MathCastle

USAMO

Day I and Day II

6 problems • 9 hours

Name _____

INSTRUCTIONS

1. Two days of 3 problems each, 4.5 hours per day.
2. Each problem is worth 7 points. Full credit needs a complete, rigorous proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Day I is handed in before Day II is opened.
5. Two sittings of 3 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Problem 1.

Let $\mathbb{Z}_{12}$ denote the integers modulo 12, that is, the set $\{0, 1, 2, \ldots, 11\}$ with addition taken modulo 12. How many functions $f\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ satisfy $f(x + f(y)) = f(x) + f(y)$ for all $x, y \in \mathbb{Z}_{12}$?

Problem 2.

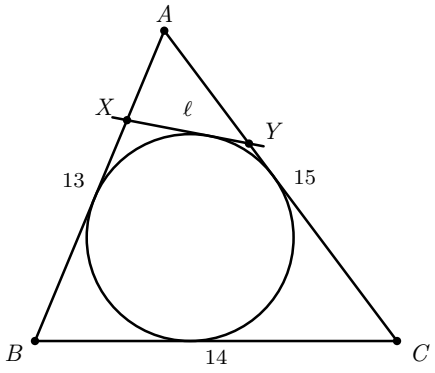
A binary string w of length n is called *bordered* if for some k with $1 \le k \le n-1$ the first k characters of w, read in order, are the same as its last k characters, read in order. For example, 0110 is bordered (take $k = 1$), 10110 is bordered (take $k = 2$), and 0011 is not bordered. A string that is not bordered is *unbordered*. How many binary strings of length 16 are unbordered?

Problem 3.

A set S of residues modulo 45 is called *product-free* if there do not exist $x, y, z \in S$ (not necessarily distinct) with $xy \equiv z \pmod{45}$. In particular, if $x \in S$ then $x^2 \bmod 45$ is not in S. What is the largest possible number of elements of a product-free set of residues modulo 45?

Problem 4.

Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. A line ℓ is tangent to the incircle of triangle ABC, with vertex A and the incircle on opposite sides of ℓ, and ℓ meets side $\overline{AB}$ at X and side $\overline{AC}$ at Y. As ℓ varies, what is the largest possible area of triangle AXY? Give the answer in simplest radical form.



Problem 5.

A set S of cells of a 5×5 grid is called *closed* if the following holds: whenever two distinct rows r, r' and two distinct columns c, c' are such that three of the four cells (r, c), (r, c'), (r', c), (r', c') lie in S, the fourth cell also lies in S. How many closed sets are there? Count the empty set and the full grid.

Problem 6.

How many sequences $(a_1, a_2, a_3, \ldots)$ of integers with $0 \le a_n \le 4$ for every $n \ge 1$ have the property that every coefficient of the power series $\exp\left(\sum_{n\ge1} \frac{a_n}{n}\, x^n\right)$ is an integer? (Here $\exp(S) = \sum_{k\ge0} \frac{S^k}{k!}$ is taken as a formal power series in x; the series S has no constant term, so every coefficient of $\exp(S)$ is a finite sum.)

Answer Key — USAMO Practice Contest 4

Problem 1. 258

Problem 2. 17622

Problem 3. 20

Problem 4. $\frac{324-32\sqrt{65}}{7}$

Problem 5. 48032

Problem 6. 27

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Solutions — USAMO Practice Contest 4

Problem 1. **Answer:** 258. **The idea.** The equation says: adding a value t of f to the input adds t to the output. So $f(x) - x$ cannot change when x moves by anything in the subgroup H generated by the image. Then f is "the identity plus a constant" on each coset of H, and the requirement that all values lie in H pins down which constants are allowed. The answer is a sum over the subgroups of $\mathbb{Z}_{12}$.

Work it out. Let I be the image of f and H the subgroup it generates. In a finite cyclic group, $-t = 11t$, so H consists of all sums $t_1 + \cdots + t_k$ of elements of I (repeats allowed).

Step 1: $f(x + h) = f(x) + h$ for every $h \in H$. For $t = f(y) \in I$ the equation reads $f(x + t) = f(x) + t$ for all x. Applying it repeatedly gives $f(x + t_1 + \cdots + t_k) = f(x) + t_1 + \cdots + t_k$, which covers every $h \in H$.

Step 2: the shape of f. By Step 1, $g(x) = f(x) - x$ is constant on each coset C of H; call the constant g_C. Every value $f(x) = x + g_C$ lies in $I \subseteq H$, so for $x \in C$ we need $g_C \in -x + H = -C$. Conversely, if $g_C \in -C$ then f maps C onto $C + g_C = H$. So the image of f is exactly H.

Step 3: every such choice works. Fix a subgroup H and, for each coset C, any $g_C \in -C$, and put $f(x) = x + g_C$ for $x \in C$. Its image is H. For any x, y we have $f(y) = h \in H$, so $x + h$ lies in the same coset as x and $f(x + h) = x + h + g_C = f(x) + f(y)$. The equation holds.

Step 4: count. Different subgroups give different functions, because H is the image. A subgroup of order d has $12/d$ cosets and d choices of g_C for each, giving $d^{12/d}$ functions. The subgroups of $\mathbb{Z}_{12}$ have orders $d = 1, 2, 3, 4, 6, 12$, one of each, so the total is $1^{12} + 2^6 + 3^4 + 4^3 + 6^2 + 12^1 = 1 + 64 + 81 + 64 + 36 + 12 = 258$.

Check. $H = \{0\}$ gives only $f \equiv 0$. $H = \mathbb{Z}_{12}$ gives the 12 translations $f(x) = x + c$, and each clearly works. For $\mathbb{Z}_4$ the same count is $1 + 2^2 + 4 = 9$, and a brute-force check of all $4^4 = 256$ functions finds exactly 9. The answer is $\boxed{258}$.

Problem 2. **Answer:** 17622. **The idea.** Classify each bordered word by its *shortest* border. That border is itself unbordered and fits in the first half of the word. Conversely, any unbordered word placed at both ends of a longer word is that word's shortest border. So the bordered words of length n are counted exactly by the unbordered words of length at most $n/2$, which gives a recursion.

Work it out. A *border* of w is a word that is both a proper prefix and a proper suffix of w. Let u_n be the number of unbordered words of length n.

Lemma 1: a border of a border of w is a border of w. If v is a prefix and a suffix of w, then any prefix of v is a prefix of w, and any suffix of v is a suffix of w.

Lemma 2: the shortest border v of a bordered word w is unbordered, and $|v| \le n/2$. By Lemma 1, a border of v would be a shorter border of w, so v is unbordered. Suppose $k = |v| > n/2$ and put $p = n - k < k$. The first copy of v occupies positions $0, \ldots, k - 1$ of w and the second copy positions $p, \ldots, n - 1$. The prefix of v of length $2k - n$ is the start of the second copy, namely $w_p w_{p+1} \cdots w_{k-1}$. The suffix of v of length $2k - n$ is the

end of the first copy, which is the same stretch $w_p \cdots w_{k-1}$. Since $1 \le 2k-n < k$, this makes v bordered, a contradiction.

Lemma 3: if v is unbordered, $|v| = k \le n/2$ and x is any string of length $n-2k$, then v is the shortest border of $w = v\,x\,v$. Clearly v is a border of w. A shorter border u (length $j < k$) would be a prefix of w of length less than k, hence a prefix of v, and likewise a suffix of v. Then v would be bordered.

So sending a bordered word of length n to (its shortest border v, the middle part x) is a bijection onto the pairs with v unbordered of length $k \le n/2$ and x arbitrary of length $n-2k$. Hence $u_n = 2^n - \sum_{k=1}^{\lfloor n/2 \rfloor} u_k\, 2^{\,n-2k}$. Comparing this formula for $n+1$ and for n gives $u_{n+1} = 2u_n$ when $n+1$ is odd, and $u_{n+1} = 2u_n - u_{(n+1)/2}$ when $n+1$ is even. Starting from $u_1 = 2$: $u_1, \ldots, u_{16} = 2, 2, 4, 6, 12, 20, 40, 74, 148, 284, 568, 1116, 2232, 4424, 8848, 17622$. For instance $u_{14} = 2 \cdot 2232 - u_7 = 4464 - 40 = 4424$, $u_{15} = 8848$ and $u_{16} = 2 \cdot 8848 - u_8 = 17696 - 74 = 17622$.

Check. Length 4: the unbordered words must start and end with different letters, so they are $0{*}{*}1$ or $1{*}{*}0$. Of these, 0101 and 1010 have the border of length 2, which leaves $8 - 2 = 6 = u_4$. A computer run over all 2^{16} strings also finds 17622. The answer is $\boxed{17622}$.

Problem 3. Answer: 20. **The idea.** The natural guess is to copy a product-free set from a prime factor. The residues that are 2 or 3 mod 5 (the non-squares mod 5) give 18. Modulo 9 one can do better: besides the three units $2, 5, 8$ (the units that are 2 mod 3), the residue $6 = 3 \cdot 2$ can be added too. This gives the 4-element product-free set $\{2, 5, 6, 8\}$, a density of $4/9 > 2/5$. For the upper bound: if S contains a unit s, then $x \mapsto sx$ sends S into its complement and keeps $\gcd(x, 45)$ fixed, so S has at most half of each "gcd class". The only class where this still leaves room is the multiples of 9, and the proof shows that using them costs more elsewhere than it gains.

Work it out. *Construction.* Let $S = \{x : x \bmod 9 \in \{2, 5, 6, 8\}\}$, which has $4 \cdot 5 = 20$ elements. For $x, y \in S$ the residue $xy \bmod 9$ is a product of two numbers from $\{2, 5, 6, 8\}$ modulo 9. Two of $2, 5, 8$ multiply to a unit that is 1 mod 3, so to 1, 4 or 7. Next, $6 \cdot 2, 6 \cdot 5, 6 \cdot 8 \equiv 3$ and $6 \cdot 6 \equiv 0$. None of $1, 4, 7, 3, 0$ lies in $\{2, 5, 6, 8\}$, so $xy \notin S$.

Upper bound. Let S be product-free. Note $0 \notin S$, since $0 \cdot 0 = 0$. Sort the residues by $d = \gcd(x, 45)$ into the classes D_1 (the 24 units), D_3 (8 elements), D_5 (6), $D_9 = \{9, 18, 27, 36\}$, $D_{15} = \{15, 30\}$ and $D_{45} = \{0\}$.

Case A: S has no unit. Then $S \subseteq D_3 \cup D_5 \cup D_9 \cup D_{15}$, which has 20 elements.

Case B: S contains a unit s. Multiplication by s permutes each class D_d, and $s \cdot (S \cap D_d)$ is disjoint from S. Hence $|S \cap D_d| \le |D_d|/2$, so $|S \cap D_1| \le 12$, $|S \cap D_3| \le 4$, $|S \cap D_5| \le 3$, $|S \cap D_9| \le 2$ and $|S \cap D_{15}| \le 1$. If $S \cap D_9 = \emptyset$, then $|S| \le 12 + 4 + 3 + 1 = 20$.

So suppose $z \in S \cap D_9$. A key observation: if $x \in S$ and $x \equiv 1 \pmod 5$, then $xz \equiv 0 \equiv z \pmod 9$ and $xz \equiv z \pmod 5$, so $xz \equiv z \in S$, which is impossible. *Thus no element of S is* 1 mod 5. More generally, for $x \in S$ with $5 \nmid x$, the product xz lies in D_9 and $xz \equiv x \cdot z \pmod 5$. The elements of D_9 are $36, 27, 18, 9 \equiv 1, 2, 3, 4 \pmod 5$.

Case B1: $|S \cap D_9| = 2$. Neither element is $36 \equiv 1$. The pair $\{9, 27\}$ fails because $27^2 \equiv 9$, and $\{9, 18\}$ fails because $18^2 \equiv 9$. So $S \cap D_9 = \{18, 27\}$. If some $x \in S$ had $x \equiv 4 \pmod 5$, then $27x \in D_9$ and $27x \equiv 2 \cdot 4 \equiv 3 \pmod 5$, so $27x \equiv 18 \in S$, which is

impossible. So every element of S not divisible by 5 is 2 or 3 mod 5. For $c \in \{2,3\}$, let S_c be the set of elements of S that are c mod 5. Let Z be the set of multiples of 5 in S. Then $|S| = |S_2| + |S_3| + |Z|$ with $|S_c| \le 9$. Suppose Z contains some $w \in D_5$, so that w is a unit mod 9. The 9 residues that are c mod 5 are distinct mod 9, so $x \mapsto wx$ is injective on them, with values among the 9 multiples of 5. Since wS_c avoids Z, we get $|S_c| \le 9 - |Z|$, and so $|S| \le 18 - |Z| < 20$. Otherwise $Z \subseteq D_{15}$, so $|Z| \le 1$ and $|S| \le 19$.

Case B2: $|S \cap D_9| = 1$. The class bounds give $|S| \le 12+4+3+1+1 = 21$, and $|S| \le 20$ unless $|S \cap D_1| = 12$. So assume $A = S \cap D_1$ has 12 elements, exactly half of the unit group U. Every $a \in A$ gives $aA \subseteq U \setminus A$ with both sides of size 12, so $aA = U \setminus A$. For $a, b \in A$ this gives $aA = bA$, so ab^{-1} lies in the subgroup $K = \{k \in U : kA = A\}$. Then $A \subseteq bK$, and A is a union of cosets of K, so $A = bK$ is a single coset of a subgroup of order 12. Moreover $A \neq K$, since $1 \in A$ would give $1 \cdot 1 = 1 \in S$. So $A = U \setminus K$, where K is one of the three subgroups of index 2 in $U \cong \mathbb{Z}_9^\times \times \mathbb{Z}_5^\times$: the units that are 1 mod 3, the units that are ± 1 mod 5, or the units for which both conditions hold or neither does. The first and third choices make A contain units that are 1 mod 5, which we have excluded. So A consists of all units that are 2 or 3 mod 5, and these cover every unit residue mod 9. Now if $w \in S \cap D_5$, then $\{wa : a \in A\}$ consists of multiples of 5 that are prime to 3, with every unit residue mod 9, so it is all of D_5, including w itself, which is impossible. Hence $S \cap D_5 = \emptyset$ and $|S| \le 12 + 4 + 0 + 1 + 1 = 18$.

In every case $|S| \le 20$, and the construction attains 20.

Check. The same method gives 4 for modulus 9 (the set $\{2,5,6,8\}$) and 2 for modulus 5. So 45 inherits the better density 4/9, giving $4 \cdot 5 = 20$ and beating the $2 \cdot 9 = 18$ of the non-squares mod 5. An exhaustive computer search over subsets of $\mathbb{Z}_{45}$ confirms the maximum. The answer is $\boxed{20}$.

Problem 4. Answer: $\frac{324-32\sqrt{65}}{7}$**. The idea.** Seen from triangle AXY, the incircle of ABC is the excircle opposite A. That fixes the perimeter of AXY (it is twice the tangent length from A). It also makes the area a decreasing function of XY alone: area $= r(s' - XY)$. So we need the shortest possible XY for a fixed perimeter and a fixed angle A, and the law of cosines shows this happens for the isosceles cut.

Work it out. Here $s = 21$, the area is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, and $r = 84/21 = 4$. The incircle touches AB at F and AC at E with $AF = AE = s - BC = 7$.

Perimeter. Let ℓ touch the incircle at P. Since ℓ separates A from the circle, it crosses segments AF and AE, so X lies between A and F, and Y between A and E. Equal tangents give $XP = XF$ and $YP = YE$, so $AX + XY + AY = AX + XF + YE + AY = AF + AE = 14$.

Area. Let I be the incentre. It lies beyond XY as seen from A, and its distances to lines AB, AC and ℓ all equal r. So $[AXY] = [AXI] + [AYI] - [XYI] = \frac{r}{2}(AX + AY - XY) = \frac{r}{2}(14 - 2XY) = 4(7 - XY)$.

Minimising XY. Here $\cos A = \frac{13^2+15^2-14^2}{2 \cdot 13 \cdot 15} = \frac{33}{65}$, so $\sin^2 \frac{A}{2} = \frac{1-\cos A}{2} = \frac{16}{65}$ and $\sin \frac{A}{2} = \frac{4}{\sqrt{65}}$. Put $p = AX + AY$ and $q = XY$. By the law of cosines and $AX \cdot AY \le p^2/4$, $q^2 = p^2 - 2\,AX \cdot AY\,(1 + \cos A) \ge p^2 - \frac{p^2}{2}(1 + \cos A) = p^2 \sin^2 \frac{A}{2}$, with equality exactly when $AX = AY$. With $p = 14 - q$ this gives $q \ge (14 - q) \cdot \frac{4}{\sqrt{65}}$, so $q \ge \frac{56}{\sqrt{65}+4} = \frac{56(\sqrt{65}-4)}{49} = \frac{8(\sqrt{65}-4)}{7}$. Equality is attained by the tangent perpendicular to AI at the

point of the incircle nearest A. By symmetry it gives $AX = AY = 7 - q/2$, which is between 0 and 7, so X and Y really lie on the sides.

Answer. The maximum area is $4\left(7 - \frac{8(\sqrt{65}-4)}{7}\right) = \frac{4(81-8\sqrt{65})}{7} = \frac{324-32\sqrt{65}}{7} \approx 9.43$.

Check. Compute the isosceles case directly. $AI = \sqrt{7^2+4^2} = \sqrt{65}$, so the distance from A to ℓ is $h = \sqrt{65} - 4$. The area of the isosceles triangle with apex angle A and height h is $h^2 \tan\frac{A}{2} = (81 - 8\sqrt{65}) \cdot \frac{4}{7}$, the same value. A 60-digit numerical search over all admissible tangent lines agrees. The answer is $\boxed{\dfrac{324 - 32\sqrt{65}}{7}}$.

Problem 5. **Answer:** 48032. **The idea.** Describe S row by row: row r contributes the set R_r of columns where it has a cell of S. Closure turns out to say exactly that any two of these column sets are either *equal or disjoint*. So a closed set is a family of disjoint "column blocks", each copied into some rows. That is a pair of set partitions matched up, counted with Stirling numbers.

Work it out. *Characterisation.* Suppose S is closed and rows $r \neq r'$ share a column c, meaning $c \in R_r \cap R_{r'}$. For any other $c' \in R_r$, the cells (r,c), (r,c') and (r',c) lie in S, so (r',c') does too, that is, $c' \in R_{r'}$. So $R_r \subseteq R_{r'}$, and by symmetry $R_r = R_{r'}$. Conversely, suppose any two row sets are equal or disjoint, and three corners $(r,c),(r,c'),(r',c)$ lie in S. Then R_r and $R_{r'}$ share c, so they are equal, and $c' \in R_{r'}$ puts the fourth corner in S. Hence S is closed exactly when the nonempty row sets are pairwise equal or disjoint.

Counting. Let the distinct nonempty row sets be $C_1, \ldots, C_k$ with $0 \le k \le 5$. These are pairwise disjoint nonempty sets of columns, with some columns possibly unused. Adding a phantom element $*$ to the 5 columns, such a family corresponds to a partition of the 6 elements into $k+1$ blocks (the block containing $*$ holds the unused columns). So there are $S(6,k+1)$ choices, where $S(n,j)$ is a Stirling number of the second kind. Next, each row gets one of $C_1, \ldots, C_k$ or the empty set, and each C_i must be used. Adding a phantom $*$ to the rows, this is a partition of 6 elements into $k+1$ blocks (the $*$ block holds the empty rows), together with a matching of the other k blocks to $C_1, \ldots, C_k$. That gives $k!\,S(6,k+1)$ ways. So the number of closed sets is $\sum_{k=0}^{5} k!\,S(6,k+1)^2$. With $S(6,1), \ldots, S(6,6) = 1, 31, 90, 65, 15, 1$ this is $1 + 961 + 2\cdot 8100 + 6\cdot 4225 + 24\cdot 225 + 120 = 1 + 961 + 16200 + 25350 + 5400 + 120 = 48032$.

Check. For a 2×2 grid the formula gives $\sum_k k!\,S(3,k+1)^2 = 1 + 9 + 2 = 12$. Directly, of the 16 subsets only the four 3-cell sets fail, leaving 12. A computer that builds the 5×5 grid row by row and tests every rectangle also finds 48032. The answer is $\boxed{48032}$.

Problem 6. **Answer:** 27. **The idea.** Rewrite the exponential as an Euler product $\prod_{d \ge 1} (1 - x^d)^{-c_d}$. Integrality turns out to mean exactly that all the exponents c_d are integers, that is, $a_n = \sum_{d \mid n} d\, c_d$ with $c_d \in \mathbb{Z}$. (If $c_d \ge 0$ counts the cycles of length d of a permutation, then a_n counts its points fixed by the n-th power; here the c_d may be negative.) Boundedness is then very restrictive: $a_{p^j k} - a_{p^{j-1} k}$ is a multiple of p^j, so every c_n vanishes except for $n \mid 12$. The sequence then depends only on $\gcd(n, 12)$, and a finite count finishes.

Work it out. *Step 1: the Euler product.* For integers c_d, $-\sum_d c_d \log(1 - x^d) = \sum_d c_d \sum_{k \ge 1} \frac{x^{dk}}{k} = \sum_{n \ge 1} \left(\sum_{d \mid n} d\, c_d \right) \frac{x^n}{n}$. So if $a_n = \sum_{d \mid n} d\, c_d$ with all c_d integers, then $\exp\left(\sum a_n x^n / n\right) = \prod_d (1 - x^d)^{-c_d}$. Each factor has integer coefficients and constant term

1 (for $c_d < 0$ it is a polynomial, for $c_d > 0$ a power of $1 + x^d + x^{2d} + \cdots$), and only the factors with $d \le N$ affect the coefficient of x^N. So the series has integer coefficients.

Conversely, let $F = \exp\left(\sum a_n x^n/n\right)$ have integer coefficients. Build integers $c_1, c_2, \ldots$ one at a time. Suppose $G = F\prod_{d<n}(1-x^d)^{c_d}$ has integer coefficients and $G \equiv 1 \pmod{x^n}$, so $G \equiv 1 + gx^n \pmod{x^{n+1}}$ with $g \in \mathbb{Z}$. Put $c_n = g$. Then $G(1-x^n)^{c_n} \equiv (1+gx^n)(1-gx^n) \equiv 1 \pmod{x^{n+1}}$, and it still has integer coefficients. In the limit $F = \prod_d (1-x^d)^{-c_d}$ with all c_d integers. Taking logarithms, $\sum a_n \frac{x^n}{n} = -\sum_d c_d \log(1-x^d)$, and comparing coefficients gives $a_n = \sum_{d|n} d\, c_d$. Since $a_n = nc_n + \sum_{d|n,\, d<n} d\, c_d$, the c_d are determined by the a_n and conversely. So we must count the integer sequences (c_d) for which every $a_n = \sum_{d|n} d\, c_d$ lies in $\{0,1,2,3,4\}$.

Step 2: only divisors of 12 *survive.* Fix a prime p and $j \ge 1$ with $p^j \ge 5$, and let k be prime to p. The divisors of $p^j k$ that do not divide $p^{j-1}k$ are the numbers $p^j e$ with $e \mid k$, so $a_{p^j k} - a_{p^{j-1}k} = p^j \sum_{e|k} e\, c_{p^j e}$. The left side lies between -4 and 4 and is a multiple of $p^j \ge 5$, so it is 0. Hence $\sum_{e|k} e\, c_{p^j e} = 0$ for every k prime to p, and strong induction on k gives $c_{p^j k} = 0$. Consequently, if $c_n \ne 0$, every prime power exactly dividing n is at most 4, that is, it is 2, 4 or 3, and $n \in \{1,2,3,4,6,12\}$. Then $a_n = \sum_{d|n,\ d|12} d\, c_d = a_{\gcd(n,12)}$ for every n.

Conversely, integers $c_1, c_2, c_3, c_4, c_6, c_{12}$ (all other $c_d = 0$) with the six values $a_1, a_2, a_3, a_4, a_6, a_{12}$ in $[0,4]$ give a valid sequence, because every $a_n = a_{\gcd(n,12)}$. So we count the 6-tuples of values in $\{0,\ldots,4\}$ for which $c_1 = a_1$, $2c_2 = a_2 - a_1$, $3c_3 = a_3 - a_1$, $4c_4 = a_4 - a_2$, $6c_6 = a_6 - a_3 - a_2 + a_1$, $12c_{12} = a_{12} - a_6 - a_4 + a_2$ are integers. So $a_2 \equiv a_1 \pmod 2$, $a_3 \equiv a_1 \pmod 3$, $a_4 \equiv a_2 \pmod 4$, then $a_6 \equiv s \pmod 6$ with $s = a_2 + a_3 - a_1$, and $a_{12} \equiv t \pmod{12}$ with $t = a_6 + a_4 - a_2$. Since $\{0,\ldots,4\}$ has only 5 elements, a_6 exists (and is unique) exactly when $s \not\equiv 5 \pmod 6$. As $-4 \le t \le 8$, a_{12} exists exactly when $0 \le t \le 4$, and then $a_{12} = t$.

Step 3: the count. Go through a_1. Write $(a_2, a_3) \to a_6$.

$a_1 = 0$: $a_2 \in \{0,2,4\}$, $a_3 \in \{0,3\}$. The pair $(2,3)$ gives $s = 5$ and fails; the others give $(0,0) \to 0$, $(0,3) \to 3$, $(2,0) \to 2$, $(4,0) \to 4$, $(4,3) \to 1$. Now $a_4 \in \{0,4\}$ when $a_2 \in \{0,4\}$ and $a_4 = 2$ when $a_2 = 2$. The condition $0 \le t \le 4$ leaves $2,1,1,2,1$ choices respectively ($t = 0,4$; $t = 3$ only; $t = 2$; $t = 0,4$; $t = 1$ only). Total 7.

$a_1 = 1$: $a_2 \in \{1,3\}$, $a_3 \in \{1,4\}$, $s = 1,4,3,6$, so $a_6 = 1,4,3,0$, all fine. Here $a_4 = a_2$ is forced, so $t = a_6$ is always allowed. Total 4.

$a_1 = 2$: $a_3 = 2$ and $a_6 = a_2 \in \{0,2,4\}$. Then $t = a_4$ for $a_2 \in \{0,4\}$ (both $a_4 \in \{0,4\}$ work) and $t = 2$ for $a_2 = 2$. Total $2 + 1 + 2 = 5$.

$a_1 = 3$: $a_2 \in \{1,3\}$, $a_3 \in \{0,3\}$, $s = -2,1,0,3$, so $a_6 = 4,1,0,3$. Again $a_4 = a_2$ and $t = a_6$. Total 4.

$a_1 = 4$: $a_2 \in \{0,2,4\}$, $a_3 \in \{1,4\}$. The pair $(2,1)$ gives $s = -1$ and fails; the others give $(0,1) \to 3$, $(0,4) \to 0$, $(2,4) \to 2$, $(4,1) \to 1$, $(4,4) \to 4$, with $1,2,1,1,2$ choices of a_4. Total 7.

Altogether $7 + 4 + 5 + 4 + 7 = 27$.

Check. Replacing every a_n by $4 - a_n$ replaces c_1 by $4 - c_1$ and every other c_d by $-c_d$, so it maps solutions to solutions. This matches the symmetric counts $7,4,5,4,7$. A sample non-obvious solution is $a_n = 1,3,4,3,0,0$ for $\gcd(n,12) = 1,2,3,4,6,12$. It comes from $c_1 = c_2 = c_3 = 1$ and $c_6 = -1$, that is, from the series $\frac{1-x^6}{(1-x)(1-x^2)(1-x^3)}$. An exact

computer search over all prefixes $a_1, \ldots, a_N$ that keep the first N coefficients integral finds 27 for every N from 12 to 60. The answer is $\boxed{27}$.