a circle with diagonal $AC = 25$ a diameter, and sides $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$. **Ptolemy's theorem** says $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Step 1: $7 \cdot 20 + 15 \cdot 24 = 140 + 360 = 500$. Step 2: $BD = 500 \div 25 = 20$.

1.5 Problem-Solving Playbook

Add a line

When a figure gives you nothing to compute with, **add an auxiliary line**: a radius to a tangent point, an altitude, a diagonal. Each new line creates triangles — and triangles bring the Pythagorean theorem, similarity, and area formulas into play.

Worked: Heron's formula

A triangle has sides 13, 14, 15. **Heron's formula**: with $s = \dfrac{13+14+15}{2} = 21$, the area is $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84$.

2 Transformations

Slides, flips, turns, and resizes on the coordinate plane

2.1 Transformation Vocabulary

A **transformation** moves or changes a figure according to a rule.

- **Pre-image:** the original figure (points A, B, C).
- **Image:** the new figure after the transformation (points A', B', C', read "A prime").
- **Mapping:** the pairing of each pre-image point with its image point. We write $A \to A'$.
- **Rigid motion** (also called an **isometry**): a transformation that keeps the same size and shape. Translations, reflections, and rotations are rigid.
- **Non-rigid** transformation: changes the size. A **dilation** is non-rigid.

Translations, reflections and rotations are the rigid motions — they preserve distances and angles, so a figure and its image are congruent. Dilations scale distances by a factor and preserve angles, so the image is similar. Composing transformations gives new ones, and the rules in coordinates make every transformation a formula.

A rigid motion moves triangle ABC to triangle $A'B'C'$ without stretching it. Every side keeps its length and every angle keeps its measure, so the two triangles are **congruent**.

Where it goes wrong

Rotating 90° counterclockwise with the wrong rule. About the origin, $(x, y) \to (-y, x)$; check with

$(1,0) \to (0,1)$, which is a quarter turn up.

Tip: If you can slide, flip, or turn a figure onto its image, the two are congruent. If you must also grow or shrink it, they are only *similar*.

Try it: Quick check

A transformation that slides every point the same distance and direction is called a _____.

Work it on paper before revealing the answer.

Answer. Answer: translation

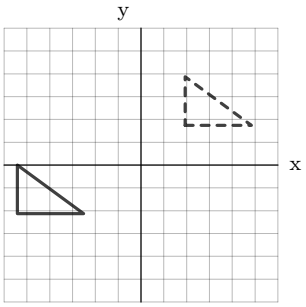
2.2 Translations (Slides)

A **translation** slides every point the same distance in the same direction. If you move a units horizontally and b units vertically:

$$(x, y) \to (x + a,\ y + b).$$

The pair $\langle a, b \rangle$ is the **translation vector**. Positive a is right, negative is left; positive b is up, negative is down.

A translation slides.



Translate triangle $A(1,1)$, $B(4,1)$, $C(1,3)$ by the rule $(x, y) \to (x + 2,\ y + 1)$.

$$A'(3,2), \quad B'(6,2), \quad C'(3,4).$$

Tip: "Right/left" changes only x; "up/down" changes only y. A slide never turns or flips the figure.

Try it: Quick check

In a transformation, the original figure is the _____ and the result is the _____.

Work it on paper before revealing the answer.

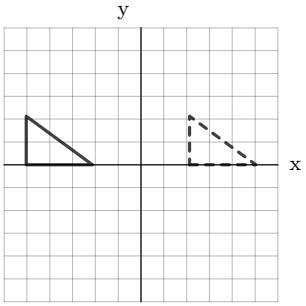
Answer. Answer: pre-image; image

2.3 Reflections (Flips)

A **reflection** flips a figure over a **line of reflection**, producing a mirror image. Coordinate rules:

$$\begin{aligned} \text{over the } x\text{-axis:} &\quad (x, y) \to (x, -y) \\ \text{over the } y\text{-axis:} &\quad (x, y) \to (-x, y) \\ \text{over } y = x: &\quad (x, y) \to (y, x) \\ \text{over } y = -x: &\quad (x, y) \to (-y, -x) \end{aligned}$$

A reflection flips across a line.



Reflect triangle $A(1, 1)$, $B(4, 1)$, $C(1, 3)$ over the x-axis using $(x, y) \to (x, -y)$.

$$A'(1, -1), \quad B'(4, -1), \quad C'(1, -3).$$

Reflect $P(3, 5)$ over the line $y = x$ using $(x, y) \to (y, x)$: the coordinates swap, giving $P'(5, 3)$. Over $y = -x$, the rule $(x, y) \to (-y, -x)$ gives $P''(-5, -3)$.

Tip: A point on the line of reflection does not move. Watch the signs: over the x-axis only y changes sign; over the y-axis only x changes sign.

Try it: Quick check

True or False: a rigid motion (isometry) changes the size of a figure.

Work it on paper before revealing the answer.

Answer. Answer: False

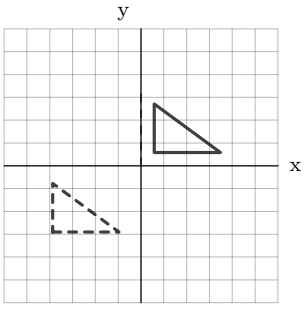
2.4 Rotations (Turns) About the Origin

A **rotation** turns a figure about a fixed point (the **center**). Rotating **counterclockwise** about the origin:

$$\begin{aligned} 90^\circ: &\quad (x, y) \to (-y,\, x) \\ 180^\circ: &\quad (x, y) \to (-x,\, -y) \\ 270^\circ: &\quad (x, y) \to (y,\, -x) \end{aligned}$$

A 90° *clockwise* turn equals a 270° counterclockwise turn (and vice versa). A 180° turn is the same either direction.

A rotation turns about a point.



Rotate triangle $A(1, 1)$, $B(4, 1)$, $C(1, 3)$ by 90° counterclockwise using $(x, y) \to (-y, x)$.

$$A'(-1, 1), \quad B'(-1, 4), \quad C'(-3, 1).$$

Tip: For 90° turns, swap the coordinates first, then fix signs. Counterclockwise is the positive direction in math.

Try it: Quick check

Which one is *not* a rigid motion: translation, reflection, rotation, or dilation?

Work it on paper before revealing the answer.

Answer. Answer: dilation

2.5 Compositions of Transformations

A **composition** applies one transformation and then another, in order. Do the *first* transformation, write the image, then apply the *second* transformation to that image.

Reflect $(1, 2)$ over the x-axis, *then* translate by $(x, y) \to (x + 3, y)$.

$$\begin{aligned} \text{Step 1 (reflect):} \quad & (1, 2) \to (1, -2) \\ \text{Step 2 (translate):} \quad & (1, -2) \to (4, -2) \end{aligned}$$

The final image is $(4, -2)$.

Two reflections over the x-axis and then the y-axis have the same effect as one 180° rotation about the origin, because $(x, y) \to (x, -y) \to (-x, -y)$.

Tip: Order matters! Reflecting then rotating usually gives a different image than rotating then reflecting. Always finish step 1 completely before starting step 2.

Try it: Quick check

Translate the point $P(2, 3)$ using the rule $(x, y) \to (x + 4,\ y - 1)$.

Work it on paper before revealing the answer.

Answer. Answer: $P'(6, 2)$

2.6 Dilations (Resizing) About the Origin

A **dilation** centered at the origin multiplies every coordinate by the same **scale factor** k:

$$(x, y) \to (kx,\ ky).$$

- $k > 1$: an **enlargement** (the figure grows).
- $0 < k < 1$: a **reduction** (the figure shrinks).
- $k = 1$: the figure is unchanged.

A dilation is *not* a rigid motion, but it keeps the same shape, so the image is **similar** to the pre-image.

Dilate triangle $A(1, 1)$, $B(2, 1)$, $C(1, 2)$ by scale factor $k = 2$ using $(x, y) \to (2x, 2y)$.

$$A'(2, 2), \quad B'(4, 2), \quad C'(2, 4).$$

Tip: The scale factor of the *lengths* is k, but the *area* scales by k^2. Doubling the sides ($k = 2$) makes the area 4 times as large.

Try it: Quick check

Translate the point $Q(-1, 5)$ using the rule $(x, y) \to (x + 2,\ y + 3)$.

Work it on paper before revealing the answer.

Answer. Answer: $Q'(1, 8)$

2.7 Symmetry

Line (reflectional) symmetry: a figure has line symmetry if a reflection over some line maps the figure exactly onto itself. That line is a **line of symmetry**.

Rotational symmetry: a figure has rotational symmetry if a rotation of less than 360° about its center maps it onto itself. The **order** is how many times it matches in one full turn; the **angle** of rotation is 360° ÷ order.

A regular hexagon has 6 lines of symmetry. Its rotational symmetry has **order** 6, with smallest angle $360° \div 6 = 60°$.

A square has 4 lines of symmetry (two diagonals and two through opposite side midpoints) and rotational symmetry of order 4 (90°). A non-square rectangle has only 2 lines of symmetry and rotational symmetry of order 2 (180°).

Tip: Every regular polygon with n sides has exactly n lines of symmetry and rotational symmetry of order n.

Try it: Quick check

Write the rule for a translation that moves a figure 5 units right and 2 units down.

Work it on paper before revealing the answer.

Answer. Answer: $(x, y) \to (x + 5,\ y - 2)$

2.8 Congruence vs. Similarity

Rigid Motions vs. Dilations

- **Rigid motions** (translations, reflections, rotations, and their compositions) preserve size and shape. The image is **congruent** to the pre-image.
- **Dilations** (scale factor $k \neq 1$) preserve shape but change size. The image is **similar**, but not congruent, to the pre-image.

Two figures are **congruent** if a sequence of rigid motions maps one onto the other. They are **similar** if a sequence of rigid motions *and* dilations does so.

Tip: Congruent $\Rightarrow$ similar (with ratio 1), but similar does not always mean congruent. Ask: "Did the size change?" If yes, it is similar only.

Try it: Quick check

Reflect the point $(4, 2)$ over the x-axis.

Work it on paper before revealing the answer.

Answer. Answer: $(4, -2)$

2.9 Going Deeper: Advanced Transformations

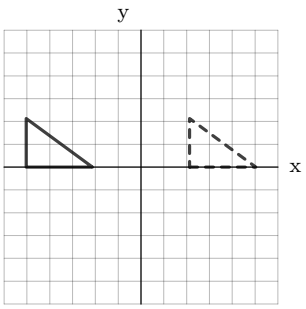
Composing Two Reflections

Reflecting twice in a row is always a *single* rigid motion. Which one depends on the two mirror lines:

- **Parallel mirrors** $\rightarrow$ a **translation**. The shift is *perpendicular* to the mirrors, in the direction from the first mirror to the second, and its distance is $2d$, where d is the gap between the mirrors.
- **Intersecting mirrors** $\rightarrow$ a **rotation** about the point where they cross. If the angle from the first mirror to the second is θ, the rotation angle is 2θ (same turning direction).

Either way, two flips restore the original **orientation**, so the result is an "even" (direct) isometry.

A reflection flips across a line.



Parallel mirrors give a translation. Reflect $P(0,2)$ over the line $x=1$, then over $x=3$. The mirrors are $d=2$ apart, so the net move is a translation of $2d=4$ units to the right.

$$\begin{aligned} \text{Reflect over } x=1: &\quad (0,2) \rightarrow (2,2) \\ \text{Reflect over } x=3: &\quad (2,2) \rightarrow (4,2) \end{aligned}$$

Net effect: $(0,2) \rightarrow (4,2)$, i.e. $(x,y) \rightarrow (x+4,y)$.

Intersecting mirrors give a rotation. Reflect $P(3,1)$ over the x-axis, then over the line $y=x$. These mirrors cross at the origin, and the angle from the x-axis to $y=x$ is $45°$, so the result is a rotation of $2(45°)=90°$ about the origin.

$$\begin{aligned} \text{Reflect over } x\text{-axis}: &\quad (3,1) \rightarrow (3,-1) \\ \text{Reflect over } y=x: &\quad (3,-1) \rightarrow (-1,3) \end{aligned}$$

Indeed $(3,1) \rightarrow (-1,3)$ is exactly the $90°$ counterclockwise rule $(x,y) \rightarrow (-y,x)$.

Rotations About Any Center & General Lines

Rotation about a center (a, b): shift the center to the origin, rotate, then shift back.

$$(x, y) \xrightarrow{\text{subtract}} (x - a,\, y - b) \xrightarrow{\text{rotate}} \cdots \xrightarrow{\text{add}} \text{(result)}.$$

For a 90° counterclockwise turn about (a, b) this collapses to

$$(x, y) \to \big(a - (y - b),\;\; b + (x - a)\big).$$

Reflection over a general line $y = mx + c$: it still sends each point to the opposite side at equal perpendicular distance. A clean way is to compose known moves — translate so the line passes through the origin, rotate the line onto the x-axis, reflect over the x-axis, then undo the rotation and translation.

Rotate about a non-origin center. Rotate $P(4, 1)$ by 90° counterclockwise about the center $O(2, 1)$.

$$\begin{aligned} \text{Subtract center:} \quad & (4, 1) - (2, 1) = (2, 0) \\ \text{Rotate } 90^\circ\ (x, y) \to (-y, x): \quad & (2, 0) \to (0, 2) \\ \text{Add center back:} \quad & (0, 2) + (2, 1) = (2, 3) \end{aligned}$$

So $P'(2, 3)$: the point swings a quarter turn around O, not around the origin.

Transformations as Matrices (Preview)

Every rotation, reflection, and dilation *about the origin* can be written as multiplication by a 2×2 matrix acting on the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$\text{Rotate } 90^\circ \text{ CCW}: \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{Reflect over } x\text{-axis}: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{Reflect over } y = x: \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{Dilation, factor } k: \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

For example, the 90° rotation of $(3, 1)$ is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

matching $(x, y) \to (-y, x)$. A *translation* is not linear, so it needs a separate added vector $\begin{bmatrix} a \\ b \end{bmatrix}$ (or a 3×3 "homogeneous" matrix you will meet later). **Composing** transformations then corresponds to *multiplying* their matrices.

Glide Reflections

A **glide reflection** is a reflection over a line followed by a translation *parallel* to that line (the order does not matter). It is the fourth basic rigid motion of the plane, alongside translations, rotations, and reflections.

- It *reverses* orientation (like a single reflection), so it is an "odd" isometry.
- It has *no* fixed points and no fixed line pointwise — the classic footprint pattern of a walking person is a glide reflection.

Here T is reflected over the x-axis and slid 2 units right to land on T'.

Symmetry Groups & Order of Rotational Symmetry

The set of all rigid motions that map a figure onto itself forms its **symmetry group**.

- **Cyclic group** C_n: only rotational symmetry — n rotations (including the 360° "do nothing"), and *no* lines of symmetry. A pinwheel or "S" shape is like this.
- **Dihedral group** D_n: rotational *and* reflective symmetry — n rotations plus n reflections, giving $2n$ symmetries in all.

A regular n-gon has symmetry group D_n: order-n rotational symmetry (smallest angle $360°/n$) *and* n lines of symmetry, for $2n$ total symmetries. An equilateral triangle is D_3 (6 symmetries); a square is D_4 (8 symmetries).

Dilation — area & orientation: Under a dilation with scale factor k, lengths scale by $|k|$, *areas* scale by k^2, and angles are unchanged. When $k > 0$ orientation is preserved; a *negative* k acts like a positive dilation of factor $|k|$ combined with a 180° rotation about the center, which also preserves orientation. So a dilation never produces a mirror image — only a resized, similar copy.

2.10 Why It Works

A reflection across the y-axis sends (x, y) to $(-x, y)$: the mirror keeps the height and negates the horizontal position. Rotation by 90° counterclockwise is two reflections in a row (across $y = x$, then across the y-axis): $(x, y) \to (y, x) \to (-y, x)$. Every rigid motion of the plane is a composition of at most three reflections — a fact that explains why they all preserve distance: each reflection does, so any composition of them does too.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line

forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

3 Points, Lines & Planes

Foundations: undefined terms, segments, midpoints & distance

3.1 Undefined Terms: Point, Line, Plane

The Three Building Blocks

In geometry we start with three ideas so basic that we do not even define them. We describe them instead.

- **Point** — a single exact location. It has no size. We draw a dot and name it with a capital letter, like P.
- **Line** — a straight path of points that goes on forever in both directions. Name it by any two of its points with a line symbol, $\overleftrightarrow{AB}$, or by a lowercase letter, ℓ.
- **Plane** — a perfectly flat surface that goes on forever. Name it with a capital script letter or by three points that are not all on one line, like plane ABC.

Collinear points lie on the same line. **Coplanar** points lie on the same plane.

Geometry starts with undefined terms — point, line, plane — and builds everything from axioms about them: two points determine a line, three non-collinear points determine a plane. Segment addition and the midpoint formula turn positions into arithmetic. Precise vocabulary matters because proofs are built from it.

Naming a line and its points

The points A, B, and C below are *collinear* because they all lie on one line. That line can be named $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$, or simply ℓ.

Where it goes wrong

Naming a line by two points that are not on it, or treating a ray $\overrightarrow{AB}$ as the same as $\overrightarrow{BA}$. A ray's first letter is its endpoint.

Collinear or not?

In the figure, D, E, and F lie on line m, so they are collinear. Point G is off the line, so D,

E, and G are *not* collinear. All four points, however, lie on the same flat page, so they are **coplanar**.

Tip: Any *two* points determine exactly one line. Any *three* non-collinear points determine exactly one plane.

Try it: Quick check

Name the geometric term described: "an exact location with no size."

Work it on paper before revealing the answer.

Answer. Answer: Point.

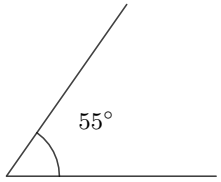
3.2 Segments, Rays, and Opposite Rays

Segments and Rays

- A **segment** $\overline{AB}$ is the part of a line between two **endpoints** A and B, including both endpoints. It has a definite length.
- A **ray** $\overrightarrow{AB}$ starts at endpoint A and goes forever through B. Order matters: $\overrightarrow{AB}$ and $\overrightarrow{BA}$ point opposite ways.
- **Opposite rays** are two rays with the *same endpoint* that together form a straight line.

Notation: $\overline{AB}$ names the segment (a figure); AB with no bar means its *length* (a number).

An angle measures a turn.



Reading a figure of rays

Here $\overrightarrow{BA}$ and $\overrightarrow{BC}$ share endpoint B and form a straight line, so they are **opposite rays**. The segment $\overline{AC}$ is the piece between A and C.

Remember: A bar means a *figure* ($\overline{AB}$, $\overrightarrow{AB}$, $\overleftrightarrow{AB}$); no bar means a *number* (AB = the length).

Try it: Quick check

True or false: any two points can be connected by exactly one line.

Work it on paper before revealing the answer.

Answer. Answer: True.

3.3 Measuring Segments & the Segment Addition Postulate

Segment Addition Postulate

If point B lies *on* $\overline{AC}$ (between A and C), then the two shorter lengths add to the whole:

$$AB + BC = AC.$$

This lets you find any one of the three lengths when you know the other two.

Adding segment lengths

Point B is between A and C with $AB = 6$ and $BC = 4$. Then

$$AC = AB + BC = 6 + 4 = 10.$$

Solving for a missing length

If $AC = 15$ and $AB = 9$ with B between A and C, then

$$BC = AC - AB = 15 - 9 = 6.$$

Tip: The point named in the middle of the sentence ("B is between A and C") is the one that splits the segment. It appears on *both* short pieces.

Try it: Quick check

Using the figure above, is point D collinear with A and B? Explain in one sentence.

Work it on paper before revealing the answer.

Answer. Answer: No — D is off the line, so it is not collinear with A and B.

3.4 Congruent Segments & Midpoints

Congruence and Midpoint

Two segments are **congruent** ($\overline{AB} \cong \overline{CD}$) when they have equal length ($AB = CD$). We mark congruent segments with matching tick marks.

The **midpoint** M of $\overline{AB}$ is the point exactly halfway between A and B. It splits the segment into two congruent halves:

$$AM = MB = \tfrac{1}{2}\,AB.$$

Using the midpoint

M is the midpoint of $\overline{AB}$ and $AB = 14$. Then each half is

$$AM = MB = \tfrac{1}{2}(14) = 7.$$

The tick marks show the two halves are congruent.

Remember: A midpoint makes two *equal* halves, so if you know the whole, halve it; if you know a half, double it.

Try it: Quick check

Give two different names for the line drawn in the figure above (use the point names).

Work it on paper before revealing the answer.

Answer. Answer: Any two of: $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$.

3.5 Midpoint Formula & Distance Formula

Formulas on the Coordinate Plane

For points $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\textbf{Midpoint } M = \left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right)$$

$$\textbf{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The midpoint *averages* the coordinates. The distance formula is the Pythagorean Theorem in disguise.

Midpoint of two points

Find the midpoint of $A(1,1)$ and $B(4,3)$.

$$M = \left(\frac{1+4}{2},\ \frac{1+3}{2}\right) = \left(\frac{5}{2},\ 2\right) = (2.5,\ 2).$$

Distance between two points

Find the distance from $A(1,1)$ to $B(4,5)$.

$$AB = \sqrt{(4-1)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

This is the famous 3-4-5 right triangle: the horizontal leg is 3, the vertical leg is 4, and the segment is the hypotenuse.

Tip: In the distance formula the subtractions are squared, so it does not matter which point you call first — a negative squared is positive. Just be careful and neat with the arithmetic.

Try it: Quick check

Write the symbol for the segment with endpoints R and S.

Work it on paper before revealing the answer.

Answer. Answer: $\overline{RS}$.

3.6 Segment Bisectors & Basic Constructions

Segment Bisector

A **segment bisector** is any line, ray, or segment that passes through the *midpoint* of a segment, cutting it into two congruent halves. A line that bisects a segment *and* meets it at a right angle is a **perpendicular bisector**.

A perpendicular bisector

Line k passes through M, the midpoint of $\overline{PQ}$, and meets it at 90°. So $PM = MQ$ (equal tick marks) and k is the perpendicular bisector of $\overline{PQ}$.

Two Classic Compass Constructions

Copying a segment $\overline{AB}$: Draw a ray with endpoint C. Open your compass to the length

AB. Without changing that opening, place the point on C and swing a small arc across the ray; label the crossing D. Then $\overline{CD} \cong \overline{AB}$.

Bisecting a segment $\overline{AB}$: Open the compass wider than half of AB. From A, draw arcs above and below the segment; from B with the *same* opening, draw two more arcs. Connect the two crossing points. That line is the perpendicular bisector, and where it meets $\overline{AB}$ is the midpoint.

Remember: Good constructions never measure with a ruler — they use only a compass and straightedge. Keep the compass opening *unchanged* for each step.

Try it: Quick check

Write the symbol for the ray that starts at R and goes through S.

Work it on paper before revealing the answer.

Answer. Answer: $\overrightarrow{RS}$.

3.7 Going Deeper: Advanced Ideas

The Section (Partition) Formula

The midpoint splits a segment in the ratio $1:1$. To find a point P that divides $\overline{AB}$ in *any* ratio $m:n$ (so that $AP:PB = m:n$), with $A(x_1, y_1)$ and $B(x_2, y_2)$, use the **section formula**:

$$P = \left(\frac{m\,x_2 + n\,x_1}{m+n},\ \ \frac{m\,y_2 + n\,y_1}{m+n} \right).$$

Notice the "cross" pattern: the weight m (nearest to B) multiplies the B-coordinates, and n multiplies the A-coordinates. Setting $m = n = 1$ recovers the midpoint formula, since each coordinate becomes a plain average.

Dividing a segment in the ratio $2:3$

Find the point P on $\overline{AB}$ with $A(1,2)$ and $B(6,7)$ such that $AP:PB = 2:3$. Here $m = 2$ and $n = 3$, so $m + n = 5$:

$$P = \left(\frac{2(6)+3(1)}{5},\ \ \frac{2(7)+3(2)}{5} \right) = \left(\frac{15}{5},\ \frac{20}{5} \right) = (3,\ 4).$$

Check: P is closer to A than to B, as expected when the first part of the ratio (2) is the smaller one.

Distance and Midpoint in Three Dimensions

A point in space needs three coordinates, $A(x_1, y_1, z_1)$. The plane formulas simply grow a z-term:

$$\textbf{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\textbf{Midpoint } M = \left(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2}, \ \frac{z_1 + z_2}{2} \right)$$

This is still just the Pythagorean Theorem, applied twice: once in the base plane and once up the vertical direction.

Distance between two points in space

Find the distance from $A(1, 2, 3)$ to $B(3, 5, 15)$.

$$AB = \sqrt{(3-1)^2 + (5-2)^2 + (15-3)^2} = \sqrt{4 + 9 + 144} = \sqrt{157}.$$

Because 157 is not a perfect square, we leave the answer as the exact radical $\sqrt{157}$ (about 12.53).

Counting Lines, Planes, and Regions

When points and lines are in "general position" (no needless overlaps), the counts follow neat patterns. For n points with *no three collinear*, the number of distinct lines through pairs of them is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

For n points with *no four coplanar*, the number of distinct planes through triples of them is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

And n lines drawn in a plane, no two parallel and no three meeting at one point, cut the plane into

$$1 + n + \binom{n}{2}$$

regions — each new line adds one more region than the line before it.

How many lines and planes from 5 points?

Take 5 points, no three collinear and no four coplanar. The number of lines is

$$\binom{5}{2} = \frac{5 \cdot 4}{2} = 10,$$

and the number of planes is

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{6} = 10.$$

Reflection for Shortest Paths

To find the shortest path from point A to a line ℓ and then on to point B (both A and B on the same side of ℓ), **reflect** one point across the line. If A' is the mirror image of A across ℓ, then for any point X on the line $AX = A'X$. So the bent path $A \to X \to B$ has the same length as $A' \to X \to B$, which is shortest when A', X, and B are *collinear*. Draw the straight segment $\overline{A'B}$; where it crosses ℓ is the best point X.

The Idea of a Locus

A **locus** is the set of *all* points that satisfy a given condition, and nothing else. Two rules of thumb turn everyday shapes into loci:

- The locus of points a fixed distance r from a point O is a **circle** of radius r centered at O.
- The locus of points *equidistant* from two fixed points A and B is the **perpendicular bisector** of $\overline{AB}$.

To describe a locus, ask: what shape do *all* the qualifying points trace out?

A coordinate proof using the locus idea

Prove that the point equidistant from $A(0,0)$ and $B(6,0)$ with y-coordinate 0 is the midpoint. A point $P(x,0)$ is equidistant when $AP = BP$, so $AP^2 = BP^2$:

$$x^2 = (x-6)^2 = x^2 - 12x + 36.$$

Subtracting x^2 from both sides gives $0 = -12x + 36$, so $x = 3$. Thus $P(3,0)$, which is exactly the midpoint $\left(\frac{0+6}{2}, \frac{0+0}{2}\right)$. This is a **coordinate proof**: by placing the figure on convenient axes, an algebra step settles a geometric claim.

Remember: Coordinate proofs work best when you place a key point at the origin and a key segment along an axis — the zeros make the distance and midpoint arithmetic short and clean.

Try it: Quick check

Does $\overrightarrow{XY}$ mean the same ray as $\overrightarrow{YX}$? Answer yes or no.

Work it on paper before revealing the answer.

Answer. Answer: No.

3.8 Why It Works

The midpoint formula averages coordinates because the midpoint is equally far from both ends along each axis: halfway from x_1 to x_2 is $x_1 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2}{2}$. Segment addition ($AB + BC = AC$ when B is between) is an axiom about lengths on a line, and it is what lets algebra enter geometry: an unknown length becomes a variable in an equation.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

4 Angles

Naming, classifying, and solving for angle measures

4.1 Naming Angles and Their Parts

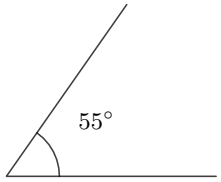
What is an angle?

An **angle** is formed by two **rays** that share a common endpoint. That shared endpoint is the **vertex**, and the two rays are the **sides** of the angle. We measure the "opening" between the sides in **degrees** (°).

There are three ways to name an angle:

- **Three points:** $\angle AOB$ or $\angle BOA$ — the *vertex letter always goes in the middle.*
- **The vertex alone:** $\angle O$ — only when there is no confusion about which angle you mean.
- **A number:** $\angle 1$ — when a small number is written inside the angle.

An angle measures a turn.



Angles are measured in degrees and classified by size; pairs of angles get names for their relationships — complementary (sum 90°), supplementary (sum 180°), vertical (opposite at a crossing, always equal), adjacent, linear pair. Most angle problems are one of these relationships written as an equation.

Naming the angle below

The rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ meet at the vertex O. We can call this angle $\angle AOB$, $\angle BOA$, or simply $\angle O$.

Where it goes wrong

Assuming two angles that look equal in a diagram are equal. Diagrams are not to scale; use only the given marks and the theorems.

Protractor tip. To measure an angle, place the protractor's center hole on the **vertex** and line up one side with the 0° mark. Read where the *other* side crosses the scale. Use the scale that starts at 0° on the side you lined up — most protractors have two rows of numbers!

Try it: Quick check

In $\angle XYZ$, which point is the **vertex**?

Work it on paper before revealing the answer.

Answer. Answer: Y

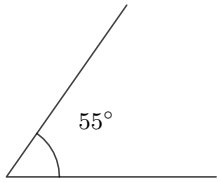
4.2 Classifying Angles

Five kinds of angles

- **Acute:** greater than 0° and less than 90°.
- **Right:** exactly 90° (marked with a small square).

- **Obtuse:** greater than 90° and less than 180°.
- **Straight:** exactly 180° (a straight line).
- **Reflex:** greater than 180° and less than 360°.

An angle measures a turn.



Remember. A right angle is your reference. If an angle looks "smaller than the corner of a page," it is acute; if it looks "more open than a corner," it is obtuse.

Try it: Quick check

Name the two rays (sides) that form $\angle RST$.

4.3 Classifying Angles

Work it on paper before revealing the answer.

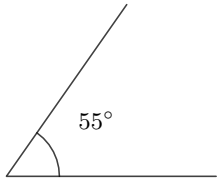
Answer. Answer: $\overrightarrow{SR}$ and $\overrightarrow{ST}$

4.4 Adjacent Angles & the Angle Addition Postulate

Adjacent angles

Two angles are **adjacent** when they share a common vertex and a common side, but do *not* overlap (no interior points in common). Think of two angles sitting side by side.

An angle measures a turn.



Angle Addition Postulate

If point B is in the interior of $\angle AOC$, then the two small angles add up to the big angle:

$$m\angle AOB + m\angle BOC = m\angle AOC.$$

Adding adjacent angles

Ray $\overrightarrow{OB}$ lies inside $\angle AOC$. If $m\angle AOB = 35°$ and $m\angle BOC = 40°$, then

$$m\angle AOC = 35° + 40° = 75°.$$

Tip. The Angle Addition Postulate works both ways: you can add two pieces to get the whole, or subtract a known piece from the whole to find the missing piece.

Try it: Quick check

An angle that measures 30° is called _____.

Work it on paper before revealing the answer.

Answer. Answer: acute

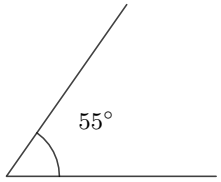
4.5 Angle Bisectors

What is a bisector?

A ray that **bisects** an angle divides it into **two congruent (equal) angles**. If $\overrightarrow{OB}$ bisects $\angle AOC$, then

$$m\angle AOB = m\angle BOC = \tfrac{1}{2}\, m\angle AOC.$$

An angle measures a turn.



Using a bisector

$\overrightarrow{OB}$ bisects $\angle AOC$, and $m\angle AOC = 80°$. Then each half is

$$m\angle AOB = m\angle BOC = \tfrac{1}{2}(80°) = 40°.$$

Remember. "Bisect" means "cut into two equal parts." The two halves are always *congruent*, so you can set them equal to each other when solving with algebra.

Try it: Quick check

A right angle measures exactly _____ degrees.

Work it on paper before revealing the answer.

Answer. Answer: 90

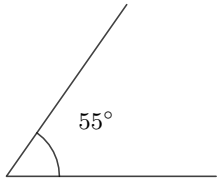
4.6 Complementary & Supplementary Angles

Two special sums

- **Complementary** angles have measures that add to **90°**.
- **Supplementary** angles have measures that add to **180°**.

The angles do *not* have to be next to each other — only their measures matter.

An angle measures a turn.



Finding a complement and a supplement

The **complement** of 30° is $90° - 30° = 60°$.

The **supplement** of 30° is $180° - 30° = 150°$.

Together the two angles above form a right angle, so they are complementary.

Memory trick. **C** comes before **S** in the alphabet, and 90 comes before 180: **C**omplementary = 90°, **S**upplementary = 180°. (Or: "**C**orner" is 90°, "**S**traight" is 180°.)

Try it: Quick check

Classify an angle that measures 178°.

Work it on paper before revealing the answer.

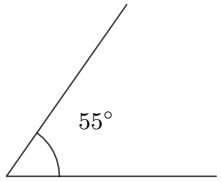
Answer. **Answer:** obtuse

4.7 Linear Pairs & Vertical Angles

Linear pairs

When two angles are **adjacent** and their outer sides form a **straight line**, they make a **linear pair**. The angles of a linear pair are always **supplementary** (they add to 180°).

An angle measures a turn.



A linear pair

$\angle 1$ and $\angle 2$ below form a linear pair, so $m\angle 1 + m\angle 2 = 180°$. If $m\angle 1 = 110°$, then $m\angle 2 = 180° - 110° = 70°$.

Vertical angles

When two straight lines cross, they form two pairs of **vertical angles** — the angles "across" from each other. **Vertical angles are always congruent (equal).**

Vertical angles are equal

Two lines cross at O. Angles $\angle 1$ and $\angle 3$ are vertical (so $m\angle 1 = m\angle 3$), and $\angle 2$ and $\angle 4$ are vertical (so $m\angle 2 = m\angle 4$). If $m\angle 1 = 65°$, then $m\angle 3 = 65°$, while $\angle 2$ and $\angle 4$ each measure $180° - 65° = 115°$.

Key facts. Linear pair ⇒ angles are *supplementary* (180°). Vertical angles ⇒ angles are *congruent* (equal). These two rules solve most crossing-line puzzles!

Try it: Quick check

Classify an angle that measures 180°.

Work it on paper before revealing the answer.

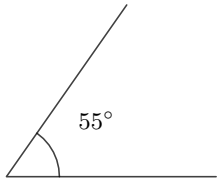
Answer. Answer: straight

4.8 Finding Unknown Angles with Algebra

Set up an equation

When an angle relationship gives you a total (90°, 180°, or a bisected angle), write an equation, then solve for the variable. Finally, *substitute back* to find each actual angle measure.

An angle measures a turn.



Complementary angles with algebra

Two complementary angles measure x and $2x$. Find each angle.

$$\begin{aligned} x + 2x &= 90 \\ 3x &= 90 \\ x &= 30 \end{aligned}$$

So the angles are $x = 30°$ and $2x = 60°$. **Check:** $30° + 60° = 90°$. ✓

Vertical angles with algebra

Two vertical angles measure $(3x + 10)°$ and $(x + 50)°$. Because vertical angles are congruent, set them equal:

$$\begin{aligned} 3x + 10 &= x + 50 \\ 2x &= 40 \\ x &= 20 \end{aligned}$$

Each angle measures $3(20) + 10 = 70°$ (and $20 + 50 = 70°$). ✓

Always check! After solving, plug your answer back in. Complementary angles should total 90°, supplementary and linear pairs should total 180°, and vertical angles should come out equal.

Try it: Quick check

Classify an angle that measures 300°.

Work it on paper before revealing the answer.

Answer. Answer: reflex

4.9 Going Deeper: Advanced Angle Ideas

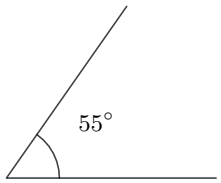
Angle chasing: work one step at a time

Angle chasing means finding an unknown angle by filling in *other* angles one at a time, using the rules you already know, until the one you want is forced. At every step, ask which single rule turns something you know into something new:

- angles on a straight line add to 180° (linear pair);
- angles around a point add to 360°;
- vertical angles are equal;
- a bisector splits an angle into two equal halves.

Mark each angle on the figure as soon as you find it — one known angle usually unlocks the next.

An angle measures a turn.



Two counting rules that act as constraints

These two facts are the "budget" every angle problem must respect:

$$\text{along a line: } \sum \text{angles} = 180^\circ, \qquad \text{around a point: } \sum \text{angles} = 360^\circ.$$

If three rays leave a single point O and go all the way around, the three angles between neighboring rays must total 360°. If several angles sit in a row on one side of a straight line, they must total 180°. Turning a picture into one of these equations is the whole game.

Angle chasing around a point

Four rays leave point O. Going around, the angles between neighbors are 90°, x, 130°, and $2x$. Find x and the two unknown angles.

Everything around a point must add to 360°:

$$\begin{aligned} 90 + x + 130 + 2x &= 360 \\ 3x + 220 &= 360 \\ 3x &= 140 \\ x &= \tfrac{140}{3} \approx 46.7. \end{aligned}$$

So $x \approx 46.7°$ and $2x \approx 93.3°$. **Check:** $90 + 46.7 + 130 + 93.3 = 360°$. ✓

The Exterior Angle Theorem

Extend one side of a triangle to make an **exterior angle**. That exterior angle equals the **sum of the two remote (non-adjacent) interior angles**:

$$m\angle \text{ext} = m\angle A + m\angle B.$$

This follows from two facts working together: the three interior angles add to 180°, and the exterior angle forms a linear pair (180°) with the interior angle beside it. Subtracting the shared interior angle from both leaves exactly the two remote angles.

Using the exterior angle theorem

In the triangle below, the two remote interior angles are 50° and 65°, and $\angle 4$ is an exterior angle. Instead of finding the third interior angle first, add the remote angles directly:

$$m\angle 4 = 50° + 65° = 115°.$$

Check the long way: the interior angle beside $\angle 4$ is $180 - 115 = 65°$, and $50 + 65 + 65 = 180°$. ✓

Bisectors and the angle between them

Angle bisectors set up equal pieces you can chain together. A useful pattern: if two rays bisect neighboring angles that sit on a straight line, the angle *between the two bisectors* is always 90°. Reason: the four half-angles come in two equal pairs, and all four together fill the 180° line, so the middle two halves total exactly half of 180°. Whenever a problem mentions a bisector, replace the whole angle by "two equal halves" and the algebra usually falls out.

A bisector condition with algebra

$\overrightarrow{OB}$ bisects $\angle AOC$. The two halves are given as $m\angle AOB = (4x - 5)°$ and $m\angle BOC =$

$(2x + 15)°$. Because a bisector makes the halves equal, set them equal:

$$\begin{aligned} 4x - 5 &= 2x + 15 \\ 2x &= 20 \\ x &= 10. \end{aligned}$$

Each half is $4(10) - 5 = 35°$, so the whole angle is $m\angle AOC = 70°$. ✓

Clock hands as angles

A clock face is a circle of 360° split into 12 hours, so each hour mark is $360/12 = 30°$ apart. The two hands move at different, *steady* speeds:

$$\text{minute hand: } \tfrac{360°}{60 \text{ min}} = 6° \text{ per minute}, \qquad \text{hour hand: } \tfrac{30°}{60 \text{ min}} = 0.5° \text{ per minute}.$$

The minute hand gains on the hour hand at $6 - 0.5 = 5.5°$ per minute. To find the angle between the hands at a given time, compute each hand's position from the top (12) and subtract. If the gap is more than 180°, use 360° minus that gap for the smaller angle.

The angle at 3:40, and when the hands overlap

Part 1 — angle at 3:40. Measure each hand clockwise from the 12:

$$\begin{aligned} \text{minute hand} &= 40 \times 6 = 240°, \\ \text{hour hand} &= 3 \times 30 + 40 \times 0.5 = 90 + 20 = 110°. \end{aligned}$$

The gap is $240 - 110 = 130°$, which is already less than 180°, so the angle between the hands is **130°**.

Part 2 — when do the hands overlap after 12:00? At 12:00 both hands sit at 0°. The minute hand closes the gap at 5.5° per minute, and it must make up a full 360° to lap the hour hand and line up again:

$$t = \frac{360}{5.5} = \frac{360}{11/2} = \frac{720}{11} \approx 65.45 \text{ minutes}.$$

So the hands first overlap again at about $1{:}05\frac{5}{11}$. In general, consecutive overlaps are exactly $\dfrac{720}{11} = \dfrac{360}{5.5}$ minutes apart, which is why the hands line up only 11 times every 12 hours — not 12.

An angle condition that leads to a quadratic

An angle measures x degrees. Its *supplement* is numerically equal to the *square* of its complement divided by 10. Find x.

The complement is $(90 - x)$ and the supplement is $(180 - x)$, so

$$180 - x = \frac{(90 - x)^2}{10}.$$

Multiply through by 10 and expand:

$$\begin{aligned} 1800 - 10x &= (90 - x)^2 \\ 1800 - 10x &= 8100 - 180x + x^2 \\ 0 &= x^2 - 170x + 6300. \end{aligned}$$

Factor (or use the quadratic formula): $x^2 - 170x + 6300 = (x - 50)(x - 120)$, so

$$x = 50 \qquad \text{or} \qquad x = 120.$$

Reject the extra root. A complement only exists for an angle *less than* $90°$, and $x = 120$ has no complement (and a negative $90 - x$). So we discard $x = 120$ and keep $\boxed{x = 50°}$. **Check:** complement $= 40$, supplement $= 130$, and $40^2/10 = 1600/10 = 130$. ✓

Advanced checklist. (1) Chase angles one rule at a time and mark each result on the figure. (2) Treat "on a line $= 180°$" and "around a point $= 360°$" as equations. (3) An exterior angle equals the two remote interior angles. (4) Replace any bisected angle by two equal halves. (5) For clocks, use $6°$/min and $0.5°$/min, with overlaps every $\frac{720}{11}$ minutes. (6) When algebra gives two roots, keep only the one that makes every angle valid (complements need angles below $90°$; all measures must be positive).

Try it: Quick check

Point B is in the interior of $\angle AOC$. If $m\angle AOB = 25°$ and $m\angle BOC = 30°$, find $m\angle AOC$.

Work it on paper before revealing the answer.

Answer. Answer: $55°$

4.10 Why It Works

Vertical angles are equal: at a crossing, angle 1 and angle 2 form a linear pair, so $\angle 1 + \angle 2 = 180°$; angle 3 and angle 2 also form a linear pair, so $\angle 3 + \angle 2 = 180°$. Both $\angle 1$ and $\angle 3$ equal $180° - \angle 2$, hence each other. Two facts about straight lines, one subtraction — that is the whole proof, and it is the template for dozens of angle-chasing arguments.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line

forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

5 Parallel & Perpendicular Lines

Everything you need to name angles, find them, and work with slopes

5.1 Kinds of Lines: Parallel, Perpendicular, and Skew

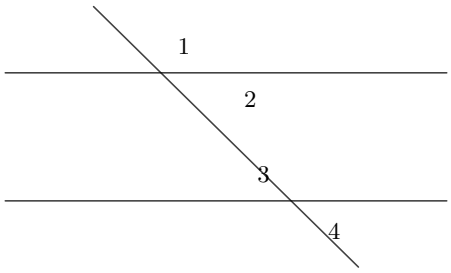
The three relationships

Parallel lines lie in the same flat surface (plane) and never meet, no matter how far you stretch them. We write $\ell \parallel m$.

Perpendicular lines cross to form a perfect square corner, a 90° angle. We write $\ell \perp m$.

Skew lines do *not* lie in the same plane. They never meet, but they are not parallel either — think of an airplane's path crossing high above a straight road.

Two parallel lines cut by a transversal.



When a transversal crosses two parallel lines it creates equal angles (corresponding, alternate interior, alternate exterior) and supplementary ones (same-side interior). The converses give ways to prove lines parallel. In coordinates, parallel lines share a slope and perpendicular slopes multiply to -1.

Transversal

A **transversal** is a line that crosses two (or more) other lines. When a transversal cuts two lines it creates **eight angles** — four at each crossing — and those angles come in special matching pairs.

Quick check: Parallel lines stay the exact same distance apart forever. Perpendicular is a *special* case of intersecting — the angle just happens to be exactly 90°.

Where it goes wrong

Using the parallel-line angle facts before knowing the lines are parallel. The facts are IF-THEN: parallel implies equal corresponding angles, and the converse needs its own justification.

Try it: Quick check

Two lines in the same plane that never meet are _____ lines.

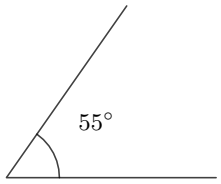
Work it on paper before revealing the answer.

Answer. Answer: parallel

5.2 The Eight Angles and Their Pairs

When a transversal t crosses lines m and n, number the angles like this.

An angle measures a turn.



Name the four pairs

Corresponding angles sit in the *same corner* at each crossing (one up, one down). Pairs: $(1,5)$, $(2,6)$, $(3,7)$, $(4,8)$.

Alternate interior angles are *between* the two lines and on *opposite* sides of the transversal. Pairs: $(3,6)$, $(4,5)$.

Alternate exterior angles are *outside* the two lines and on *opposite* sides of the transversal. Pairs: $(1,8)$, $(2,7)$.

Same-side interior angles (also called co-interior or consecutive interior) are *between* the lines and on the *same* side of the transversal. Pairs: $(3,5)$, $(4,6)$.

Memory helper: *Interior* means *inside* (between the two lines: angles 3, 4, 5, 6). *Exterior* means *outside* (angles 1, 2, 7, 8). *Alternate* always means *opposite sides* of the transversal; *same-side* means the *same* side.

Naming a pair

Look at the diagram above. What is the relationship between angle 4 and angle 5?

Both angles are *between* lines m and n (interior), and angle 4 is on the right of t while angle 5 is on the left (opposite sides). **They are alternate interior angles.**

Try it: Quick check

Two lines that cross to form a 90° angle are _____ lines.

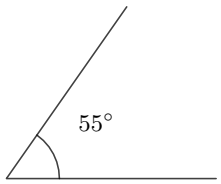
Work it on paper before revealing the answer.

Answer. Answer: perpendicular

5.3 Angle Relationships When the Lines Are Parallel

Everything above works for any two lines. The magic happens when the two lines are *parallel*.

An angle measures a turn.



The parallel-line angle rules

If two **parallel** lines are cut by a transversal, then:

- **Corresponding angles are congruent** (equal).
- **Alternate interior angles are congruent.**
- **Alternate exterior angles are congruent.**
- **Same-side interior angles are supplementary** (they add to 180°).

Also, **vertical angles** (the ones directly across an X) are always congruent, and a straight line gives a **linear pair** that adds to 180°.

One angle unlocks them all

In the figure, $m \parallel n$ and angle $1 = 70°$. Find every other angle.

Angle $1 = 70°$. Its *vertical* angle is 4, so angle $4 = 70°$. Angles 2 and 3 form linear pairs with angle 1, so each is $180° - 70° = 110°$.

Now jump to the bottom crossing using the parallel rules: angle 5 corresponds to angle 1, so angle $5 = 70°$. Then angle $8 = 70°$ (vertical to 5), and angles 6 and 7 are $110°$ each.

Result: angles $1, 4, 5, 8 = 70°$ and angles $2, 3, 6, 7 = 110°$.

Same-side interior are supplementary

Suppose $m \parallel n$ and one same-side interior angle measures $115°$. Its partner satisfies $115° + x = 180°$, so $x = 65°$.

Shortcut: With parallel lines, every one of the eight angles is either equal to the angle you know or is its supplement ($180°$ minus it). There are really only *two* different numbers in the whole picture!

Try it: Quick check

Two lines that are not in the same plane and never meet are _____ lines.

Work it on paper before revealing the answer.

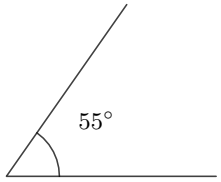
Answer. Answer: skew

5.4 Finding Unknown Angles with Algebra

Turn a relationship into an equation

Decide whether the two labeled angles are **congruent** (equal) or **supplementary** (add to $180°$). Then write the matching equation and solve for the variable. Finally, plug back in to find the actual angle measures.

An angle measures a turn.



Congruent angles (equal)

Lines $m \parallel n$. Two *corresponding* angles measure $(3x + 15)^\circ$ and $(5x - 25)^\circ$. Find x and each angle.

Corresponding angles are congruent, so set them equal:

$$\begin{aligned} 3x + 15 &= 5x - 25 \\ 40 &= 2x \\ x &= 20 \end{aligned}$$

Each angle is $3(20) + 15 = 75^\circ$ (and $5(20) - 25 = 75^\circ$ as a check). $x = 20$**, both angles** $= 75^\circ$**.**

Supplementary angles (add to 180)

Lines $m \parallel n$. Two *same-side interior* angles measure $(2x + 20)^\circ$ and $(3x - 10)^\circ$. Find x and each angle.

$$\begin{aligned} (2x + 20) + (3x - 10) &= 180 \\ 5x + 10 &= 180 \\ 5x &= 170 \\ x &= 34 \end{aligned}$$

The angles are $2(34) + 20 = 88^\circ$ and $3(34) - 10 = 92^\circ$. Check: $88 + 92 = 180$. ✓

Watch the word: "congruent" ⇒ set the expressions *equal*. "Supplementary" ⇒ set their *sum equal to* 180.

Try it: Quick check

A line that crosses two other lines is called a _____.

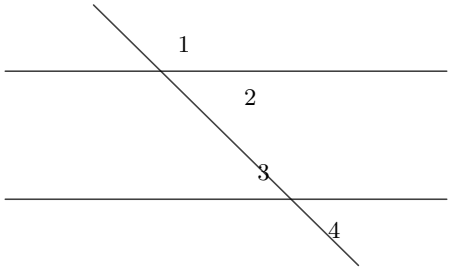
Work it on paper before revealing the answer.

Answer. Answer: transversal

5.5 Proving Lines Are Parallel (the Converses)

The rules above run *backwards* too. If you can find one of the special angle relationships, you can *conclude* the lines are parallel.

Two parallel lines cut by a transversal.



The converse statements

Two lines cut by a transversal are **parallel** if any one of these is true:

- a pair of **corresponding** angles is congruent, *or*
- a pair of **alternate interior** angles is congruent, *or*
- a pair of **alternate exterior** angles is congruent, *or*
- a pair of **same-side interior** angles is supplementary.

Are these lines parallel?

A transversal crosses two lines. A pair of alternate interior angles measure 63° and 63°. Because these alternate interior angles are *congruent*, the converse rule tells us the two lines **are parallel**.

If instead they had measured 63° and 70°, they would *not* be congruent, so the lines would **not** be parallel.

Careful: Same-side interior angles do the opposite — the lines are parallel when those two angles *add to* 180°, not when they are equal.

Try it: Quick check

The symbol $\parallel$ means _____, and the symbol $\perp$ means _____.

Work it on paper before revealing the answer.

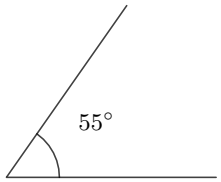
Answer. Answer: parallel; perpendicular

5.6 Perpendicular Lines and Right Angles

Perpendicular means right angles

When two lines are **perpendicular** ($\ell \perp m$), they cross to make four 90° angles. A small square drawn in the corner is the symbol for a right angle. If *one* of the four angles is 90°, all four are.

An angle measures a turn.



Splitting a right angle

A right angle is split by a ray into two smaller angles of $x°$ and 52°. Since the whole angle is 90°:

$$x + 52 = 90 \quad \Rightarrow \quad x = 38°.$$

Remember: Perpendicular is the same idea whether you are looking at drawn lines (right angles) or graphed lines (slopes that are negative reciprocals — next section).

Try it: Quick check

Angles 1 and 5: _____

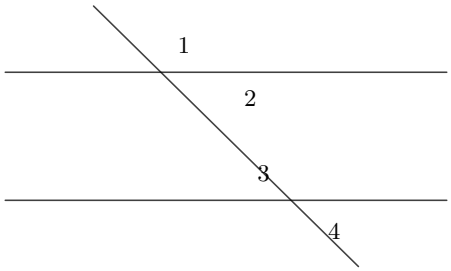
Work it on paper before revealing the answer.

Answer. Answer: corresponding

5.7 Slopes of Parallel and Perpendicular Lines

On the coordinate plane, **slope** measures steepness: $\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{change in } y}{\text{change in } x}$.

Two parallel lines cut by a transversal.



The slope rules

Parallel lines have equal slopes. If line 1 has slope m, then any line parallel to it also has slope m.

Perpendicular lines have slopes that are negative reciprocals. Flip the fraction and change the sign: if one slope is m, the perpendicular slope is $-\dfrac{1}{m}$. A quick test: two slopes are perpendicular exactly when their *product is* -1.

Read off the slopes

Line A has slope $\dfrac{3}{4}$.

- A line **parallel** to A has slope $\dfrac{3}{4}$ (same).
- A line **perpendicular** to A has slope $-\dfrac{4}{3}$ (flip $\frac{3}{4}$ to $\frac{4}{3}$, change the sign). Check: $\dfrac{3}{4}\cdot\left(-\dfrac{4}{3}\right) = -1$. ✓

Parallel, perpendicular, or neither?

Line A has slope -2 and line B has slope $\dfrac{1}{2}$. Their product is $-2 \cdot \dfrac{1}{2} = -1$, so the lines are **perpendicular**.

Special cases: A *horizontal* line has slope 0; a *vertical* line has an undefined slope. A horizontal line and a vertical line are perpendicular to each other.

Try it: Quick check

Angles 3 and 6: _____

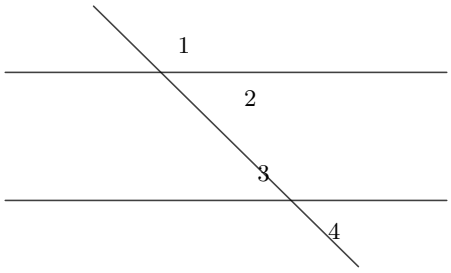
Work it on paper before revealing the answer.

Answer. Answer: alternate interior

5.8 Writing Equations of Parallel and Perpendicular Lines

We use **slope-intercept form** $y = mx + b$ and **point-slope form** $y - y_1 = m(x - x_1)$.

Two parallel lines cut by a transversal.



The three-step recipe

1. Find the slope of the given line (read m from $y = mx + b$).
2. Choose the new slope: *same* m for parallel, *negative reciprocal* for perpendicular.
3. Plug that slope and the given point into point-slope form, then simplify to $y = mx + b$.

A parallel line through a point

Write the equation of the line through $(1, 4)$ that is **parallel** to $y = 2x + 1$.
The given slope is 2; parallel means the new slope is also 2. Point-slope:

$$y - 4 = 2(x - 1) \;\Rightarrow\; y - 4 = 2x - 2 \;\Rightarrow\; y = 2x + 2.$$

Check the point: $2(1) + 2 = 4$. ✓

A perpendicular line through a point

Write the equation of the line through $(4, 3)$ that is **perpendicular** to $y = 2x + 1$.

The given slope is 2; the perpendicular slope is $-\dfrac{1}{2}$. Point-slope:

$$y - 3 = -\tfrac{1}{2}(x - 4) \;\Rightarrow\; y - 3 = -\tfrac{1}{2}x + 2 \;\Rightarrow\; y = -\tfrac{1}{2}x + 5.$$

Check the point: $-\frac{1}{2}(4) + 5 = 3$. ✓

Final tip: Always test your equation by plugging the given point back in. If it makes the equation true, your line really does pass through that point — and the slope guarantees it is parallel or perpendicular as required.

Try it: Quick check

Angles 1 and 8: _____

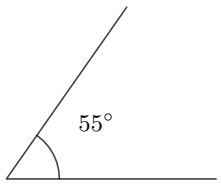
Work it on paper before revealing the answer.

Answer. Answer: alternate exterior

5.9 Going Deeper: Advanced Ideas

Up to now we have *used* the parallel-line rules. Here we see *why* they are true, chase angles through several transversals at once, and push the slope ideas into real coordinate computations: distances, feet of perpendiculars, and equations of lines.

An angle measures a turn.



Why the transversal theorems are true

The whole system rests on *one* assumed fact, the **Corresponding Angles Postulate**: if two parallel lines are cut by a transversal, corresponding angles are congruent. Everything else is a short deduction from it, using two facts you already know:

- **Vertical angles** (across an X) are congruent.
- A **linear pair** (two angles on a straight line) is supplementary.

Chain them together: a corresponding angle is congruent to the one directly across from it (vertical), which *is* the alternate interior angle — so **alternate interior angles are congruent**. Swap the vertical step for a linear pair and you get **same-side interior angles are supplementary**. No new assumptions are needed.

Worked proof: alternate interior angles are congruent

Using the eight-angle numbering from earlier, suppose $m \parallel n$. Prove $\angle 3 \cong \angle 6$.

The angle you *start* with ($\angle 3$) hops to its corresponding partner ($\angle 7$), then across the X to its vertical partner ($\angle 6$). Two congruences in a row give the result. The same two-step hop, run backwards, proves the *converse* used to show lines are parallel.

The auxiliary parallel line trick (zigzag chases)

When a path *bends* between two parallel lines, there is no single transversal to use. The fix: **draw a new line through the bend point, parallel to the two given lines.** This splits the bend angle into two pieces, and each piece is now an alternate-interior angle with one of the original lines. Add (or subtract) the pieces to finish. This "build your own parallel line" move turns an impossible-looking figure into two ordinary transversal problems.

Worked zigzag angle chase

Lines r and s are parallel. A path goes from A on r down to a corner P, then to B on s. The path makes a 35° angle with r at A and a 25° angle with s at B. Find $\angle APB$.

Draw the dashed line through P parallel to r and s. It splits $\angle APB$ into an upper piece and a lower piece.

- Upper piece $= 35^\circ$: alternate interior angles with A's angle across transversal AP.
- Lower piece $= 25^\circ$: alternate interior angles with B's angle across transversal BP.

So $\angle APB = 35^\circ + 25^\circ = \mathbf{60}^\circ$. (Whenever the bend points *away* from the region between the parallels you add the pieces; if the path stays on one side you subtract them instead.)

Distance between two parallel lines

Parallel lines never meet, so they stay a fixed distance apart — but that distance is measured *perpendicular* to the lines, not vertically. For two lines written as

$$y = mx + b_1 \qquad \text{and} \qquad y = mx + b_2 \quad \text{(same slope } m\text{)},$$

the perpendicular distance between them is

$$d = \frac{|b_2 - b_1|}{\sqrt{1 + m^2}}.$$

The numerator is the vertical gap between the intercepts; dividing by $\sqrt{1+m^2}$ tilts that vertical gap into the true perpendicular gap. (If both lines are given in the form $ax+by+c=0$ with matching a, b, the distance is $|c_2 - c_1|/\sqrt{a^2+b^2}$.)

Worked distance between parallels

Find the distance between $y = 2x + 1$ and $y = 2x + 6$.

Same slope $m = 2$, so they are parallel. Here $b_1 = 1$ and $b_2 = 6$:

$$d = \frac{|6 - 1|}{\sqrt{1 + 2^2}} = \frac{5}{\sqrt{5}} = \sqrt{5} \approx 2.24.$$

Notice this is *less* than the vertical gap of 5 — the perpendicular route between two slanted lines is always the shortest one.

The foot of a perpendicular

The **foot of the perpendicular** from a point P to a line ℓ is the single point F on ℓ where the perpendicular dropped from P lands. It is the closest point on ℓ to P, and the distance PF is the true distance from the point to the line. To find it:

1. Take the slope of ℓ and use its negative reciprocal as the slope of the perpendicular through P.
2. Write that perpendicular line in point-slope form.
3. Solve the two equations together — the solution is the foot F.

Worked foot of a perpendicular

Find the foot of the perpendicular from $P = (4, 1)$ to the line $\ell:\ y = x + 1$, and the distance from P to ℓ.

Line ℓ has slope 1, so the perpendicular has slope -1. Through $P = (4, 1)$:

$$y - 1 = -1(x - 4) \ \Rightarrow\ y = -x + 5.$$

Set the two lines equal to find where they cross:

$$\begin{aligned} x + 1 &= -x + 5 \\ 2x &= 4 \\ x &= 2, \qquad y = 2 + 1 = 3. \end{aligned}$$

The foot is $F = (2, 3)$. The distance is

$$PF = \sqrt{(4-2)^2 + (1-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \approx 2.83.$$

No other point on ℓ is closer to P than F.

Why the slope criteria are consistent: a line with slope $\frac{b}{a}$ points in direction (a, b). Rotating that direction a quarter turn ($90°$) sends $(a, b) \mapsto (-b, a)$, whose slope is $\dfrac{a}{-b} = -\dfrac{a}{b}$ — exactly the *negative reciprocal*, and their product is -1. So the algebra (negative reciprocals) and the geometry (a right angle) are the *same* statement. Likewise, two lines have *equal slopes* precisely when they point the same direction, which is precisely when they never cross — the definition of parallel.

5.10 Why It Works

Take the parallel postulate as the axiom (through a point not on a line there is exactly one parallel). Corresponding angles equal follows: if they were unequal, the second line could be rotated slightly to make them equal, producing a second parallel through the same point — impossible. Alternate interior angles equal then comes from combining corresponding angles with vertical angles, and same-side interior angles supplementary from combining with a linear pair. One axiom, three theorems.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

6 Triangles & Congruence

Classifying, angle rules, congruence shortcuts, and a first proof

6.1 Classifying Triangles

Two ways to sort a triangle

Every triangle can be described in **two** ways at once: by its **sides** and by its **angles**.

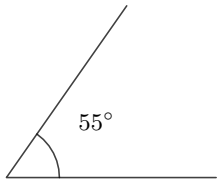
By sides:

- **Scalene** — all three sides different lengths.
- **Isosceles** — at least two sides equal.
- **Equilateral** — all three sides equal.

By angles:

- **Acute** — all three angles less than 90°.
- **Right** — exactly one 90° angle.
- **Obtuse** — one angle greater than 90°.
- **Equiangular** — all three angles equal (60° each).

An angle measures a turn.



Two triangles are congruent when they match exactly, and you never need all six parts to know it: SSS, SAS, ASA, AAS (and HL for right triangles) each suffice. Once triangles are proved congruent, every remaining pair of corresponding parts is equal — CPCTC — which is how most geometry proofs finish.

Example: name a triangle two ways

The triangle below has a square corner mark (a 90° angle) and its two short sides carry matching tick marks, meaning they are equal.

By sides: two equal sides ⇒ **isosceles**. **By angles:** one 90° angle ⇒ **right**.

So this is an **isosceles right triangle**.

Where it goes wrong

Using SSA (two sides and a non-included angle) as a congruence test. It fails: the same data can build two different triangles. Only HL, the right-angle case of SSA, works.

Tip: "Equilateral" and "equiangular" are partners — if all three sides are equal, then all three angles are equal (60°), and the other way around too.

Try it: Quick check

Classify by its **sides**: a triangle with side lengths 6, 6, 6.

Work it on paper before revealing the answer.

Answer. Answer: Equilateral.

6.2 The Triangle Angle-Sum Theorem

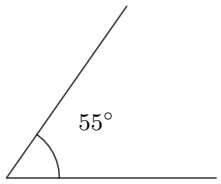
The angles always add to $180°$

In **any** triangle, the three interior angles add up to exactly $180°$.

$$\angle A + \angle B + \angle C = 180°$$

If you know two angles, subtract their sum from $180°$ to find the third.

An angle measures a turn.



Example: find the missing angle

Add the two known angles: $65° + 40° = 105°$. Then $x = 180° - 105° = \boxed{75°}$.

Tip: A triangle can have *at most one* right or obtuse angle. Two right angles alone would already use up $180°$, leaving nothing for the third!

Try it: Quick check

Classify by its **sides**: a triangle with side lengths 5, 8, 11.

Work it on paper before revealing the answer.

Answer. Answer: Scalene.

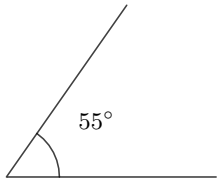
6.3 The Exterior Angle Theorem

An outside angle equals two inside angles

If you extend one side of a triangle, you create an **exterior angle**. That exterior angle equals the **sum of the two remote (far-away) interior angles**.

$$\text{exterior angle} = \text{remote angle}_1 + \text{remote angle}_2$$

An angle measures a turn.



Example: use the exterior angle

Side AB is extended to point D. The two remote interior angles are $A = 70°$ and $C = 50°$.

The exterior angle $y = 70° + 50° = \boxed{120°}$.

Tip: The exterior angle and its neighbor interior angle always add to 180° (a straight line). That is a handy way to check your work.

Try it: Quick check

Classify by its **sides**: a triangle with side lengths 9, 9, 4.

Work it on paper before revealing the answer.

Answer. Answer: Isosceles.

6.4 Congruent Figures and Corresponding Parts

Same shape, same size

Two figures are **congruent** (≅) if they have exactly the same shape and size — one could be slid, turned, or flipped onto the other perfectly. When we write $\triangle ABC \cong \triangle DEF$, the **order**

of the letters tells us which parts match:

$$A \leftrightarrow D, \quad B \leftrightarrow E, \quad C \leftrightarrow F.$$

These matching pieces are called **corresponding parts**.

Example: read off corresponding parts

Given $\triangle ABC \cong \triangle DEF$, the matching sides and angles are:

$$\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \overline{AC} \cong \overline{DF},$$

$$\angle A \cong \angle D, \quad \angle B \cong \angle E, \quad \angle C \cong \angle F.$$

So if $BC = 9$, then $EF = 9$ too, because they correspond.

CPCTC stands for "**C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent." Once you prove two triangles congruent, *every* pair of matching parts is automatically congruent.

Try it: Quick check

Classify by its **angles**: a triangle with angles 40°, 60°, 80°.

Work it on paper before revealing the answer.

Answer. Answer: Acute (all angles $< 90°$).

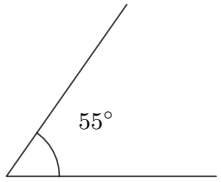
6.5 Triangle Congruence Shortcuts

Five ways to prove congruence

You do not need to check all six parts (3 sides + 3 angles). Any **one** of these shortcuts is enough:

- **SSS** — three pairs of sides.
- **SAS** — two sides and the *included* angle (the angle *between* them).
- **ASA** — two angles and the *included* side.
- **AAS** — two angles and a *non-included* side.
- **HL** — (right triangles only) the hypotenuse and one leg.

An angle measures a turn.



Example: a congruent pair marked with SAS

Matching tick marks show equal sides; the arc shows the equal included angle. Two sides and the angle *between* them match, so this is **SAS**.

Therefore $\triangle ABC \cong \triangle DEF$ by **SAS**.

The two that do NOT work: SSA (two sides and a non-included angle) is unreliable — it can produce two different triangles (the "ambiguous case"). **AAA** (three angles) only guarantees the *same shape*, not the same *size* — think of a small triangle and a giant one with equal angles.

Try it: Quick check

Classify by its **angles**: a triangle with angles 90°, 30°, 60°.

Work it on paper before revealing the answer.

Answer. Answer: Right.

6.6 Isosceles and Equilateral Triangles

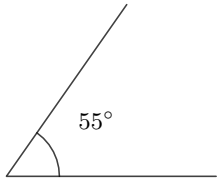
Base Angles Theorem

In an **isosceles** triangle, the two sides that are equal are the *legs*, and the third side is the *base*. The theorem says:

The angles opposite the equal sides (the base angles) are congruent.

And the reverse is true too: if two angles are equal, the sides opposite them are equal.

An angle measures a turn.



Example: find the vertex angle

The two legs are marked equal, so the two base angles are equal (50° each).

Vertex angle $= 180° - (50° + 50°) = \boxed{80°}$.

Equilateral facts: every equilateral triangle is also isosceles, is equiangular, and has three 60° angles. Its perimeter is simply 3× (one side).

Try it: Quick check

Two angles of a triangle are 55° and 65°. Find the third angle.

Work it on paper before revealing the answer.

Answer. Answer: $180° - 120° = 60°$.

6.7 The Triangle Inequality

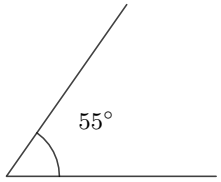
Can these lengths make a triangle?

Three lengths form a triangle only if the two shorter ones can "reach across." The rule:

The sum of any two sides must be greater than the third side.

A quick check: add the two **smallest** sides. If that sum beats the largest side, you have a triangle.

An angle measures a turn.



Example: test some lengths

Lengths $5, 7, 9$**:** smallest two are $5 + 7 = 12$, and $12 > 9$. **Yes**, a triangle.

Lengths $2, 3, 6$**:** smallest two are $2 + 3 = 5$, but $5 < 6$. **No** triangle — the short sides cannot reach.

Range for a third side: if two sides are 8 and 5, the third side s must satisfy

$$8 - 5 < s < 8 + 5, \quad \text{that is} \quad 3 < s < 13.$$

Tip: The third side is always *between* the difference and the sum of the other two: $|a - b| < s < a + b$.

Try it: Quick check

Two angles of a triangle are $90°$ and $25°$. Find the third angle.

Work it on paper before revealing the answer.

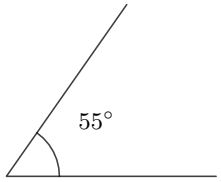
Answer. Answer: $180° - 115° = 65°$.

6.8 A First Two-Column Proof

What a proof looks like

A **two-column proof** lists **Statements** on the left and the **Reason** for each on the right. You start from what is *Given* and end at what you want to *Prove*, one logical step at a time.

An angle measures a turn.



Example: prove two triangles congruent

Given: $\overline{AB} \cong \overline{CB}$, and $\overline{BM}$ bisects $\angle ABC$ (so $\angle ABM \cong \angle CBM$). **Prove:** $\triangle ABM \cong \triangle CBM$.

Notice the pattern: **Side** (AB), **Angle** (at B), **Side** (BM) — that is SAS!

Tip: The **Reflexive Property** ($\overline{BM} \cong \overline{BM}$) is your best friend in proofs. Whenever two triangles *share* a side or an angle, that shared part is congruent to itself.

Try it: Quick check

Two angles of a triangle are 33° and 33°. Find the third angle.

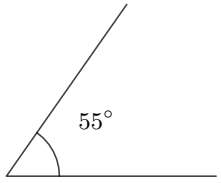
Work it on paper before revealing the answer.

Answer. Answer: $180° - 66° = 114°$.

6.9 Going Deeper: Advanced Triangle Ideas

You have met the everyday rules of triangles. Now we push further into the ideas that professional geometers actually use — special points inside a triangle, a clever "inner" triangle, a theorem that slices sides in a neat ratio, proofs that add a line of their own, and the full story of which lengths can form a triangle. Sketch along and enjoy the extra depth.

An angle measures a turn.



The four centers of a triangle

Every triangle has four famous "centers," each built from a different set of special lines. All three lines of a kind always meet at a single point (they are *concurrent*).

- **Centroid** — where the three **medians** meet (a median joins a vertex to the *midpoint* of the opposite side). It is the triangle's balance point, and it always sits *inside*.
- **Incenter** — where the three **angle bisectors** meet. It is the center of the inscribed circle that just touches all three sides, and it is always *inside*.
- **Circumcenter** — where the three **perpendicular bisectors** of the sides meet. It is the center of the circle through all three vertices; it may fall outside for an obtuse triangle.
- **Orthocenter** — where the three **altitudes** (perpendicular heights from each vertex) meet. It too can land outside an obtuse triangle.

The centroid's 2:1 rule: the centroid divides each median so that the piece from the *vertex* is **twice** as long as the piece to the *midpoint*. So the vertex-to-centroid part is $\frac{2}{3}$ of the whole median, and the centroid-to-midpoint part is $\frac{1}{3}$.

Example: use the centroid's 2:1 ratio

In $\triangle ABC$ below, M is the midpoint of $\overline{BC}$, so $\overline{AM}$ is a median. The three medians meet at the centroid G. Suppose the whole median $AM = 18$. Find AG and GM.

The centroid splits the median in the ratio $2 : 1$ from the vertex. So the vertex part is $\frac{2}{3}$ of 18 and the midpoint part is $\frac{1}{3}$ of 18:

$$AG = \tfrac{2}{3} \times 18 = \boxed{12}, \qquad GM = \tfrac{1}{3} \times 18 = \boxed{6}.$$

Check: $12 + 6 = 18$, and 12 is indeed twice 6.

The medial triangle and the Midsegment Theorem

Join the **midpoints** of the three sides of a triangle and you get a smaller triangle inside, called the **medial triangle**. Each of its sides is a **midsegment** of the original. The **Midsegment Theorem** says a midsegment is:

- **parallel** to the third side of the triangle, and
- exactly **half** as long as that side.

Because all three sides are halved, the medial triangle has the same shape as the original, half the perimeter, and one quarter of the area.

Example: a midsegment length

P, Q, R are the midpoints of the sides of $\triangle ABC$. Side $\overline{BC}$ has length 10. Find the midsegment $\overline{PQ}$ that joins the midpoints of $\overline{AB}$ and $\overline{AC}$.

$\overline{PQ}$ joins the midpoints of the two sides meeting at A, so it is the midsegment parallel to $\overline{BC}$. By the Midsegment Theorem it is half of BC:

$$PQ = \tfrac{1}{2} \times 10 = \boxed{5}, \quad \text{and} \quad \overline{PQ} \parallel \overline{BC}.$$

The Angle Bisector Theorem

Draw the bisector of one angle of a triangle down to the opposite side. It cuts that side into two pieces whose lengths are in the **same ratio as the two sides** of the angle. In $\triangle ABC$, if the bisector from A meets $\overline{BC}$ at D, then

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

The bisector divides the far side in proportion to the two nearby sides.

Example: split a side with the Angle Bisector Theorem

In $\triangle ABC$, the bisector from A meets $\overline{BC}$ at D. The two sides are $AB = 8$ and $AC = 6$, and the whole base is $BC = 14$. Find BD and DC.

By the theorem, $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{8}{6} = \dfrac{4}{3}$. So the base of 14 splits into $4 + 3 = 7$ equal shares, each of length $14/7 = 2$:

$$BD = 4 \times 2 = \boxed{8}, \qquad DC = 3 \times 2 = \boxed{6}.$$

Check: $8 + 6 = 14$, and $8 : 6 = 4 : 3$ as required.

Congruence proofs with an auxiliary line

Sometimes a proof seems stuck because there are not yet two triangles to compare. The trick is to **add your own line** — called an **auxiliary line** — that creates the triangles you need. Popular choices are an *angle bisector*, a *median*, an *altitude*, or a segment joining two existing points. Draw it, justify why it exists, and the congruence shortcuts (SSS, SAS, ASA, AAS, HL) do the rest.

Example: prove the base angles of an isosceles triangle are equal

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$. **Prove:** $\angle B \cong \angle C$.

Auxiliary line: draw the bisector of $\angle A$, meeting $\overline{BC}$ at D. This single extra segment splits

the triangle into two triangles we can compare.

The auxiliary bisector turned one triangle into two matching ones, and CPCTC finished the job.

The full triangle-inequality range

For three sides a, b, c *all three* sums must beat the remaining side:

$$a+b>c, \qquad b+c>a, \qquad a+c>b.$$

Turned around, if two sides are a and b, the third side s is squeezed between their **difference** and their **sum**:

$$|a-b| \;<\; s \;<\; a+b.$$

Both ends are *strict* — landing exactly on $|a-b|$ or $a+b$ would flatten the triangle into a straight line.

Example: count the integer third sides

Two sides of a triangle are 8 and 5. How many **whole-number** lengths are possible for the third side s?

Apply the range with $a=8$, $b=5$:

$$|8-5|<s<8+5 \quad\Longrightarrow\quad 3<s<13.$$

Since the ends are strict, s cannot equal 3 or 13. The allowed integers are

$$s\in\{4,5,6,7,8,9,10,11,12\}.$$

Counting them (or using $12-4+1$) gives $\boxed{9}$ possible integer lengths.

Exterior angles in an angle chase: in a longer figure, keep reusing "exterior angle = sum of the two remote interior angles." Each time you meet an extended side, you can jump straight to a new angle without first finding the interior one — a fast shortcut when hunting an unknown angle several steps away.

6.10 Why It Works

Why SAS is enough: place the two triangles so the equal angles coincide and one pair of equal sides lies along the same ray. The second pair of equal sides must then lie along the other ray of the angle, ending at the same point. Three vertices now coincide, so the triangles coincide. SSA fails because the second side's length pins its endpoint to a circle, and a circle can meet a ray twice — two triangles, one set of data.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

7 Similarity

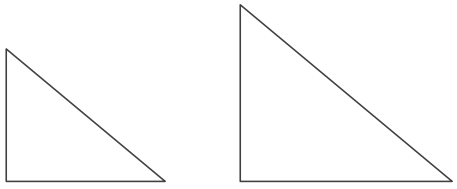
Ratios, similar polygons & triangles, indirect measurement, midsegments & scaling

7.1 Ratios, Proportions, and Scale Factor

The Language of Comparison

- A **ratio** compares two quantities by division. The ratio of 6 to 4 can be written $6 : 4$, "6 to 4", or $\frac{6}{4} = \frac{3}{2}$.
- A **proportion** is an equation saying two ratios are equal, like $\frac{a}{b} = \frac{c}{d}$.
- To solve a proportion, use **cross-multiplication**: if $\frac{a}{b} = \frac{c}{d}$, then $a \cdot d = b \cdot c$.
- The **scale factor** is the number you multiply every length of a small figure by to get the matching length of the larger figure. It is just the ratio of a pair of corresponding sides.

Similar triangles: same shape, scaled sides.



Similar figures have the same shape: equal angles and proportional sides. For triangles, AA is enough (two equal angles force the third). Similarity gives the proportions behind scale drawings, indirect measurement (shadows, mirrors), and the side-splitter theorem for a line parallel to one side.

Solving a proportion

Solve $\dfrac{x}{6} = \dfrac{10}{4}$.

$$\frac{x}{6} = \frac{10}{4} \implies 4x = 6 \cdot 10 = 60 \implies x = 15.$$

Check: $\frac{15}{6} = 2.5$ and $\frac{10}{4} = 2.5$. ✓

Where it goes wrong

Matching sides in the wrong order when writing the proportion. Write the correspondence of vertices first ($\triangle ABC \sim \triangle DEF$) and pair sides by position in the names.

Tip: A scale factor greater than 1 means an *enlargement*; a scale factor between 0 and 1 means a *reduction*. The scale factor is always (new length) ÷ (old length).

Try it: Quick check

Write the ratio 8 : 12 in simplest form.

Work it on paper before revealing the answer.

Answer. Answer: 2 : 3

7.2 Similar Polygons

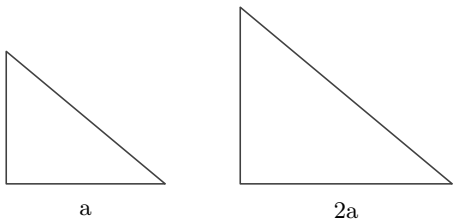
What "Similar" Means

Two polygons are **similar** ($\sim$) when they have the same shape but not necessarily the same size. This requires *both*:

- All pairs of **corresponding angles** are congruent (equal).
- All pairs of **corresponding sides** are proportional (same scale factor).

A **similarity statement** lists vertices in matching order. Writing $\triangle ABC \sim \triangle DEF$ tells you $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$, so $\angle A \cong \angle D$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$.

Similar triangles: same shape, scaled sides.



Reading a similarity statement

Here $\triangle ABC \sim \triangle DEF$. The small triangle has sides 3, 4, 5; the large one 6, 8, 10. Each large side is exactly twice its matching small side, so the scale factor from $\triangle ABC$ to $\triangle DEF$ is 2.

Corresponding sides: $\frac{AB}{DE} = \frac{4}{8} = \frac{1}{2}$, $\frac{BC}{EF} = \frac{5}{10} = \frac{1}{2}$, $\frac{AC}{DF} = \frac{3}{6} = \frac{1}{2}$. All equal, so the triangles are similar.

Tip: Match corresponding vertices *in order*. $\triangle ABC \sim \triangle DEF$ is not the same claim as $\triangle ABC \sim \triangle EFD$. Line up the letters, then line up the sides.

Try it: Quick check

Solve the proportion for x: $\frac{x}{5} = \frac{12}{10}$.

Work it on paper before revealing the answer.

Answer. Answer: $x = 6$

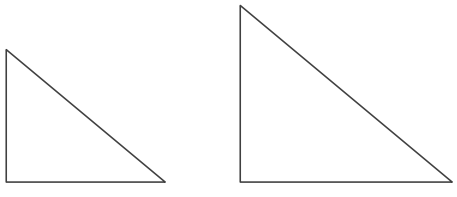
7.3 Triangle Similarity Criteria: AA, SSS∼, SAS∼

Three Shortcuts for Triangles

You do not need to check everything. For triangles, any one of these guarantees similarity:

- **AA** (Angle-Angle): two pairs of corresponding angles are congruent. (The third pair is then automatic.)
- **SSS∼** (Side-Side-Side): all three pairs of corresponding sides are proportional.
- **SAS∼** (Side-Angle-Side): two pairs of corresponding sides are proportional *and* the included angles (between those sides) are congruent.

Similar triangles: same shape, scaled sides.



Using AA

The two triangles share a pair of congruent marked angles, and both have a right angle. That is two pairs of congruent angles, so by **AA** the triangles are similar.

Using SAS∼

Two sides of a big triangle are 10 and 6; two sides of a small triangle are 5 and 3, and the included angles are equal. Since $\frac{10}{5} = \frac{6}{3} = 2$ and the angle between them matches, the triangles are similar by **SAS∼**.

Tip: For SAS∼ the equal angle must be *between* the two proportional sides. An angle in the wrong spot does not count.

Try it: Quick check

Solve the proportion for x: $\frac{3}{4} = \frac{9}{x}$.

Work it on paper before revealing the answer.

Answer. Answer: $x = 12$

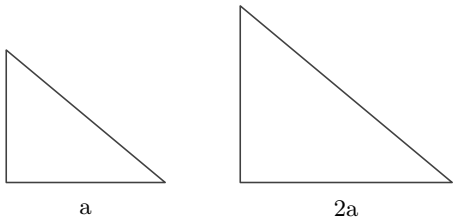
7.4 Finding Missing Sides with Similar Triangles

Set Up a Proportion, Then Solve

When two triangles are similar, corresponding sides are proportional. To find a missing length:

1. Write the similarity statement so vertices match.
2. Build a proportion using one *known* pair of sides and the pair that contains the unknown.
3. Cross-multiply and solve.

Similar triangles: same shape, scaled sides.



Solving for a missing side

Given $\triangle ABC \sim \triangle DEF$ with $AB = 4$, $DE = 6$, and $BC = 5$, find EF.

$$\frac{AB}{DE} = \frac{BC}{EF} \implies \frac{4}{6} = \frac{5}{x} \implies 4x = 30 \implies x = 7.5.$$

So $EF = 7.5$.

Tip: Keep the same figure "on top" in every ratio. If small-triangle sides are numerators on one side of the equation, keep them numerators on the other side too.

Try it: Quick check

A small figure has a side of length 4. The matching side on the larger, similar figure is 12. What is the scale factor from small to large?

Work it on paper before revealing the answer.

Answer. Answer: 3

7.5 Indirect Measurement (Shadows & Mirrors)

Measuring the Unreachable

You can find a height you cannot reach by using similar triangles.

- **Shadow method:** At the same time of day, an object and its shadow form a triangle similar to a nearby object and its shadow (the sun's rays hit at the same angle). So $\frac{\text{height}_1}{\text{shadow}_1} = \frac{\text{height}_2}{\text{shadow}_2}$.
- **Mirror method:** Place a mirror on the ground. The angle of sight into the mirror equals the angle back out, creating two similar triangles.

The shadow of a tree

A 6-ft person casts a 4-ft shadow. At the same moment a tree casts a 20-ft shadow. How tall is the tree? The two triangles (person + shadow, tree + shadow) are similar by AA, so:

$$\frac{6}{4} = \frac{h}{20} \implies 4h = 120 \implies h = 30 \text{ ft}.$$

Tip: Line up the ratios the same way every time — height over shadow on *both* sides. Matching units matters too; convert first if one length is in inches and another in feet.

Try it: Quick check

Given $\triangle ABC \sim \triangle DEF$, which angle is congruent to $\angle B$?

Work it on paper before revealing the answer.

Answer. Answer: $\angle E$

7.6 The Triangle Midsegment Theorem

The Midsegment

A **midsegment** of a triangle joins the midpoints of two sides. The **Triangle Midsegment Theorem** says a midsegment is:

- **parallel** to the third side, and
- exactly **half** the length of that third side.

This creates a small triangle similar to the whole triangle with scale factor $\frac{1}{2}$.

Finding a midsegment length

M and N are midpoints of $\overline{AB}$ and $\overline{AC}$. Since $\overline{MN}$ is a midsegment, $MN = \frac{1}{2}\,BC$. If $BC = 14$, then $MN = 7$.

Tip: The midsegment is *half* the base — so the base is *twice* the midsegment. Read the problem carefully to see which one you are given.

Try it: Quick check

Given $\triangle ABC \sim \triangle DEF$, which side of $\triangle DEF$ corresponds to $\overline{AB}$?

Work it on paper before revealing the answer.

Answer. Answer: $\overline{DE}$

7.7 Side-Splitter & Proportional Segments

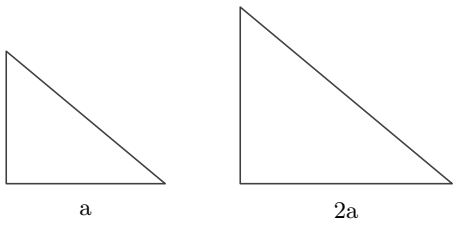
The Side-Splitter Theorem

If a line is drawn **parallel** to one side of a triangle and crosses the other two sides, it divides those two sides **proportionally**.

$$\text{If } \overline{DE} \parallel \overline{BC}, \quad \text{then } \frac{AD}{DB} = \frac{AE}{EC}.$$

This happens because the parallel line creates a smaller similar triangle (by AA).

Similar triangles: same shape, scaled sides.



Splitting the sides

In $\triangle ABC$, $\overline{DE} \parallel \overline{BC}$ with $AD = 3$, $DB = 6$, and $AE = 4$. Find EC.

$$\frac{AD}{DB} = \frac{AE}{EC} \implies \frac{3}{6} = \frac{4}{EC} \implies 3 \cdot EC = 24 \implies EC = 8.$$

Tip: The side-splitter compares the two *pieces* of each split side ($\frac{\text{top}}{\text{bottom}}$), not a piece to the whole. Keep top-with-top and bottom-with-bottom.

Try it: Quick check

Two triangles have two pairs of corresponding angles congruent. Which criterion proves they

are similar?

Work it on paper before revealing the answer.

Answer. Answer: AA

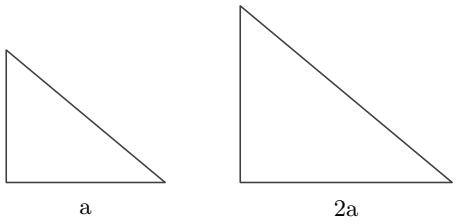
7.8 Scale Factor: Perimeter and Area

How Scaling Changes Size

If two similar figures have scale factor k (each length of the big figure is k times the matching length of the small one), then:

- The ratio of **perimeters** equals the scale factor: k.
- The ratio of **areas** equals the scale factor *squared*: k^2.

Similar triangles: same shape, scaled sides.



Perimeter and area under scaling

Two similar rectangles have scale factor $k = 3$. The small one has perimeter 12 and area 8.

Big perimeter $= 3 \times 12 = 36$. Big area $= 3^2 \times 8 = 9 \times 8 = 72$.

Tip: Perimeter grows like the scale factor; area grows like its *square*. If you triple the sides, the area becomes 9 times as big — not 3 times.

Try it: Quick check

Two triangles have all three pairs of corresponding sides proportional. Which criterion proves they are similar?

Work it on paper before revealing the answer.

Answer. Answer: SSS$\sim$

7.9 Going Deeper: Advanced Similarity

The Altitude-on-Hypotenuse Geometric-Mean Relations

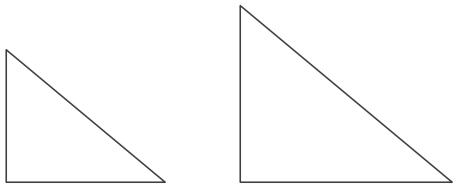
Drop the altitude from the right angle of a right triangle to the hypotenuse. It splits the big triangle into *two smaller triangles, each similar to the original and to each other* (all three share the same angles, by AA). Label the hypotenuse pieces p and q, the altitude h, and the legs a and b.

Matching corresponding sides across the similar triangles gives three "geometric mean" (mean proportional) relations:

$$h = \sqrt{p\,q}, \qquad a = \sqrt{p\,(p+q)}, \qquad b = \sqrt{q\,(p+q)}.$$

In words: the altitude is the geometric mean of the two hypotenuse pieces, and each leg is the geometric mean of its adjacent piece and the whole hypotenuse.

Similar triangles: same shape, scaled sides.



Using the geometric-mean relations

The altitude to the hypotenuse divides it into pieces $p = 4$ and $q = 9$. Find the altitude h and the shorter leg a.

$$h = \sqrt{p\,q} = \sqrt{4 \cdot 9} = \sqrt{36} = 6.$$

The whole hypotenuse is $p + q = 4 + 9 = 13$, so the leg adjacent to the piece $p = 4$ is

$$a = \sqrt{p\,(p+q)} = \sqrt{4 \cdot 13} = \sqrt{52} = 2\sqrt{13} \approx 7.2.$$

Check with the Pythagorean theorem on the small left triangle: $a^2 = h^2 + p^2 = 36 + 16 = 52$. ✓

Why the Side-Splitter Theorem Is True (A Proof)

Claim. In $\triangle ABC$, if $\overline{DE} \parallel \overline{BC}$ with D on $\overline{AB}$ and E on $\overline{AC}$, then $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Proof. Because $\overline{DE} \parallel \overline{BC}$, the parallel lines cut off equal corresponding angles: $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Triangle ADE and triangle ABC also share $\angle A$. So by **AA**,

$\triangle ADE \sim \triangle ABC$. Similar triangles have proportional corresponding sides, giving

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

Now flip each ratio to compare whole-to-part, then subtract 1 from both sides:

$$\frac{AB}{AD} = \frac{AC}{AE} \implies \frac{AB}{AD} - 1 = \frac{AC}{AE} - 1 \implies \frac{AB - AD}{AD} = \frac{AC - AE}{AE}.$$

Since $AB - AD = DB$ and $AC - AE = EC$, this is exactly $\dfrac{DB}{AD} = \dfrac{EC}{AE}$, and taking reciprocals gives $\dfrac{AD}{DB} = \dfrac{AE}{EC}$. ■

Area Scales by k^2, Volume Scales by k^3

When two figures are similar with scale factor k (every length of the large figure is k times the matching length of the small one), the effect compounds with each dimension:

- **Lengths** (sides, perimeters, radii, heights) scale by $k^1 = k$.
- **Areas** (surface areas too) scale by k^2, because area multiplies two lengths.
- **Volumes** scale by k^3, because volume multiplies three lengths.

Read backwards, this is powerful: if you *know* the area or volume ratio, take a square or cube root to recover the length scale factor. If $\dfrac{\text{Area}_1}{\text{Area}_2} = \dfrac{9}{25}$, then $k = \sqrt{9/25} = \dfrac{3}{5}$.

From volume ratio back to a length

Two similar cylinders hold 54 cm^3 and 128 cm^3. The smaller has radius 3 cm. Find the larger radius.

The volume ratio equals k^3, so

$$k^3 = \frac{128}{54} = \frac{64}{27} \implies k = \sqrt[3]{\tfrac{64}{27}} = \frac{4}{3}.$$

Radii scale by k, so the larger radius is $3 \times \dfrac{4}{3} = 4$ cm.

Nested and Overlapping Similar Triangles

Similar triangles often *share* a vertex or side, so they overlap on the page. Two habits keep them straight:

- **Redraw them separately**, pulling the small triangle out of the big one and turning it so

matching vertices sit in the same positions.

- **Use the whole, not the piece.** When a small triangle sits inside a big one along a shared side, the big triangle's side is the *entire* length, not just the leftover part. Mixing a piece with a whole is the most common error.

A shared angle plus a pair of parallel sides (or a pair of equal marked angles) is the usual AA setup that makes the overlap similar.

The Angle Bisector Theorem — via Similarity

Claim. If a ray from vertex A bisects $\angle A$ and meets $\overline{BC}$ at D, then D divides $\overline{BC}$ in the ratio of the two adjacent sides:

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Proof idea. Through C draw a line parallel to the bisector $\overline{AD}$, meeting line $\overline{BA}$ extended at a point E. Since $\overline{AD} \parallel \overline{CE}$, the side-splitter theorem in $\triangle BCE$ gives $\dfrac{BD}{DC} = \dfrac{BA}{AE}$. The parallel lines also force $\triangle ACE$ to be isosceles ($\angle AEC$ and $\angle ACE$ equal the two bisected halves of $\angle A$), so $AE = AC$. Substituting yields $\dfrac{BD}{DC} = \dfrac{AB}{AC}$. ■

Applying the Angle Bisector Theorem

In $\triangle ABC$, $\overline{AD}$ bisects $\angle A$ with $AB = 8$, $AC = 6$, and $BC = 14$. Find BD and DC. Let $BD = x$; then $DC = 14 - x$. The theorem gives

$$\frac{BD}{DC} = \frac{AB}{AC} \implies \frac{x}{14 - x} = \frac{8}{6}.$$

Cross-multiply: $6x = 8(14 - x) = 112 - 8x$, so $14x = 112$ and $x = 8$. Thus $BD = 8$ and $DC = 14 - 8 = 6$.

Indirect Measurement with Two Unknowns

Sometimes a single similar-triangle setup hides *two* unknowns — for example an unknown height *and* an unknown distance. The fix is to write **two** proportions (or one proportion plus one length equation) so you have as many equations as unknowns, then solve the system.

A classic case: an observer walks toward a tall object, and you know the two viewing distances but neither the object's height nor the observer's starting gap. Set the height as one variable and the hidden distance as another, build a proportion at each observation, and eliminate.

Two sightings, two unknowns

A student wants the height h of a flagpole. A vertical stake of height 2 m casts a shadow

whose tip lines up with the top of the pole. From the first spot the stake stands x m from the pole's base with a 3 m shadow; the student steps 4 m farther back and now the same-height stake needs a 5 m shadow to line up. Set up similar triangles (sight-line from shadow tip over stake top to pole top) at each position:

$$\text{First: } \frac{h}{x+3} = \frac{2}{3}, \qquad \text{Second: } \frac{h}{x+4+5} = \frac{2}{5}.$$

From the first equation, $3h = 2(x+3) = 2x+6$. From the second, $5h = 2(x+9) = 2x+18$. Subtract the first from the second to eliminate x:

$$5h - 3h = (2x+18) - (2x+6) \implies 2h = 12 \implies h = 6 \text{ m}.$$

Back-substitute: $3(6) = 2x+6 \Rightarrow 18 = 2x+6 \Rightarrow x = 6$ m. The flagpole is 6 m tall, and the first stake stood 6 m from its base.

Tip: Count your unknowns *before* solving. Two unknowns need two independent equations. Line every proportion up the same way — height over base-distance on both sides — and eliminate one variable by subtraction or substitution.

7.10 Why It Works

Side-splitter: if $DE \parallel BC$ in triangle ABC then triangles ADE and ABC share angle A and have equal corresponding angles at D and B (parallel lines), so they are similar by AA, giving $\frac{AD}{AB} = \frac{AE}{AC}$. Areas of similar figures scale by the square of the ratio because area is a product of two lengths, each scaled by the same factor; volumes scale by the cube for the same reason with three lengths.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

8 Right Triangles & Trig

Pythagoras, special triangles, SOH-CAH-TOA, and angles of elevation

8.1 The Pythagorean Theorem

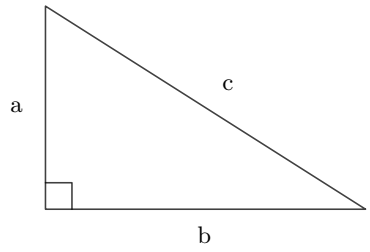
The Pythagorean Theorem

In *any* right triangle, the two shorter sides are the **legs** (a and b) and the longest side — always across from the right angle — is the **hypotenuse** (c). The theorem says:

$$a^2 + b^2 = c^2.$$

The two legs squared and added always equal the hypotenuse squared. Because c is the biggest side, it sits *alone* on its own side of the equation.

The legs and the hypotenuse.



The Pythagorean theorem $a^2+b^2=c^2$ is the engine of right-triangle work, together with the special triangles (45-45-90 and 30-60-90) whose side ratios you should know cold. The trigonometric ratios sine, cosine and tangent are the side ratios of a right triangle as functions of an angle, so an angle and one side determine everything.

Finding the hypotenuse

The legs are 6 and 8. Find the hypotenuse c.

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100, \qquad c = \sqrt{100} = 10.$$

Where it goes wrong

Putting the hypotenuse in the wrong place. It is the longest side, opposite the right angle, and it is always the c in $a^2 + b^2 = c^2$ — never a leg.

Finding a leg

The hypotenuse is 13 and one leg is 5. Find the other leg b. The hypotenuse stays alone:

$$5^2 + b^2 = 13^2 \;\Rightarrow\; 25 + b^2 = 169 \;\Rightarrow\; b^2 = 144 \;\Rightarrow\; b = 12.$$

Tip: To find a *leg*, subtract: $b = \sqrt{c^2 - a^2}$. To find the *hypotenuse*, add: $c = \sqrt{a^2 + b^2}$. If you ever get the hypotenuse smaller than a leg, you added when you should have subtracted.

Try it: Quick check

The legs of a right triangle are 6 and 8. Find the hypotenuse.

Work it on paper before revealing the answer.

Answer. Answer: $c = 10$

8.2 The Converse: Right, Acute, or Obtuse?

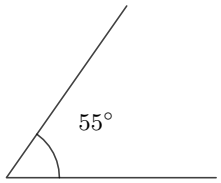
Converse of the Pythagorean Theorem

Take the three sides of a triangle and let c be the *longest*. Compare $a^2 + b^2$ with c^2:

- If $a^2 + b^2 = c^2$, the triangle is **right**.
- If $a^2 + b^2 > c^2$, the triangle is **acute** (all angles under 90°).
- If $a^2 + b^2 < c^2$, the triangle is **obtuse** (one angle over 90°).

A bigger c^2 "pulls" the longest side apart into an obtuse angle.

An angle measures a turn.



Classifying a triangle

Sides 7, 8, 12. The longest is 12, so

$$7^2 + 8^2 = 49 + 64 = 113, \qquad 12^2 = 144.$$

Since $113 < 144$, we have $a^2 + b^2 < c^2$, so the triangle is **obtuse**.

Remember: Always square the *longest* side for c^2. "Sum of squares of the two short sides" versus "square of the long side" tells the whole story.

Try it: Quick check

The legs of a right triangle are 9 and 12. Find the hypotenuse.

Work it on paper before revealing the answer.

Answer. Answer: $c = 15$

8.3 Pythagorean Triples

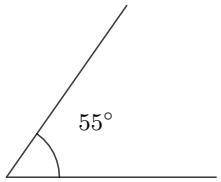
Whole-number right triangles

A **Pythagorean triple** is a set of three whole numbers that fit $a^2+b^2 = c^2$ exactly. Memorizing a few lets you skip the arithmetic. The common ones are:

$$3\text{-}4\text{-}5, \qquad 5\text{-}12\text{-}13, \qquad 8\text{-}15\text{-}17, \qquad 7\text{-}24\text{-}25.$$

Any multiple of a triple is also a triple. Multiply 3-4-5 by 2 to get 6-8-10; by 3 to get 9-12-15; by 10 to get 30-40-50.

An angle measures a turn.



Spotting a triple

The legs are 10 and 24. These are 2×5 and 2×12, so this is the 5-12-13 triple doubled. The hypotenuse is $2 \times 13 = 26$. Check: $10^2 + 24^2 = 100 + 576 = 676 = 26^2$. ✓

Tip: A triple only works when you match the right roles — the largest number is always the hypotenuse. In 8-15-17, the legs are 8 and 15 and the hypotenuse is 17.

Try it: Quick check

The hypotenuse is 13 and one leg is 5. Find the other leg.

Work it on paper before revealing the answer.

Answer. Answer: $b = 12$

8.4 Special Right Triangle: 45-45-90

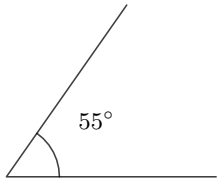
The 45-45-90 Triangle

This is an isosceles right triangle: the two legs are **equal**, and each base angle is 45°. The hypotenuse is always the leg times $\sqrt{2}$:

$$\text{leg} = x, \qquad \text{leg} = x, \qquad \text{hypotenuse} = x\sqrt{2}.$$

Going *up* from a leg, multiply by $\sqrt{2}$. Going *down* from the hypotenuse, divide by $\sqrt{2}$.

An angle measures a turn.



Legs equal, hypotenuse is leg times $\sqrt{2}$

Each leg is 5. Then the hypotenuse is $5\sqrt{2}$ (about 7.07). We keep the exact radical $5\sqrt{2}$ as the answer.

Remember: In a 45-45-90 triangle the hypotenuse is the *only* side with a $\sqrt{2}$. Both legs match.

Try it: Quick check

The hypotenuse is 17 and one leg is 8. Find the other leg.

Work it on paper before revealing the answer.

Answer. Answer: $b = 15$

8.5 Special Right Triangle: 30-60-90

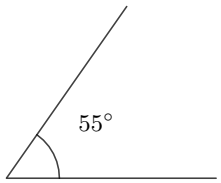
The 30-60-90 Triangle

The sides are always in the ratio

$$\text{short leg} = x, \qquad \text{long leg} = x\sqrt{3}, \qquad \text{hypotenuse} = 2x.$$

The **short leg** (across from 30°) is the key. The hypotenuse (across from 90°) is *twice* the short leg, and the long leg (across from 60°) is the short leg times $\sqrt{3}$.

An angle measures a turn.



Using the x, $x\sqrt{3}$, $2x$ pattern

The short leg (opposite 30°) is 4. Then the hypotenuse is $2 \cdot 4 = 8$ and the long leg is $4\sqrt{3}$ (about 6.93). We keep $4\sqrt{3}$ exact.

Tip: Always find the *short leg* first. If you are given the hypotenuse, halve it to get the short leg; then multiply the short leg by $\sqrt{3}$ for the long leg.

Try it: Quick check

Sides 6, 8, 10.

Work it on paper before revealing the answer.

Answer. Answer: Right ($36 + 64 = 100$)

8.6 The Three Trig Ratios: SOH-CAH-TOA

Sine, Cosine, Tangent

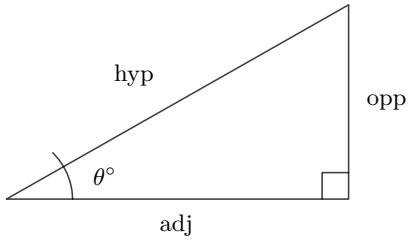
Pick one of the two *acute* angles and call it θ. Relative to θ, name the sides:

- **opposite** — the leg across from θ
- **adjacent** — the leg touching θ (that is not the hypotenuse)
- **hypotenuse** — across from the right angle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Remember it as **SOH-CAH-TOA**.

Opposite, adjacent and hypotenuse are named from the angle.



Reading the three ratios

For angle θ at B: the opposite leg is 3, the adjacent leg is 4, and the hypotenuse is 5. So

$$\sin\theta = \tfrac{3}{5}, \qquad \cos\theta = \tfrac{4}{5}, \qquad \tan\theta = \tfrac{3}{4}.$$

Remember: "Opposite" and "adjacent" switch when you switch which angle you look at, but the hypotenuse never moves. Always decide which angle is θ *first*.

Try it: Quick check

Sides 4, 5, 6.

Work it on paper before revealing the answer.

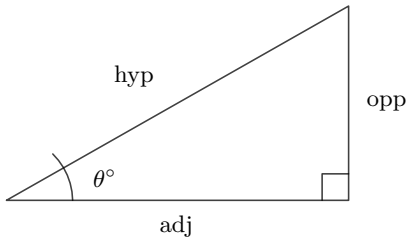
Answer. Answer: Acute ($41 > 36$)

8.7 Using a Trig Ratio to Find a Missing Side

Set up, then solve

When you know one angle and one side and want another side, pick the ratio that uses your two sides (SOH, CAH, or TOA), then solve for the unknown. Your calculator must be in **degree** mode.

Opposite, adjacent and hypotenuse are named from the angle.



Finding a side with cosine

Find x, the side adjacent to the $40°$ angle, when the hypotenuse is 10. Adjacent and hypotenuse means cosine (CAH):

$$\cos 40° = \frac{x}{10} \;\Rightarrow\; x = 10\cos 40° \approx 10(0.766) \approx 7.7.$$

Rounded to the nearest tenth, $x \approx 7.7$.

When the unknown is on the bottom

Find the hypotenuse h when the side opposite $35°$ is 6. Opposite and hypotenuse means sine (SOH):

$$\sin 35° = \frac{6}{h} \;\Rightarrow\; h = \frac{6}{\sin 35°} \approx \frac{6}{0.574} \approx 10.5.$$

When the unknown sits in the denominator, swap it with the trig value.

Tip: If the unknown is on *top*, multiply. If the unknown is on the *bottom*, divide the known side by the trig value.

Try it: Quick check

Sides 5, 6, 10.

Work it on paper before revealing the answer.

Answer. Answer: Obtuse ($61 < 100$)

8.8 Using Inverse Trig to Find a Missing Angle

The inverse buttons

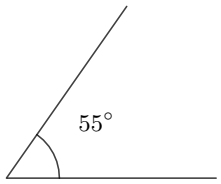
When you know two sides but want the *angle*, use the inverse functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$

(often the "2nd" + sin/cos/tan keys):

$$\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right), \quad \theta = \cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right), \quad \theta = \tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right).$$

The inverse "undoes" the ratio and gives back the angle in degrees.

An angle measures a turn.



Finding an angle with tangent

The leg opposite θ is 5 and the leg adjacent is 8. Two legs means tangent (TOA):

$$\tan\theta = \frac{5}{8} = 0.625 \;\Rightarrow\; \theta = \tan^{-1}(0.625) \approx 32.0^\circ.$$

Rounded to the nearest tenth of a degree, $\theta \approx 32.0^\circ$.

Remember: A plain trig button turns an angle into a ratio; an inverse button ($^{-1}$) turns a ratio back into an angle. Choose the ratio SOH-CAH-TOA from the two sides you have.

Try it: Quick check

Sides 9, 12, 15.

Work it on paper before revealing the answer.

Answer. Answer: Right ($225 = 225$)

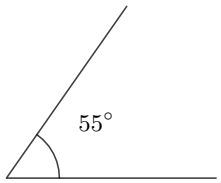
8.9 Angles of Elevation & Depression

Looking up and looking down

An **angle of elevation** is measured *up* from the horizontal to an object above you. An **angle of depression** is measured *down* from the horizontal to an object below you. Because the two horizontal lines are parallel, the angle of elevation from the ground equals the angle of

depression from above (alternate interior angles).

An angle measures a turn.



Height from an angle of elevation

From 50 ft away, the angle of elevation to the top of a tree is 32°. Find the height h. Opposite and adjacent means tangent (TOA):

$$\tan 32^\circ = \frac{h}{50} \;\Rightarrow\; h = 50\tan 32^\circ \approx 50(0.625) \approx 31.2 \text{ ft}.$$

Rounded to the nearest tenth, the tree is about 31.2 ft tall.

An angle of depression

A lifeguard 10 ft up sees a swimmer at an angle of depression of 20°. The angle of depression equals the angle of elevation from the swimmer, so at the tower base the angle inside the triangle is 20°. With the height 10 opposite that angle and the horizontal distance d adjacent:

$$\tan 20^\circ = \frac{10}{d} \;\Rightarrow\; d = \frac{10}{\tan 20^\circ} \approx \frac{10}{0.364} \approx 27.5 \text{ ft}.$$

Tip: Always sketch the right triangle and label horizontal, vertical, and line of sight. The elevation/depression angle sits between the *horizontal* and the *line of sight* — never at the top of a vertical side.

8.10 Going Deeper: Advanced Right-Triangle & Trig Ideas

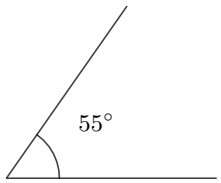
Where the 45-45-90 ratio comes from

The ratio is not memorized magic — it *follows* from the Pythagorean Theorem. Start with an isosceles right triangle whose two equal legs are each x. The hypotenuse c obeys

$$c^2 = x^2 + x^2 = 2x^2 \;\Rightarrow\; c = \sqrt{2x^2} = x\sqrt{2}.$$

Because both legs are equal, the two acute angles are equal, and since they must sum to 90° each is 45°. That is exactly the pattern x, x, $x\sqrt{2}$.

An angle measures a turn.



Where the 30-60-90 ratio comes from

Take an equilateral triangle with every side $2x$ and every angle 60°. Drop an altitude from the top vertex: it splits the base into two halves of length x and cuts the top angle into two 30° pieces, making a right triangle. The short leg (opposite 30°) is x and the hypotenuse is $2x$, so the altitude (the long leg, opposite 60°) is

$$h^2 = (2x)^2 - x^2 = 4x^2 - x^2 = 3x^2 \;\Rightarrow\; h = x\sqrt{3}.$$

That gives the pattern x, $x\sqrt{3}$, $2x$.

Preview: the identity $\sin^2\theta + \cos^2\theta = 1$

In a right triangle with hypotenuse c, opposite side a, and adjacent side b, we have $\sin\theta = a/c$ and $\cos\theta = b/c$. Squaring and adding:

$$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

because $a^2+b^2 = c^2$. So this famous **Pythagorean identity** is just the Pythagorean Theorem divided by c^2. (Here $\sin^2\theta$ means $(\sin\theta)^2$.) It holds for *every* angle, so if you know one of $\sin\theta$ or $\cos\theta$ you can find the other.

Using the identity

Suppose $\sin\theta = \frac{3}{5}$ for an acute angle θ. Find $\cos\theta$ without drawing the triangle:

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \tfrac{9}{25} = \tfrac{16}{25} \;\Rightarrow\; \cos\theta = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}.$$

We take the positive root because θ is acute. (This is just the 3-4-5 triangle in disguise.)

Beyond right triangles: Law of Sines & Law of Cosines

SOH-CAH-TOA only works in right triangles. For *any* triangle with angles A, B, C opposite sides a, b, c:

$$\text{Law of Sines:}\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$$\text{Law of Cosines:}\quad c^2 = a^2 + b^2 - 2ab\cos C.$$

The Law of Sines pairs each angle with its opposite side (use it when you know an angle-side pair). The Law of Cosines is the Pythagorean Theorem with a correction term $-2ab\cos C$; when $C = 90°$, $\cos C = 0$ and it collapses back to $c^2 = a^2 + b^2$.

Law of Cosines for a non-right triangle

A triangle has sides $a = 7$ and $b = 10$ with the included angle $C = 40°$ between them. Find the third side c:

$$c^2 = 7^2 + 10^2 - 2(7)(10)\cos 40° = 49 + 100 - 140(0.766) \approx 149 - 107.2 = 41.8.$$

$$c = \sqrt{41.8} \approx 6.5.$$

Rounded to the nearest tenth, $c \approx 6.5$.

Right triangles in 3D: the space diagonal

A rectangular box with length ℓ, width w, and height h has a **space diagonal** d running corner to opposite corner. Use the Pythagorean Theorem *twice*: first the diagonal of the base is $\sqrt{\ell^2 + w^2}$, then that base diagonal and the height h form a second right triangle:

$$d = \sqrt{\ell^2 + w^2 + h^2}.$$

The one clean formula just adds the squares of all three dimensions.

Space diagonal of a box

Find the space diagonal of a $3 \times 4 \times 12$ box.

$$d = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

Notice the base diagonal is $\sqrt{3^2 + 4^2} = 5$, and then $\sqrt{5^2 + 12^2} = 13$ — two nested triples (3-4-5 and 5-12-13).

Two observers, one balloon

Two people stand 200 ft apart on level ground, a balloon directly between them on the line joining them. One measures an angle of elevation of 50°, the other 40°. Find the height h.

Let the foot of the balloon be x ft from the first person, so it is $200 - x$ ft from the second. Each person sees the same height:

$$h = x \tan 50^\circ, \qquad h = (200 - x) \tan 40^\circ.$$

Set them equal: $x \tan 50^\circ = (200 - x) \tan 40^\circ$, so $1.19x = (200 - x)(0.839)$, giving $1.19x = 167.8 - 0.839x$, hence $2.03x = 167.8$ and $x \approx 82.7$. Then

$$h = 82.7 \tan 50^\circ \approx 82.7(1.19) \approx 98.5 \text{ ft}.$$

Rounded to the nearest tenth, the balloon is about 98.5 ft high.

The altitude on the hypotenuse: geometric means

Drop the altitude from the right angle to the hypotenuse. It splits the hypotenuse into two pieces p and q and creates two smaller triangles, each *similar* to the original. Similarity gives three "geometric mean" relations:

$$(\text{altitude}) = \sqrt{p\,q}, \qquad (\text{leg}_1) = \sqrt{p\,(p+q)}, \qquad (\text{leg}_2) = \sqrt{q\,(p+q)}.$$

The altitude is the geometric mean of the two hypotenuse pieces; each leg is the geometric mean of the whole hypotenuse and the piece next to it.

Finding the altitude by geometric mean

The altitude to the hypotenuse splits it into pieces $p = 4$ and $q = 9$. Find the altitude a:

$$a = \sqrt{p\,q} = \sqrt{4 \cdot 9} = \sqrt{36} = 6.$$

As a check, the shorter leg is $\sqrt{4(4+9)} = \sqrt{52} = 2\sqrt{13}$ and the longer leg is $\sqrt{9(13)} = \sqrt{117} = 3\sqrt{13}$; indeed $(2\sqrt{13})^2 + (3\sqrt{13})^2 = 52 + 117 = 169 = 13^2$, matching the full hypotenuse $4 + 9 = 13$. ✓

Big picture: Almost every idea here is the Pythagorean Theorem wearing a costume — the special-triangle ratios, the identity $\sin^2\theta + \cos^2\theta = 1$, the Law of Cosines, and the 3D space diagonal all reduce to "add the squares." When a problem looks new, hunt for the right triangle hiding inside it.

8.11 Why It Works

One proof of Pythagoras: arrange four copies of the right triangle inside a square of side $a+b$, corners in, so they leave a tilted square of side c in the middle. The big square's area is $(a+b)^2 = a^2+2ab+b^2$; it also equals the four triangles ($4 \cdot \frac{1}{2}ab = 2ab$) plus the inner square c^2. Cancelling $2ab$ leaves $a^2 + b^2 = c^2$. The trig ratios are well defined because all right triangles with a given acute angle are similar, so the ratios do not depend on the size.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

9 Quadrilaterals & Polygons

Vocabulary, angle formulas, special quadrilaterals, and the family tree

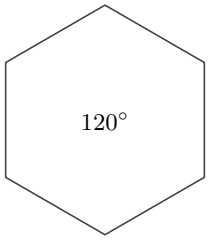
9.1 Polygon Vocabulary

What is a polygon?

A **polygon** is a closed, flat shape made of straight line segments. Each segment is a **side**, each corner where two sides meet is a **vertex**, and a segment that connects two *non-adjacent* vertices is a **diagonal**.

- **Convex:** no vertex is "pushed in"; every diagonal stays inside the shape.
- **Concave:** at least one vertex is "pushed in" (it has a dent), and at least one diagonal falls outside.
- **Regular:** all sides congruent *and* all angles congruent.
- **Irregular:** not regular (sides or angles differ).

A regular hexagon.



A polygon with n sides has interior angles summing to $(n-2)\cdot 180^\circ$ and exterior angles summing to 360°. The special quadrilaterals form a family tree: parallelograms (opposite sides parallel and equal, diagonals bisect each other), then rectangles, rhombi, and squares, plus trapezoids and kites with their own diagonal facts.

Convex vs. concave

The pentagon on the left is convex — it bulges outward at every vertex. The arrow shape on

the right is concave, because vertex D is pushed inward.

Where it goes wrong

Assuming a rhombus is a square or a rectangle is a rhombus. Both are parallelograms with an extra property — equal sides for the rhombus, right angles for the rectangle — and only the square has both.

Counting diagonals

From a single vertex you can draw $\mathbf{n-3}$ diagonals (you cannot connect a vertex to itself or to its two neighbors). Counting *every* diagonal of the whole polygon:

$$\text{number of diagonals} = \frac{n(n-3)}{2}.$$

Diagonals of a hexagon

A hexagon has $n = 6$ sides. From one vertex: $6 - 3 = 3$ diagonals. In all:

$$\frac{6(6-3)}{2} = \frac{6 \cdot 3}{2} = 9 \text{ diagonals.}$$

Memory hook. *"Concave"* has the word *cave* in it — a concave polygon has a cave (a dent). Convex shapes have no caves.

Try it: Quick check

What is the name of a polygon that has 7 sides?

Work it on paper before revealing the answer.

Answer. Answer: heptagon

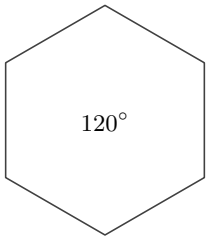
9.2 Sum of the Interior Angles

The interior-angle-sum formula

Every polygon can be split into triangles by drawing all diagonals from one vertex. A polygon with n sides splits into $n-2$ triangles, and each triangle holds $180°$. So:

$$\text{sum of interior angles} = (n-2) \cdot 180°.$$

A regular hexagon.



Interior-angle sum of a pentagon

A pentagon has $n = 5$ sides. Drawing diagonals from one vertex makes $5 - 2 = 3$ triangles.

$$(5 - 2) \cdot 180^\circ = 3 \cdot 180^\circ = 540^\circ.$$

Finding a missing angle

Four angles of a pentagon are 100°, 115°, 120°, and 90°. Find the fifth. The five angles must total 540°:

$$100 + 115 + 120 + 90 = 425, \qquad 540 - 425 = 115.$$

The missing angle is 115°.

Tip. Always find the total first with $(n - 2) \cdot 180^\circ$, then subtract the angles you already know to reveal the missing one.

Try it: Quick check

How many sides does a *decagon* have?

Work it on paper before revealing the answer.

Answer. Answer: 10 sides

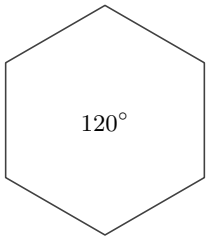
9.3 Each Interior Angle of a Regular Polygon

Sharing the total equally

In a **regular** polygon all n interior angles are equal, so each one is the total divided by n:

$$\text{each interior angle} = \frac{(n - 2) \cdot 180^\circ}{n}.$$

A regular hexagon.



Interior angle of a regular hexagon

A regular hexagon has $n = 6$.

$$\frac{(6-2)\cdot 180^\circ}{6} = \frac{4\cdot 180^\circ}{6} = \frac{720^\circ}{6} = 120^\circ.$$

Every interior angle of a regular hexagon is 120°.

Watch out. The formula $\frac{(n-2)180^\circ}{n}$ gives the measure of *one* angle only when the polygon is *regular*. In an irregular polygon the angles can be all different.

Try it: Quick check

True or False: a *regular* polygon has all sides congruent *and* all angles congruent.

Work it on paper before revealing the answer.

Answer. Answer: True

9.4 Sum of the Exterior Angles

The exterior angles always total 360°

At each vertex, extend one side; the angle between that extension and the next side is an **exterior angle**. No matter how many sides a convex polygon has,

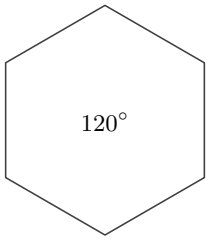
$$\text{sum of exterior angles} = 360^\circ.$$

For a **regular** polygon each exterior angle is the same:

$$\text{each exterior angle} = \frac{360^\circ}{n}.$$

An interior angle and its exterior angle form a linear pair, so they add to 180°.

A regular hexagon.



Exterior angle of a regular octagon

A regular octagon has $n = 8$, so each exterior angle is

$$\frac{360^\circ}{8} = 45^\circ.$$

Then each interior angle is $180^\circ - 45^\circ = 135^\circ$.

Working backward from an exterior angle

A regular polygon has an exterior angle of 30°. How many sides?

$$\frac{360^\circ}{n} = 30^\circ \;\Rightarrow\; n = \frac{360}{30} = 12.$$

It is a regular dodecagon (12 sides).

Fast trick. To find the number of sides of a regular polygon, it is usually easier to use the *exterior* angle: $n = \dfrac{360^\circ}{\text{exterior angle}}$.

Try it: Quick check

How many diagonals can you draw from *one* vertex of a hexagon? (Use $n - 3$.)

Work it on paper before revealing the answer.

Answer. Answer: $6 - 3 = 3$ diagonals

9.5 Properties of Parallelograms

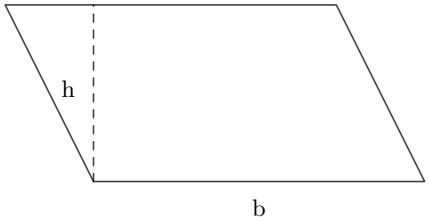
What makes a parallelogram?

A **parallelogram** is a quadrilateral with *both* pairs of opposite sides parallel. From that one

fact, four properties always follow:

- Opposite sides are **congruent**.
- Opposite angles are **congruent**.
- Consecutive (next-door) angles are **supplementary** (add to 180°).
- The diagonals **bisect each other** (cut each other in half).

Height is perpendicular to the base.



Reading a parallelogram

In parallelogram $ABCD$, side $AB \parallel DC$ and $AD \parallel BC$. Because opposite angles are equal, $m\angle A = m\angle C$ and $m\angle B = m\angle D$. Because consecutive angles are supplementary, $m\angle A + m\angle B = 180°$. If $m\angle A = 70°$, then $m\angle C = 70°$ and $m\angle B = m\angle D = 110°$.

Diagonals bisect each other

The diagonals of parallelogram $PQRS$ meet at M. Since they bisect each other, $PM = MR$ and $QM = MS$. If $PM = 5$, then $PR = 10$.

Remember. "Opposite parts match, next-door angles add to 180°." Diagonals bisect each other, but in a *general* parallelogram they are *not* equal and *not* perpendicular.

Try it: Quick check

Find the sum of the interior angles of a **pentagon** ($n = 5$). Use $(n - 2) \cdot 180°$.

Work it on paper before revealing the answer.

Answer. Answer: $(5 - 2)180° = 540°$

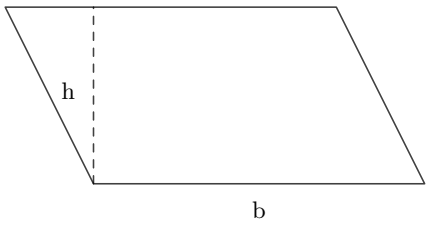
9.6 Special Parallelograms: Rectangle, Rhombus, Square

Extra powers

Each special parallelogram keeps *all* parallelogram properties and adds more:

- **Rectangle:** four right angles; diagonals are **congruent**.
- **Rhombus:** four congruent sides; diagonals are **perpendicular** and **bisect the angles**.
- **Square:** a rectangle *and* a rhombus — four right angles *and* four equal sides; diagonals are congruent, perpendicular, and bisect the angles.

Height is perpendicular to the base.



A rhombus and a square with their diagonals

The rhombus (left) has four equal sides and diagonals that cross at a right angle. The square (right) has everything: equal sides, right angles, and equal, perpendicular diagonals.

Diagonal shortcut. *Rect*angle $\rightarrow$ diagonals are equal (R for *Right* angles). *Rhombus* $\rightarrow$ diagonals are perpendicular. *Square* $\rightarrow$ both at once.

Try it: Quick check

Find the sum of the interior angles of an **octagon** ($n = 8$).

Work it on paper before revealing the answer.

Answer. Answer: $(8-2)180^\circ = 1080^\circ$

9.7 Trapezoids and Kites

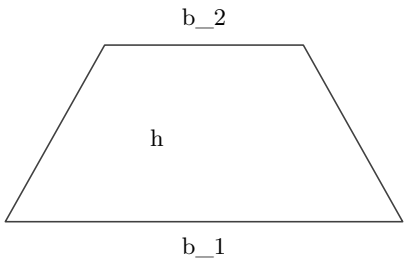
Trapezoid parts

A **trapezoid** has *exactly one* pair of parallel sides, called the **bases**. The other two sides are the **legs**. The **midsegment** connects the midpoints of the legs; it is parallel to the bases and its length is the *average* of the two bases:

$$\text{midsegment} = \frac{b_1 + b_2}{2}.$$

An **isosceles trapezoid** has congruent legs; its base angles are congruent and its diagonals are congruent.

A trapezoid has two parallel bases.



Midsegment of a trapezoid

A trapezoid has bases $b_1 = 6$ and $b_2 = 10$. The midsegment is

$$\frac{6 + 10}{2} = \frac{16}{2} = 8.$$

A kite

A **kite** has two pairs of *consecutive* congruent sides (not opposite). One pair of opposite angles is congruent, the diagonals are **perpendicular**, and one diagonal bisects the other. Tick marks show the two pairs of equal sides.

Don't mix them up. A *trapezoid* has one pair of *parallel* sides; a *kite* has two pairs of *equal adjacent* sides. Both a rhombus and a kite have perpendicular diagonals, but only the rhombus has all four sides equal.

Try it: Quick check

Three angles of a quadrilateral measure 80°, 100°, and 110°. Find the fourth angle. (The four

angles total 360°.)

Work it on paper before revealing the answer.

Answer. Answer: $360 - 290 = 70°$

9.8 The Quadrilateral Family Tree

Who is who

Every square is a rectangle and a rhombus; every rectangle and every rhombus is a parallelogram; every parallelogram is a quadrilateral. As you climb *down* the tree, shapes gain more properties; as you climb *up*, they become more general.

The hierarchy at a glance

A square sits at the bottom because it inherits every property above it.

Always, sometimes, or never?

- A square is **always** a rectangle (it has four right angles). ✓
- A rectangle is **sometimes** a square (only when its sides are all equal).
- A parallelogram is **never** a trapezoid *under the "exactly one pair"* definition, because a parallelogram has *two* pairs of parallel sides.

Reasoning tip. For "always / sometimes / never," ask: does the lower shape's definition *force* the upper property? If yes → *always*. If it can happen but isn't required → *sometimes*. If the definitions clash → *never*.

Try it: Quick check

Find the measure of each interior angle of a regular **octagon** ($n = 8$).

Work it on paper before revealing the answer.

Answer. Answer: $\frac{6 \cdot 180}{8} = 135°$

9.9 Going Deeper: Advanced Polygon Ideas

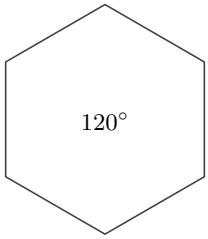
Cyclic quadrilaterals: opposite angles are supplementary

A **cyclic quadrilateral** is one whose four vertices all lie on a single circle. Its defining property is beautiful and useful:

opposite angles add to 180°.

So if $ABCD$ is cyclic, then $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$. (The reason: an inscribed angle is half its intercepted arc, and the two opposite angles together intercept the *whole* circle, or 360°; half of that is 180°.) Every rectangle and every isosceles trapezoid is cyclic; a general parallelogram is cyclic only when it is a rectangle.

A regular hexagon.



Worked example: a missing angle in a cyclic quadrilateral

Quadrilateral $ABCD$ is inscribed in a circle. Its angles are $m\angle A = 95^\circ$ and $m\angle B = 70^\circ$. Find $m\angle C$ and $m\angle D$. Opposite angles are supplementary, so $\angle A$ pairs with $\angle C$ and $\angle B$ pairs with $\angle D$:

$$m\angle C = 180^\circ - 95^\circ = 85^\circ, \qquad m\angle D = 180^\circ - 70^\circ = 110^\circ.$$

Check: all four angles total $95 + 70 + 85 + 110 = 360^\circ$, exactly the interior-angle sum of any quadrilateral.

Varignon's theorem: the midpoint quadrilateral

Take *any* quadrilateral and mark the midpoint of each of its four sides. Connect those midpoints in order. The result — called the **Varignon parallelogram** — is *always a parallelogram*, no matter how lopsided the original shape is.

The key idea is the triangle **midsegment**: in triangle ABC, the segment joining the midpoints of AB and BC is parallel to diagonal AC and half its length. The same holds on the other side, so both pairs of opposite midpoint-sides are parallel — the definition of a parallelogram.

Consequences of Varignon. The Varignon parallelogram's sides are half the lengths of the original diagonals, so its perimeter equals the *sum of the diagonals* of the outer quadrilateral, and its area is exactly *half* the area of the original quadrilateral.

The apothem and the area of a regular polygon

The **apothem** a of a regular polygon is the perpendicular distance from the center to the

middle of any side. Slice the polygon into n congruent triangles, each with base equal to a side s and height equal to the apothem. Adding the triangle areas gives

$$\text{Area} = \tfrac{1}{2}\, a\, P,$$

where $P = ns$ is the perimeter. In words: *half the apothem times the perimeter.*

Worked example: area of a regular hexagon from its apothem

A regular hexagon has side length $s = 6$ and apothem $a = 3\sqrt{3} \approx 5.196$. Find its area. The perimeter is $P = ns = 6 \cdot 6 = 36$. Then

$$\text{Area} = \tfrac{1}{2}\, a\, P = \tfrac{1}{2}\,(3\sqrt{3})(36) = 54\sqrt{3} \approx 93.5.$$

So the hexagon covers about 93.5 square units.

Area from coordinates: the shoelace formula

When you know the corner coordinates of a polygon, you can find its area *directly* — no base or height needed. List the vertices in order (all the way around) as $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then

$$\text{Area} = \tfrac{1}{2}\left|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\right|.$$

It is called the **shoelace** formula because the cross-multiplications lace diagonally like shoelaces. Go around once in a consistent direction and take the absolute value at the end.

Worked example: quadrilateral area by the shoelace method

Find the area of quadrilateral with vertices $A(1, 1)$, $B(5, 2)$, $C(4, 5)$, and $D(2, 4)$, taken in order. Set up the "down-then-up" products and subtract:

$$\begin{aligned} \text{down products } (x_iy_{i+1}):&\ 1 \cdot 2 + 5 \cdot 5 + 4 \cdot 4 + 2 \cdot 1 = 2 + 25 + 16 + 2 = 45, \\ \text{up products } (y_ix_{i+1}):&\ 1 \cdot 5 + 2 \cdot 4 + 5 \cdot 2 + 4 \cdot 1 = 5 + 8 + 10 + 4 = 27. \end{aligned}$$

Then

$$\text{Area} = \tfrac{1}{2}\,|45 - 27| = \tfrac{1}{2}\,(18) = 9 \text{ square units.}$$

Finding n from an angle condition

Angle formulas run *both* directions: given a single interior or exterior angle of a regular polygon, you can solve for the number of sides n.

- From an **exterior** angle E: $\quad n = \dfrac{360^\circ}{E}$.
- From an **interior** angle I: first get the exterior angle $E = 180^\circ - I$, then $n = \dfrac{360^\circ}{E}$; or

solve $\dfrac{(n-2)180^\circ}{n} = I$ directly.

A valid answer must be a whole number $n \geq 3$ — if the division is not exact, no regular polygon has that angle.

Worked example: which regular polygon has a 150° interior angle?

Each interior angle is $I = 150^\circ$, so each exterior angle is

$$E = 180^\circ - 150^\circ = 30^\circ, \qquad n = \frac{360^\circ}{30^\circ} = 12.$$

It is a regular dodecagon. As a check, try the direct equation:

$$\frac{(n-2)180}{n} = 150 \;\Rightarrow\; 180n - 360 = 150n \;\Rightarrow\; 30n = 360 \;\Rightarrow\; n = 12.\checkmark$$

By contrast, an interior angle of 140° gives $E = 40^\circ$ and $n = 9$ (a nonagon), while 155° gives $E = 25^\circ$ and $n = 14.4$ — not a whole number, so *no* regular polygon has a 155° interior angle.

Justifying always / sometimes / never

The family tree turns every classification claim into a logic question. Compare the *definitions* of the two shapes:

- **Always** — the first shape's definition *forces* the second's properties. *Every rhombus is a parallelogram*: having four equal sides guarantees both pairs of opposite sides are parallel.
- **Sometimes** — the property *can* hold but is not required. *A rhombus is sometimes a square*: only when its angles are also right angles.
- **Never** — the definitions *contradict*. *A square is never a (strict) trapezoid*: a square has two pairs of parallel sides, but the strict trapezoid definition demands exactly one.

The habit that never fails: state each definition, then ask whether the first *must*, *may*, or *cannot* satisfy the second.

Big picture. These advanced tools all come from the same two engines you already know: (1) angles built from 180° and 360°, and (2) breaking a polygon into triangles. Cyclic angles, the apothem area, and Varignon's theorem are just those engines pushed one step further.

9.10 Why It Works

Interior angle sum: pick one vertex and draw diagonals to the others, cutting the polygon into $n-2$ triangles; each contributes 180°. Exterior angles sum to 360° because walking around the polygon

you turn through each exterior angle in turn and end up facing the way you started — one full turn. A parallelogram's diagonals bisect each other because the two triangles they form with a pair of opposite sides are congruent by ASA (alternate interior angles from the parallel sides).

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

10 Circles

Parts, circumference, area, angles, tangents, chords, and equations

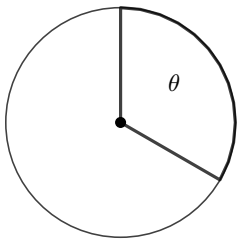
10.1 The Parts of a Circle

Naming the pieces

A **circle** is the set of all points that are the same distance from one point, the **center**. We name a circle by its center: the circle below is "circle O," written $\odot O$.

- **Radius:** a segment from the *center* to a point *on* the circle.
- **Diameter:** a chord that passes *through the center*. It is the longest chord, and $d = 2r$.
- **Chord:** a segment whose *two endpoints are on* the circle.
- **Secant:** a *line* that cuts the circle at *two* points.
- **Tangent:** a *line* that touches the circle at *exactly one* point.

A sector cut by a central angle.



Circle theorems come in three families: angles (a central angle equals its arc, an inscribed angle is half its arc, an angle in a semicircle is right), lengths (equal chords are equidistant, a tangent is perpendicular to the radius, power of a point), and measurements (arc length and sector area are fractions of the circumference and the area).

A circle with its interior parts labeled

Here $\overline{OC}$ is a radius, $\overline{AB}$ is a diameter (it goes through O), and $\overline{DE}$ is a chord (its endpoints are on the circle, but it misses the center).

Where it goes wrong

Using the inscribed angle as equal to its arc instead of half. Central angle = arc; inscribed angle = half the arc.

Secant line vs. tangent line

A **secant** slices through the circle at two points. A **tangent** just grazes it at one point, P (the *point of tangency*).

Arcs: pieces of the circle itself

An **arc** is a curved piece of the circle between two points.

- **Minor arc:** the *shorter* way around (less than 180°). Named with two letters, like $\overset{\frown}{AB}$.
- **Major arc:** the *longer* way around (more than 180°). Named with three letters to show which way you travel.
- **Semicircle:** exactly *half* the circle (180°); its endpoints are the ends of a diameter.

Minor arc and major arc

Points A and B split the circle into two arcs. The short (thick) path is minor arc $\overset{\frown}{AB}$; the long way around is the major arc.

Radius vs. diameter. The diameter is always *twice* the radius: $d = 2r$, so $r = \frac{1}{2}d$. Mixing these up is the most common circle mistake — always ask "do I have the radius or the diameter?"

Try it: Quick check

A circle has a radius of 7 cm. What is its diameter?

Work it on paper before revealing the answer.

Answer. Answer: $d = 2r = 2(7) = 14$ cm.

10.2 Circumference and Arc Length

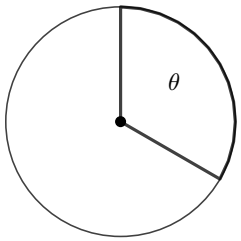
Circumference: the distance around

The **circumference** C is the perimeter of a circle:

$$C = 2\pi r = \pi d.$$

The number π ("pi") is about 3.14. We leave answers *in terms of* π unless a decimal is asked for; then we use $\pi \approx 3.14$.

A sector cut by a central angle.



Circumference from the radius

A circle has radius $r = 5$ cm.

$$C = 2\pi r = 2\pi(5) = 10\pi \text{ cm} \approx 10(3.14) = 31.4 \text{ cm}.$$

Arc length: part of the way around

An arc is a fraction of the whole circle. If its central angle is n°, then

$$\text{arc length} = \frac{n}{360} \cdot 2\pi r.$$

The fraction $\dfrac{n}{360}$ is "how much of the full 360°" the arc covers.

Arc length

Find the length of an arc with central angle $90°$ in a circle of radius 8.

$$\text{arc length} = \frac{90}{360} \cdot 2\pi(8) = \frac{1}{4} \cdot 16\pi = 4\pi \approx 12.56.$$

Watch the units. Circumference and arc length are *lengths*, so they use ordinary units (cm, m, in). Later, area will use *square* units. If you see "cm^2" on a length answer, something went wrong.

Try it: Quick check

A circle has a diameter of 18 in. What is its radius?

Work it on paper before revealing the answer.

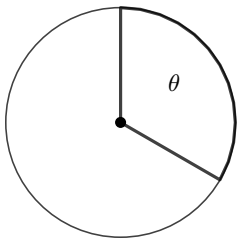
Answer. Answer: $r = \frac{1}{2}d = \frac{1}{2}(18) = 9$ in.

10.3 Central Angles and Arc Measure

A central angle equals its arc

A **central angle** has its vertex at the *center* of the circle. The **measure of a minor arc equals the measure of its central angle**. A full trip around the circle is $360°$, so all the central angles around the center add to $360°$.

A sector cut by a central angle.



Central angle = arc

The central angle $\angle AOB$ measures $60°$, so the intercepted arc $\overset{\frown}{AB}$ also measures $60°$. The major arc is $360° - 60° = 300°$.

Everything adds to 360°. If a circle is cut into central angles, their measures total 360°. To find a missing central angle, subtract the known ones from 360°.

Try it: Quick check

Fill in the blank: a segment with *both* endpoints on the circle (but not through the center) is called a _____.

Work it on paper before revealing the answer.

Answer. Answer: Chord.

10.4 Inscribed Angles

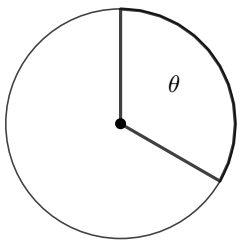
The Inscribed Angle Theorem

An **inscribed angle** has its vertex *on* the circle, and its two sides are chords. The

$$\text{inscribed angle} = \tfrac{1}{2}\,(\text{intercepted arc}).$$

So the intercepted arc is *twice* the inscribed angle. (Notice a central angle equals its arc, but an inscribed angle is only *half* of it.)

A sector cut by a central angle.



Central angle vs. inscribed angle on the same arc

Both angles below open onto arc $\overset{\frown}{AB}$. The central angle $\angle AOB = 90°$ equals the arc. The inscribed angle $\angle ACB$ is half of that arc: $\frac{1}{2}(90°) = 45°$.

Angle inscribed in a semicircle is 90°

If a chord is a *diameter*, the arc it cuts off is a semicircle (180°). Any inscribed angle on that arc is $\frac{1}{2}(180°) = 90°$. So $\angle ACB$ below is a right angle.

Two big ideas. (1) Inscribed angle = half its arc. (2) An angle inscribed in a semicircle (sitting on a diameter) is always 90°. This second fact turns diameters into right triangles!

Try it: Quick check

Find the circumference of a circle with radius $r = 3$. Leave your answer *in terms of* π.

Work it on paper before revealing the answer.

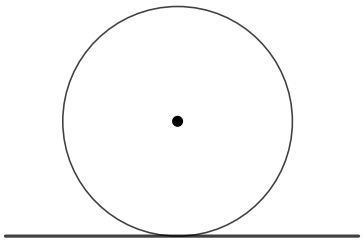
Answer. Answer: $C = 2\pi r = 2\pi(3) = 6\pi$.

10.5 Tangent Lines

Tangent ⊥ radius

A **tangent** touches a circle at exactly one point. At that **point of tangency**, the tangent line is **perpendicular to the radius**. That right angle is the key to almost every tangent problem, because it makes a right triangle you can use with the Pythagorean Theorem.

A tangent touches at exactly one point.



The tangent meets the radius at 90°

Radius $\overline{OP}$ meets tangent line t at the point of tangency P, forming a right angle.

Two tangents from one outside point are equal

From an outside point P, the two tangent segments to the circle are congruent: $PT_1 = PT_2$.

Using the right triangle

A tangent from point P touches $\odot O$ at T. If $OP = 13$ and radius $OT = 5$, find the tangent length PT. Because $\angle OTP = 90°$:

$$PT = \sqrt{OP^2 - OT^2} = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12.$$

Draw the radius. Whenever a tangent appears, draw the radius to the point of tangency. You instantly get a 90° angle — and a right triangle where OP (center to outside point) is the *hypotenuse*.

Try it: Quick check

Find the circumference of a circle with diameter $d = 10$. Leave your answer *in terms of* π.

Work it on paper before revealing the answer.

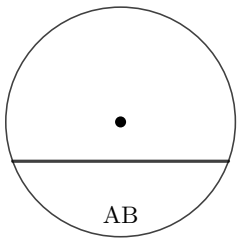
Answer. Answer: $C = \pi d = 10\pi$.

10.6 Chord Relationships

Two chord facts

- **Equal chords:** In the same circle, chords that are the same distance from the center are *congruent* (equal in length).
- **Perpendicular from the center:** A radius (or diameter) that is *perpendicular* to a chord *bisects* it — it cuts the chord into two equal halves.

A chord joins two points on the circle.



The perpendicular from the center bisects the chord

$\overline{OM} \perp \overline{AB}$, so M is the midpoint of chord $\overline{AB}$ and $AM = MB$. This also makes a right

triangle OMA: if the radius $OA = 5$ and $OM = 3$, then

$$AM = \sqrt{OA^2 - OM^2} = \sqrt{25 - 9} = \sqrt{16} = 4, \quad \text{so } AB = 2(4) = 8.$$

Half the chord, please. When you drop a perpendicular from the center to a chord, the right triangle uses *half* the chord as a leg and the *radius* as the hypotenuse. Don't plug in the whole chord by accident.

Try it: Quick check

Find the circumference of a circle with radius $r = 6$, using $\pi \approx 3.14$.

Work it on paper before revealing the answer.

Answer. Answer: $C = 2\pi(6) = 12\pi \approx 37.68$.

10.7 Area of a Circle and Area of a Sector

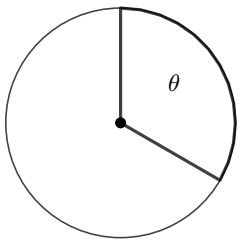
Area of a circle

The **area** inside a circle is

$$A = \pi r^2.$$

Remember it is r *squared*, then times π. Area is measured in *square* units.

A sector cut by a central angle.



Area from the radius

A circle has radius $r = 5$.

$$A = \pi r^2 = \pi(5)^2 = 25\pi \approx 25(3.14) = 78.5 \text{ square units.}$$

Area of a sector

A **sector** is a "pizza slice" bounded by two radii and an arc. If its central angle is n°,

$$\text{sector area} = \frac{n}{360} \cdot \pi r^2.$$

It is the same fraction $\dfrac{n}{360}$ of the whole circle that we used for arc length.

Area of a sector

Find the area of a 60° sector in a circle of radius 6.

$$\text{sector area} = \frac{60}{360} \cdot \pi(6)^2 = \frac{1}{6} \cdot 36\pi = 6\pi \approx 18.84.$$

Same fraction, two uses. The fraction $\dfrac{n}{360}$ scales the *whole circumference* to get arc length, and scales the *whole area* to get sector area. Learn it once, use it twice!

Try it: Quick check

A central angle measures 70°. What is the measure of the minor arc it intercepts?

Work it on paper before revealing the answer.

Answer. Answer: 70° (arc = central angle).

10.8 The Equation of a Circle

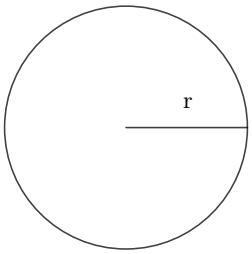
Center–radius form

A circle with center (h, k) and radius r has the equation

$$(x - h)^2 + (y - k)^2 = r^2.$$

Read it carefully: the numbers subtracted from x and y give the *center*, and the right side is r^2 (so take the *square root* to get the radius).

Everything follows from the radius.



Writing an equation

A circle has center $(3, 3)$ and radius 2. Its equation is

$$(x-3)^2 + (y-3)^2 = 2^2 = 4.$$

Reading center and radius from an equation

Given $(x-2)^2 + (y+3)^2 = 16$. Rewrite $y+3$ as $y-(-3)$, so the center is $(2, -3)$. Since $r^2 = 16$, the radius is $r = \sqrt{16} = 4$.

Mind the signs. The center uses the *opposite* sign of what you see: $(x-2)$ means $h = +2$, and $(y+3) = (y-(-3))$ means $k = -3$. And the right side is r^2, *not* r — always take the square root.

Try it: Quick check

A minor arc measures 130°. What is the measure of its central angle?

Work it on paper before revealing the answer.

Answer. Answer: 130° (central angle = arc).

10.9 Going Deeper: Advanced Circle Ideas

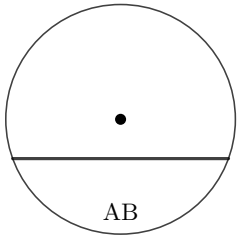
Proving the Inscribed Angle Theorem

Why is an inscribed angle exactly *half* its arc? Look at the easy case where one side of the angle is a **diameter**. Draw radius $\overline{OB}$. Since $OB = OC = r$, triangle OBC is *isosceles*, so its base angles are equal: $\angle OCB = \angle OBC$. Now $\angle AOB$ is an **exterior angle** of that triangle, so it equals the sum of the two remote interior angles:

$$\angle AOB = \angle OCB + \angle OBC = 2\,\angle OCB = 2\,\angle ACB.$$

The central angle $\angle AOB$ equals arc $\overset{\frown}{AB}$, so the inscribed angle $\angle ACB$ is half that arc. (Any inscribed angle can be split into cases like this by drawing the diameter through C.)

A chord joins two points on the circle.



Why the perpendicular from the center bisects a chord

Section on chords stated it; here is *why*. Drop $\overline{OM} \perp \overline{AB}$ and draw radii $\overline{OA}$ and $\overline{OB}$. Then right triangles OMA and OMB share leg $\overline{OM}$ and have equal hypotenuses $OA = OB = r$. By **Hypotenuse–Leg**, the triangles are congruent, so $AM = MB$. The perpendicular from the center *must* split the chord in half.

The tangent–chord angle

When a **tangent** and a **chord** meet at the point of tangency, the angle they form is *half the intercepted arc* — the same "half the arc" rule as an inscribed angle:

$$\text{tangent–chord angle} = \tfrac{1}{2}\,(\text{intercepted arc}).$$

In the picture, chord $\overline{PB}$ and tangent t meet at P; the marked angle is half of arc $\overset{\frown}{PB}$.

Cyclic quadrilaterals: opposite angles are supplementary

A **cyclic quadrilateral** has all four vertices *on* a circle. Each angle is inscribed, so it equals half of the arc across from it. The two arcs opposite a pair of angles together make the whole circle (360°), so each pair of *opposite* angles adds to half of that:

$$\angle A + \angle C = 180^\circ, \qquad \angle B + \angle D = 180^\circ.$$

The Power of a Point

If two lines through a point P each meet the circle, the products of the two distances along each line are *equal*. There are three versions of this one idea:

- **Two chords** crossing *inside* at P: $PA \cdot PB = PC \cdot PD$.
- **Two secants** from an *outside* point P: $PA \cdot PB = PC \cdot PD$ (each product is near point $\times$ far point).
- **Secant and tangent** from an outside point, with tangent length PT: $PT^2 = PA \cdot PB$.

Power of a point: two chords

Chords $\overline{AB}$ and $\overline{CD}$ cross at P inside the circle. If $PA = 4$, $PB = 6$, and $PC = 3$, find PD:

$$PA \cdot PB = PC \cdot PD \;\Rightarrow\; 4 \cdot 6 = 3 \cdot PD \;\Rightarrow\; PD = \frac{24}{3} = 8.$$

Power of a point: secant and tangent

From outside point P, a tangent of length PT and a secant hitting the circle at A (near) then B (far) are drawn. If $PA = 4$ and $PB = 9$, find PT:

$$PT^2 = PA \cdot PB = 4 \cdot 9 = 36 \;\Rightarrow\; PT = \sqrt{36} = 6.$$

Area of a segment

A **segment** is the region between a chord and its arc — what is left of a sector after you remove the triangle formed by the two radii. So

$$\text{segment area} = (\text{sector area}) - (\text{triangle area}) = \frac{n}{360}\,\pi r^2 - (\text{area of the two-radius triangle}).$$

Area of a segment

Find the area of the segment cut off by a 90° arc in a circle of radius 4. The two radii form a right triangle with legs 4 and 4.

$$\begin{aligned} \text{sector} &= \frac{90}{360}\,\pi(4)^2 = \tfrac{1}{4} \cdot 16\pi = 4\pi, \\ \text{triangle} &= \tfrac{1}{2}(4)(4) = 8, \\ \text{segment} &= 4\pi - 8 \approx 12.56 - 8 = 4.56. \end{aligned}$$

Finding the center and radius by completing the square

An equation like $x^2 + y^2 - 6x + 4y - 12 = 0$ is a circle in disguise. To find its center and radius, **complete the square** on the x-terms and the y-terms separately, adding the same amounts to *both* sides. Each time, take half the middle coefficient and square it.

Completing the square

Put $x^2 + y^2 - 6x + 4y - 12 = 0$ into center–radius form. Group the x's and y's and move the constant over:

$$(x^2 - 6x) + (y^2 + 4y) = 12.$$

Half of -6 is -3, squared is 9; half of 4 is 2, squared is 4. Add 9 and 4 to both sides:

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4,$$
$$(x - 3)^2 + (y + 2)^2 = 25.$$

So the center is $(3, -2)$ and the radius is $r = \sqrt{25} = 5$.

One rule, many faces. "Half the arc" powers inscribed angles, tangent–chord angles, and cyclic quadrilaterals. "Near × far" powers all three Power-of-a-Point setups. And "sector minus triangle" gives a segment. Spotting which familiar idea is hiding in a hard problem is most of the battle.

10.10 Why It Works

Inscribed angle theorem, first case (one side through the center): the triangle formed by the center and the chord is isosceles (two radii), so its base angles are equal; the central angle is an exterior angle of that triangle and equals the sum of the two equal base angles — twice the inscribed angle. The other cases add or subtract two such configurations. The angle-in-a-semicircle corollary follows instantly: the arc is 180°, so the inscribed angle is 90°.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

11 Area & Volume

Flat shapes, solid shapes, and the formulas that measure them

11.1 Perimeter & Area of Polygons

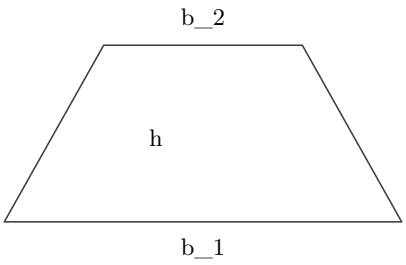
Two rules to carry through everything:

- **Perimeter** is a length, so its units are plain: cm, m, in. **Area** is a *covering*, so its units

are *squared*: cm^2. **Volume** is a *filling*, so its units are *cubed*: cm^3.

- Whenever a formula uses π, we will round with $\pi \approx 3.14$ and *say so*. Always keep the correct unit on your final answer.

A trapezoid has two parallel bases.



Area formulas are built from the rectangle by cutting and rearranging: a parallelogram is a sheared rectangle, a triangle is half a parallelogram, a trapezoid is the average of its bases times the height. Volumes of prisms and cylinders are base area times height; pyramids and cones are one third of that; the sphere is $\frac{4}{3}\pi r^3$.

Formula Reference — Flat Shapes

Let b be a base and h the matching *perpendicular* height.

- **Rectangle** (length ℓ, width w): $P = 2\ell + 2w$, $A = \ell w$.
- **Triangle**: $P = a + b + c$ (add the three sides), $A = \frac{1}{2}bh$.
- **Parallelogram**: $P = 2a + 2b$, $A = bh$.
- **Trapezoid** (parallel sides b_1 and b_2): $A = \frac{1}{2}(b_1 + b_2)\, h$.

The height is always measured *straight across*, at a right angle to the base — never along a slanted side.

Rectangle: perimeter and area

For this rectangle, $\ell = 8$ cm and $w = 5$ cm:

$$P = 2(8) + 2(5) = 26 \text{ cm}, \qquad A = 8 \times 5 = 40 \text{ cm}^2.$$

Where it goes wrong

Forgetting the $\frac{1}{3}$ for pyramids and cones, or using the slant height where the perpendicular height belongs in a volume formula (slant height belongs to surface area).

Triangle and parallelogram

Triangle with base $b = 6$ and height $h = 4$:

$$A = \tfrac{1}{2}(6)(4) = 12 \text{ cm}^2.$$

Parallelogram with base $b = 10$ and height $h = 4$:

$$A = (10)(4) = 40 \text{ cm}^2.$$

Trapezoid

The two parallel sides are $b_1 = 6$ and $b_2 = 10$; the height between them is $h = 4$:

$$A = \tfrac{1}{2}(b_1 + b_2)\, h = \tfrac{1}{2}(6 + 10)(4) = \tfrac{1}{2}(16)(4) = 32 \text{ cm}^2.$$

Tip: The $\frac{1}{2}$ in the triangle and trapezoid formulas is not a coincidence — a triangle is half of a parallelogram, and a trapezoid is the *average* of its two bases times the height.

Try it: Quick check

Find the **perimeter** of the same rectangle (9 cm by 4 cm).

Work it on paper before revealing the answer.

Answer. Answer: $P = 2(9) + 2(4) = 26$ cm.

11.2 Regular Polygons & Circles

Formula Reference — Regular Polygons and Circles

- **Regular polygon** (all sides and angles equal): the *apothem* a is the distance from the center straight out to the middle of a side. With perimeter P,

$$A = \tfrac{1}{2}\, a\, P.$$

- **Circle** with radius r (and diameter $d = 2r$):

$$\text{Circumference } C = 2\pi r = \pi d, \qquad \text{Area } A = \pi r^2.$$

- **Sector** (a "pizza slice" with central angle θ degrees):

$$\text{Arc length} = \frac{\theta}{360} \cdot 2\pi r, \qquad \text{Sector area} = \frac{\theta}{360} \cdot \pi r^2.$$

Regular hexagon using the apothem

A regular hexagon has side 6, so $P = 6 \times 6 = 36$. Its apothem is about $a \approx 5.2$:

$$A = \tfrac{1}{2}(5.2)(36) = 93.6 \text{ cm}^2.$$

Circle: circumference, area, and a sector (with $\pi \approx 3.14$)

A circle has radius $r = 10$:

$$C = 2\pi r \approx 2(3.14)(10) = 62.8 \text{ cm}, \qquad A = \pi r^2 \approx 3.14(10)^2 = 314 \text{ cm}^2.$$

A **quarter** of a circle of radius 6 (so $\theta = 90°$) has area

$$\frac{90}{360} \cdot \pi r^2 \approx \tfrac{1}{4}(3.14)(36) = \tfrac{1}{4}(113.04) = 28.26 \text{ cm}^2.$$

Remember: $A = \frac{1}{2}aP$ works for *any* regular polygon. A circle is like a regular polygon with endlessly many tiny sides — its "apothem" is the radius and its "perimeter" is the circumference, and $\frac{1}{2}r \cdot 2\pi r = \pi r^2$. Neat!

Try it: Quick check

A regular hexagon has each side 5 cm long. Find its **perimeter**.

Work it on paper before revealing the answer.

Answer. Answer: $P = 6 \times 5 = 30$ cm.

11.3 Composite (Compound) Figures

Break It Apart, Then Add or Subtract

A **composite figure** is made of familiar shapes joined together. To find its area:

1. Cut the figure into shapes you know (rectangles, triangles, half-circles, …).
2. Find each piece's area.
3. **Add** the pieces that are present; **subtract** any region that is cut out.

For perimeter, trace the *outside* edge only, and do not count any inner cut lines.

A rectangle topped with a half-circle

This shape is a 6×4 rectangle with a semicircle of radius 3 sitting on top. Add the two areas ($\pi \approx 3.14$):

$$A_{\text{rect}} = 6 \times 4 = 24, \qquad A_{\text{semi}} = \tfrac{1}{2}\pi r^2 \approx \tfrac{1}{2}(3.14)(3)^2 = 14.13.$$

$$A_{\text{total}} = 24 + 14.13 = 38.13 \text{ cm}^2.$$

Tip: When a piece is *removed* (a hole, a bite, a corner cut off), you *subtract* its area. When a piece is *added on*, you *add* it. Always ask: "Is this region part of my shape, or missing from it?"

Try it: Quick check

A regular polygon has apothem $a = 6$ cm and perimeter $P = 40$ cm. Find its **area** using $A = \frac{1}{2}aP$.

Work it on paper before revealing the answer.

Answer. Answer: $A = \frac{1}{2}(6)(40) = 120 \text{ cm}^2$.

11.4 Surface Area of Solids

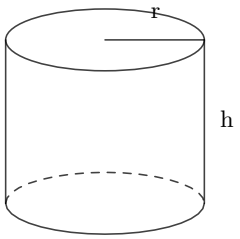
Formula Reference — Surface Area

Surface area (SA) is the total area of *all* the outside faces — the amount of wrapping paper needed. It is always in *square* units.

- **Prism** (base area B, base perimeter P, height h): $SA = 2B + Ph$.
- **Cube** (edge s): $SA = 6s^2$ (six equal square faces).
- **Cylinder** (radius r, height h): $SA = 2\pi r^2 + 2\pi rh$.
- **Pyramid** (base area B, base perimeter P, *slant height* ℓ): $SA = B + \frac{1}{2}P\ell$.
- **Cone** (radius r, slant height ℓ): $SA = \pi r^2 + \pi r\ell$.
- **Sphere** (radius r): $SA = 4\pi r^2$.

The **slant height** ℓ runs up the slanted *face* of a pyramid or cone — it is longer than the straight-up height.

A cylinder.



Rectangular prism

A box measures $\ell = 4$, $w = 3$, $h = 2$. Add the areas of all six faces (opposite faces match):

$$SA = 2(\ell w + \ell h + wh) = 2(4 \cdot 3 + 4 \cdot 2 + 3 \cdot 2) = 2(12 + 8 + 6) = 52 \text{ cm}^2.$$

Cylinder (with $\pi \approx 3.14$)

A can has radius $r = 3$ and height $h = 10$. The surface is two circular ends plus the wrapper:

$$SA = 2\pi r^2 + 2\pi rh \approx 2(3.14)(9) + 2(3.14)(3)(10) = 56.52 + 188.4 = 244.92 \text{ cm}^2.$$

Pyramid and cone (using slant height ℓ)

Square pyramid, base side 6 (so $B = 36$, $P = 24$), slant height $\ell = 5$:

$$SA = B + \tfrac{1}{2}P\ell = 36 + \tfrac{1}{2}(24)(5) = 36 + 60 = 96 \text{ cm}^2.$$

Cone, radius $r = 3$, slant height $\ell = 5$ ($\pi \approx 3.14$):

$$SA = \pi r^2 + \pi r\ell \approx 3.14(9) + 3.14(3)(5) = 28.26 + 47.1 = 75.36 \text{ cm}^2.$$

Sphere (with $\pi \approx 3.14$)

A ball has radius $r = 6$:

$$SA = 4\pi r^2 \approx 4(3.14)(6)^2 = 4(3.14)(36) = 452.16 \text{ cm}^2.$$

Remember: For a prism, Ph is the "label around the can" (the sides), and $2B$ is the two ends. For a pyramid or cone, use the *slant* height ℓ for the slanted faces — save the straight-up height for *volume*.

Try it: Quick check

Find the **area** of the same circle ($r = 5$ cm, $\pi \approx 3.14$).

Work it on paper before revealing the answer.

Answer. Answer: $A = 3.14(5)^2 = 78.5$ cm^2.

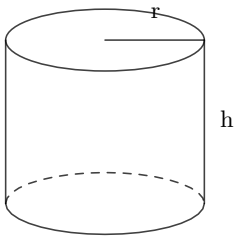
11.5 Volume of Solids

Formula Reference — Volume

Volume is how much space fills the inside — always in *cubic* units.

- **Prism or cylinder**: $V = Bh$, where B is the base area and h the height.
  - Rectangular prism: $V = \ell wh$. Cube: $V = s^3$. Cylinder: $V = \pi r^2 h$.
- **Pyramid or cone**: $V = \frac{1}{3}Bh$ (exactly one-third of the matching prism or cylinder).
  - Cone: $V = \frac{1}{3}\pi r^2 h$.
- **Sphere**: $V = \frac{4}{3}\pi r^3$.

A cylinder.



Prism, cylinder, cone, and sphere ($\pi \approx 3.14$)

Box $4 \times 3 \times 2$: $V = \ell wh = 4 \cdot 3 \cdot 2 = 24$ cm^3.

Cylinder $r = 3$, $h = 10$: $V = \pi r^2 h \approx 3.14(9)(10) = 282.6$ cm^3.

Cone $r = 3$, $h = 4$: $V = \frac{1}{3}\pi r^2 h \approx \frac{1}{3}(3.14)(9)(4) = \frac{1}{3}(113.04) = 37.68$ cm^3.

Sphere $r = 6$: $V = \frac{4}{3}\pi r^3 \approx \frac{4}{3}(3.14)(216) = \frac{4}{3}(678.24) = 904.32$ cm^3.

Pyramid: one-third of its box

A square pyramid has base 6×6 (so $B = 36$) and height $h = 10$:

$$V = \tfrac{1}{3}Bh = \tfrac{1}{3}(36)(10) = 120 \text{ cm}^3.$$

A full prism with that base and height would hold 360 cm^3 — the pyramid is exactly one-third of it.

Remember: A pyramid or cone always holds $\frac{1}{3}$ of the prism or cylinder that shares its base and height. If you forget the $\frac{1}{3}$, your answer will be three times too big!

Try it: Quick check

Find the **perimeter** of the same L-shaped figure (add up all the outside edges).

Work it on paper before revealing the answer.

Answer. Answer: $P = 8 + 3 + 5 + 2 + 3 + 5 = 26$ cm.

11.6 Scaling Dimensions & Real-World Problems

What Happens When You Resize a Shape

If you multiply *every* length of a figure by the same factor k:

- Perimeter (a length) is multiplied by k.
- Area is multiplied by k^2.
- Volume is multiplied by k^3.

So doubling the sides ($k = 2$) makes area $4\times$ and volume $8\times$ bigger.

Scaling a cube

A cube with edge 2 has volume $2^3 = 8$. Triple every edge to 6: the new volume is $6^3 = 216$. That is $216 \div 8 = 27$ times as large — and sure enough, $k^3 = 3^3 = 27$.

A real-world cost problem

A rectangular floor is 5 m by 4 m. Carpet costs \$3 per square meter. The area is $5 \times 4 = 20$ m^2,

so the cost is

$$20 \times \$3 = \$60.$$

Tip: Real-world word problems are just formula problems in a costume. Read carefully to spot which measurement you need — area for painting or carpeting, volume for filling or capacity — then round money sensibly and keep the units.

Try it: Quick check

Find the **surface area** of a rectangular prism that is 6 cm × 4 cm × 2 cm.

Work it on paper before revealing the answer.

Answer. Answer: $SA = 2(24 + 12 + 8) = 88$ cm^2.

11.7 Going Deeper: Advanced Area & Volume

Heron's Formula — Area from the Three Sides

Sometimes you know a triangle's three sides but *not* its height. **Heron's formula** finds the area directly. First compute the *semiperimeter* (half the perimeter):

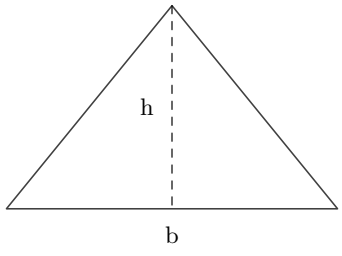
$$s = \frac{a+b+c}{2}.$$

Then the area is

$$A = \sqrt{s\,(s-a)(s-b)(s-c)}\,.$$

This always works for a valid triangle, and it never needs a perpendicular height.

Half of a rectangle with the same base and height.



Worked example: Heron's formula on a 13–14–15 triangle

A triangle has sides $a = 13$, $b = 14$, $c = 15$. First the semiperimeter:

$$s = \frac{13 + 14 + 15}{2} = \frac{42}{2} = 21.$$

Now the differences are $s - a = 8$, $s - b = 7$, $s - c = 6$, so

$$A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84 \text{ cm}^2.$$

From Coordinates: the Shoelace Formula & Pick's Theorem

When a polygon is given by the coordinates of its corners, two tools find its area without any height at all.

Shoelace formula. List the vertices in order (going around once) as $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then loop back to the first. Multiply "down the diagonals" and subtract:

$$A = \tfrac{1}{2}\left|\sum_{i=1}^{n} \left(x_i\, y_{i+1} - x_{i+1}\, y_i\right)\right|.$$

It is called the *shoelace* because the cross-multiplications criss-cross like laces.

Pick's theorem. If *every* corner sits on a grid (lattice) point, count I, the interior grid points, and B, the grid points *on* the boundary. Then

$$A = I + \frac{B}{2} - 1.$$

Two very different counts, one exact area — and they must agree.

Worked example: shoelace and Pick's theorem agree

Take the lattice triangle with corners $(0, 0)$, $(4, 0)$, and $(2, 3)$.

Shoelace (list the corners, wrap back to the start):

$$A = \tfrac{1}{2}\big|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\big|.$$

$$A = \tfrac{1}{2}\big|(0 \cdot 0 - 4 \cdot 0) + (4 \cdot 3 - 2 \cdot 0) + (2 \cdot 0 - 0 \cdot 3)\big| = \tfrac{1}{2}|0 + 12 + 0| = 6 \text{ units}^2.$$

Pick's theorem. The boundary points are the 3 corners plus the 3 extra grid points along the bottom edge from $(0, 0)$ to $(4, 0)$ — the slanted edges pass through no grid points — so $B = 6$. The interior points are $(1, 1), (2, 1), (3, 1), (2, 2)$, so $I = 4$:

$$A = I + \frac{B}{2} - 1 = 4 + \frac{6}{2} - 1 = 4 + 3 - 1 = 6 \text{ units}^2.$$

Both methods give 6.

Combined Solids — Deriving SA and Volume

A **combined (composite) solid** is glued from familiar solids: a cylinder capped by a hemisphere, an ice-cream cone (cone + hemisphere), a house (prism + pyramid), and so on. Treat it like a composite figure, but in 3D:

- **Volume:** *add* the volumes of the separate pieces. Nothing is lost where they join.
- **Surface area:** add only the surfaces that are *exposed* on the outside. The circle (or face) where two pieces meet is *inside* the solid — do *not* count it.

So for a cone stacked on a hemisphere of the same radius, the exposed surface is the cone's *slant* side $\pi r\ell$ plus the hemisphere's curved half $2\pi r^2$ — neither flat base circle is counted, because they are hidden at the join.

Worked example: cone on a hemisphere ($\pi \approx 3.14$)

A solid is a cone (radius $r = 3$, height $h = 4$, so slant height $\ell = 5$) sitting on a hemisphere of the same radius $r = 3$.

Volume = cone + hemisphere:

$$V = \tfrac{1}{3}\pi r^2 h + \tfrac{2}{3}\pi r^3 \approx \tfrac{1}{3}(3.14)(9)(4) + \tfrac{2}{3}(3.14)(27) = 37.68 + 56.52 = 94.2 \text{ cm}^3.$$

Surface area = cone's slant side + hemisphere's curved half (the shared circle is hidden):

$$SA = \pi r\ell + 2\pi r^2 \approx (3.14)(3)(5) + 2(3.14)(9) = 47.1 + 56.52 = 103.62 \text{ cm}^2.$$

Cross-Sections, Nets, and the Scaling Laws

Cross-section: the flat shape you see when you slice straight through a solid. Slicing a cylinder horizontally gives a *circle*; slicing it vertically through the middle gives a *rectangle*. Slicing a cube can give a square, a rectangle, or even a triangle or hexagon, depending on the angle.

Net: a solid unfolded flat, showing every face at once. A cube's net is 6 squares; a cylinder's net is two circles plus a rectangle whose *width equals the circumference* $2\pi r$. A net is the fastest way to *see* why a surface-area formula adds up — the total area of the net *is* the surface area.

Scaling: multiply every length by a factor k and lengths grow by k, areas by k^2, and volumes by k^3. A subtle consequence: the surface-area-to-volume ratio scales like $k^2/k^3 = 1/k$, so *bigger objects have relatively less surface* — which is why large animals hold heat and small ones lose it fast.

Worked example: an area optimization

A farmer has 40 m of fencing for a rectangular pen. Which rectangle encloses the *most* area?

With perimeter fixed at 40, the width w and length ℓ satisfy $\ell + w = 20$, so try a few:

The area is largest when $\ell = w = 10$ — a *square*. Among all rectangles of a given perimeter, the square always wins. (The same pattern holds in 3D: of all rectangular boxes with a fixed surface area, the cube holds the most volume.)

Big picture: Every advanced tool here is a shortcut around a missing height. Heron uses the three sides; the shoelace and Pick's theorem use coordinates; combined solids reuse the basic formulas piece by piece. And whenever a length is scaled by k, remember the ladder: length $\to k$, area $\to k^2$, volume $\to k^3$.

Try it: Quick check

Find the **surface area** of a sphere with radius 10 cm ($\pi \approx 3.14$).

Work it on paper before revealing the answer.

Answer. Answer: $SA = 4(3.14)(100) = 1256$ cm^2.

11.8 Why It Works

Why $\frac{1}{3}$? A cube can be cut into three identical pyramids that meet at the cube's center, each with a face of the cube as base and the center as apex — so each has volume $\frac{1}{3}$ of the cube, i.e. $\frac{1}{3}$ base $\times$ height. Cavalieri's principle (solids with equal cross-sections at every height have equal volumes) then extends the factor $\frac{1}{3}$ to every pyramid and cone. The sphere's $\frac{4}{3}\pi r^3$ comes from comparing its cross-sections with a cylinder minus a cone.

Reading a proof

Do not read a proof the way you read a rule. Cover the last line, ask what the previous line forces, and only then uncover it. If a step feels like magic, that is the step to work by hand with a concrete number.

12 Course Review

Every key definition, theorem, and formula from all ten units

12.1 Points, Lines, Planes & Segments

How to use this sheet. Each section is one unit. *Concept boxes* give the definitions and rules, *example boxes* show a worked calculation, and the red *tip box* at the end of each unit

warns you about the most common mistake. Keep answers exact (leave π and radicals) unless a decimal is requested.

The undefined terms and basic figures

- **Point:** a location, no size (named A). **Line:** extends forever in two directions ($\overleftrightarrow{AB}$). **Plane:** a flat surface extending forever.
- **Collinear** points lie on one line; **coplanar** points lie in one plane.
- **Segment** $\overline{AB}$: two endpoints and all points between (has length AB). **Ray** $\overrightarrow{AB}$: one endpoint, goes forever one way.
- **Congruent** segments have equal length: $\overline{AB} \cong \overline{CD}$ means $AB = CD$.

Segment addition, midpoint, and distance

- **Segment Addition Postulate:** if B is between A and C, then $AB + BC = AC$.
- **Midpoint** M of $\overline{AB}$: the point with $AM = MB$ (it *bisects* the segment).

On a coordinate plane, for $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right)$$

$$\text{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint and distance

For $A(1, 2)$ and $B(7, 10)$:

$$M = \left(\tfrac{1+7}{2}, \tfrac{2+10}{2}\right) = (4, 6), \qquad AB = \sqrt{(7-1)^2 + (10-2)^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

Midpoint adds, distance subtracts. The midpoint formula *averages* (add the coordinates, divide by 2); the distance formula *subtracts* coordinates first, then squares. Squaring makes signs irrelevant, so order of subtraction does not matter.

12.2 Angles & Angle Relationships

Types of angles

Measured in degrees (°): **acute** $< 90°$, **right** $= 90°$, **obtuse** between $90°$ and $180°$, **straight** $= 180°$. An **angle bisector** splits an angle into two equal halves.

Angle pairs

- **Complementary:** two angles that add to $90°$.
- **Supplementary:** two angles that add to $180°$.
- **Linear pair:** adjacent angles on a straight line — they are supplementary ($= 180°$).
- **Vertical angles:** the opposite angles formed by two crossing lines — they are *congruent* (equal).
- **Angle Addition Postulate:** if D is inside $\angle ABC$, then $\angle ABD + \angle DBC = \angle ABC$.

Vertical and linear pair

Two lines cross. One angle is $70°$. Its *vertical* angle is also $70°$. Each angle next to it (linear pair) is $180° - 70° = 110°$.

Vertical vs. linear. Vertical angles are *equal*; a linear pair is *supplementary* (adds to $180°$). Do not confuse "complementary" ($90°$) with "supplementary" ($180°$) — c comes before s, and 90 before 180.

12.3 Parallel & Perpendicular Lines

Angles formed by a transversal

When a **transversal** cuts two **parallel** lines, these pairs are formed:

- **Corresponding angles:** congruent (equal).
- **Alternate interior angles:** congruent.
- **Alternate exterior angles:** congruent.
- **Co-interior (same-side interior) angles:** supplementary ($= 180°$).

The converse also works: if any of these relationships holds, the lines *are* parallel.

Slopes of parallel and perpendicular lines

For slope m (recall $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, "rise over run"):

- **Parallel** lines have *equal* slopes: $m_1 = m_2$.
- **Perpendicular** lines have *opposite-reciprocal* slopes: $m_1 \cdot m_2 = -1$, i.e. $m_2 = -\dfrac{1}{m_1}$.

Horizontal ($m = 0$) and vertical (undefined slope) lines are perpendicular to each other.

Parallel and perpendicular slopes

A line has slope $m = \frac{2}{3}$. A line *parallel* to it also has slope $\frac{2}{3}$. A line *perpendicular* to it has slope $-\frac{3}{2}$ (flip and negate).

Flip and negate. Perpendicular slope = reciprocal *with the sign changed*. Just flipping ($\frac{2}{3} \to \frac{3}{2}$) or just negating ($\frac{2}{3} \to -\frac{2}{3}$) is not enough — you must do *both*.

12.4 Triangles & Congruence

Angle facts in triangles

- **Triangle Angle Sum:** the three interior angles add to 180°.
- **Exterior Angle Theorem:** an exterior angle equals the sum of the two *remote* (non-adjacent) interior angles.
- **Classify by sides:** scalene (none equal), isosceles (two equal), equilateral (all equal). **By angles:** acute, right, obtuse.

Proving triangles congruent

Two triangles are congruent if any one of these matches:

- **SSS** — three sides
- **SAS** — two sides + included angle
- **ASA** — two angles + included side
- **AAS** — two angles + a non-included side
- **HL** — (right triangles) hypotenuse + a leg

SSA and **AAA** do *not* prove congruence. After proving congruence, **CPCTC** lets you conclude the matching parts are equal.

Isosceles triangle & the triangle inequality

- **Isosceles Triangle Theorem:** the angles opposite the two equal sides (*base angles*) are equal — and the converse holds.
- **Triangle Inequality:** the sum of any two sides is greater than the third: $a + b > c$. The third side lies between $|a - b|$ and $a + b$.

Missing angle and side range

A triangle has angles $50°$ and $60°$; the third is $180° - 50° - 60° = 70°$. If two sides are 7 and 10, the third side x satisfies $10 - 7 < x < 10 + 7$, i.e. $3 < x < 17$.

"Included" is everything. SAS needs the angle *between* the two sides; ASA needs the side *between* the two angles. If the parts are not in that position, the shortcut does not apply (that is why SSA fails).

12.5 Similarity

Similar triangles

Similar figures ($\sim$) have **equal corresponding angles** and **proportional corresponding sides**. Prove triangles similar by:

- **AA** — two pairs of equal angles.
- **SSS$\sim$** — all three side ratios equal.
- **SAS$\sim$** — two side ratios equal with equal included angle.

The common ratio of the sides is the **scale factor** k.

Proportions, midsegment, and scaling

- Solve proportions by **cross-multiplying**: $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow ad = bc$.
- **Triangle Midsegment:** the segment joining midpoints of two sides is *parallel* to the third side and *half* its length.
- **Side-Splitter:** a line parallel to one side cuts the other two sides proportionally.

For scale factor k: ratio of **perimeters** $= k$, ratio of **areas** $= k^2$, ratio of **volumes** $= k^3$.

Scale factor and area

Two similar triangles have sides in ratio $k = \frac{3}{2}$. Their perimeters are in ratio $\frac{3}{2}$, and their areas are in ratio $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Square the scale factor for area. If lengths scale by k, area scales by k^2 (and volume by k^3). Doubling every side does *not* double the area — it makes it 4 times bigger.

12.6 Right Triangles & Trigonometry

Pythagorean Theorem and its converse

In a right triangle with legs a, b and hypotenuse c (the side opposite the right angle):

$$a^2 + b^2 = c^2.$$

Converse: if $a^2 + b^2 = c^2$, the triangle is right. If $a^2 + b^2 > c^2$ it is acute; if $a^2 + b^2 < c^2$ it is obtuse. Common **triples:** 3-4-5, 5-12-13, 8-15-17, 7-24-25 (and their multiples).

Special right triangles

$$\begin{aligned} &45°\text{-}45°\text{-}90°: && \text{legs equal } x,\ \text{hypotenuse } = x\sqrt{2} \\ &30°\text{-}60°\text{-}90°: && \text{short leg } x,\ \text{long leg } = x\sqrt{3},\ \text{hyp } = 2x \end{aligned}$$

(In 30-60-90 the short leg is opposite 30°; the long leg is opposite 60°.)

Trigonometric ratios (SOH-CAH-TOA)

For an acute angle θ in a right triangle:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

To find a missing *angle*, use **inverse trig**: $\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right)$, and likewise $\cos^{-1}$, $\tan^{-1}$.

Finding a side and an angle

A right triangle has an angle θ with opposite side 3 and hypotenuse 5. Then $\sin\theta = \frac{3}{5}$, so $\theta = \sin^{-1}(0.6) \approx 36.9°$. The adjacent leg is $\sqrt{5^2 - 3^2} = 4$.

Ratio for sides, inverse for angles. If you know the angle and want a side, use sin / cos / tan. If you know two sides and want the angle, use $\sin^{-1}$ / $\cos^{-1}$ / $\tan^{-1}$. Set your calculator to *degrees*.

12.7 Quadrilaterals & Polygons

Angle sums in any polygon

For a polygon with n sides:

$$\text{sum of interior angles} = (n-2)\cdot 180^\circ, \qquad \text{sum of exterior angles} = 360^\circ.$$

For a **regular** polygon: each interior angle $= \dfrac{(n-2)180^\circ}{n}$, each exterior angle $= \dfrac{360^\circ}{n}$.

Parallelogram properties

In every **parallelogram**: opposite sides are parallel *and* congruent; opposite angles are congruent; consecutive angles are supplementary; and the *diagonals bisect each other*.

The special quadrilaterals

- **Rectangle:** parallelogram with 4 right angles; diagonals are *congruent*.
- **Rhombus:** parallelogram with 4 equal sides; diagonals are *perpendicular* and bisect the angles.
- **Square:** both a rectangle and a rhombus (all properties of each).
- **Trapezoid:** exactly one pair of parallel sides; *isosceles* trapezoid has congruent legs, congruent base angles, and congruent diagonals.
- **Kite:** two pairs of adjacent congruent sides; diagonals are perpendicular.

Interior angle of a regular polygon

A regular hexagon has $n = 6$: interior-angle sum $= (6-2)180^\circ = 720^\circ$, so each angle $= \frac{720^\circ}{6} = 120^\circ$. Each exterior angle $= \frac{360^\circ}{6} = 60^\circ$.

Exterior angles always total 360°. No matter how many sides, the exterior angles sum to 360°. That is the fast way to find a regular polygon's angle: exterior $= \frac{360^\circ}{n}$, then interior $= 180^\circ -$ exterior.

12.8 Circles

Circumference, area, and arcs

With radius r and diameter $d = 2r$:

$$C = 2\pi r = \pi d, \qquad A = \pi r^2.$$

For a central angle of n°: arc length $= \dfrac{n}{360} \cdot 2\pi r$, sector area $= \dfrac{n}{360} \cdot \pi r^2$.

Central, inscribed, and other angles

- **Central angle** (vertex at center) = its intercepted arc.
- **Inscribed angle** (vertex on circle) = $\frac{1}{2}$ its intercepted arc. An angle inscribed in a semicircle is 90°.
- **Tangent** $\perp$ radius at the point of tangency; two tangents from one outside point are congruent.

Power of a Point

From a point, the products of the two chord/secant/tangent pieces are equal:

$$\begin{aligned} \text{Two chords:} &\quad a \cdot b = c \cdot d \\ \text{Two secants:} &\quad (\text{whole}_1)(\text{outside}_1) = (\text{whole}_2)(\text{outside}_2) \\ \text{Tangent \& secant:} &\quad t^2 = (\text{whole})(\text{outside}) \end{aligned}$$

Equation of a circle

A circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2.$$

The center uses the *opposite* sign of what appears; the right side is r^2, so take a square root for r.

Inscribed angle and a circle equation

An inscribed angle intercepting a 100° arc measures $\frac{1}{2}(100^\circ) = 50^\circ$. The equation $(x - 2)^2 + (y + 3)^2 = 25$ has center $(2, -3)$ and radius $\sqrt{25} = 5$.

Central equals, inscribed halves. A central angle *equals* its arc; an inscribed angle is *half* its arc. And length answers (circumference, arc) use plain units, while area answers (circle, sector) use *square* units.

12.9 Perimeter, Area & Volume

Area of 2D figures

- Rectangle: $A = bh$
- Parallelogram: $A = bh$
- Triangle: $A = \frac{1}{2}bh$
- Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$
- Rhombus / kite: $A = \frac{1}{2}d_1 d_2$
- Circle: $A = \pi r^2$
- Regular polygon: $A = \frac{1}{2}(\text{apothem})(\text{perimeter})$

Perimeter is the distance around; the **circumference** $C = 2\pi r$ is a circle's perimeter.

Surface area and volume of solids

Let B = area of the base, P = base perimeter, h = height, ℓ = slant height.

Prism:	$V = Bh$,	$SA = 2B + Ph$
Cylinder:	$V = \pi r^2 h$,	$SA = 2\pi r^2 + 2\pi rh$
Pyramid:	$V = \frac{1}{3}Bh$,	$SA = B + \frac{1}{2}P\ell$
Cone:	$V = \frac{1}{3}\pi r^2 h$,	$SA = \pi r^2 + \pi r\ell$
Sphere:	$V = \frac{4}{3}\pi r^3$,	$SA = 4\pi r^2$

Volume of a cylinder and a cone

A cylinder with $r = 3$, $h = 10$: $V = \pi(3)^2(10) = 90\pi$. A cone with the same base and height holds one third as much: $V = \frac{1}{3}\pi(3)^2(10) = 30\pi$.

The one-third family. Pyramids and cones each get a $\frac{1}{3}$ compared to the prism or cylinder with the same base and height. Area uses *square* units; volume uses *cubic* units — check your units to catch errors.

12.10 Transformations & Symmetry

Rigid motions (isometries)

These preserve size and shape (image $\cong$ pre-image):

$$\begin{aligned}
\text{Translation by } (a,b): &\quad (x,y) \to (x+a,\ y+b)\\
\text{Reflection over } x\text{-axis}: &\quad (x,y) \to (x,\ -y)\\
\text{Reflection over } y\text{-axis}: &\quad (x,y) \to (-x,\ y)\\
\text{Reflection over } y=x: &\quad (x,y) \to (y,\ x)\\
\text{Reflection over } y=-x: &\quad (x,y) \to (-y,\ -x)
\end{aligned}$$

Rotations and dilations

Rotations about the origin (counterclockwise):

$$\begin{aligned}
90^\circ: &\quad (x,y) \to (-y,\ x)\\
180^\circ: &\quad (x,y) \to (-x,\ -y)\\
270^\circ: &\quad (x,y) \to (y,\ -x)
\end{aligned}$$

Dilation (center origin, scale factor k): $(x,y) \to (kx,\ ky)$. This is *not* rigid — it changes size (lengths $\times k$, area $\times k^2$) but keeps shape, so figures are *similar*.

Symmetry

- **Line (reflection) symmetry:** a fold line maps the figure onto itself.
- **Rotational symmetry:** a turn of less than 360° about a center maps the figure onto itself (a regular n-gon has order n, smallest angle $\frac{360^\circ}{n}$).

Applying a rule

Reflect $P(3,5)$ over the x-axis: $(x,y) \to (x,-y)$ gives $P'(3,-5)$. Then rotate P' by 90°: $(x,y) \to (-y,x)$ gives $(5,3)$.

Rigid motions keep congruence; dilations make similarity. Translations, reflections, and rotations preserve length and angle (image $\cong$ pre-image). Only a dilation changes size — and it scales lengths by k, area by k^2.