MathCastle

USAMO

Day I and Day II

6 problems • 9 hours

Name _____

INSTRUCTIONS

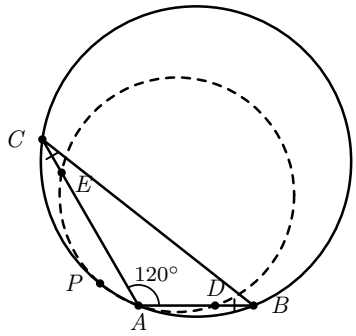
1. Two days of 3 problems each, 4.5 hours per day.
2. Each problem is worth 7 points. Full credit needs a complete, rigorous proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Day I is handed in before Day II is opened.
5. Two sittings of 3 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Problem 1.

Triangle ABC has $AB = 3$, $AC = 5$ and $\angle BAC = 120^\circ$ (so $BC = 7$). Point D lies on side $\overline{AB}$ and point E lies on side $\overline{AC}$, chosen so that $BD = CE = 1$. The circumcircle of triangle ADE and the circumcircle of triangle ABC meet at A and at a second point P. Find the length AP, in simplest radical form.



Problem 2.

For a positive integer n, let $d(n)$ be the number of positive divisors of n, let $\sigma(n)$ be the sum of the positive divisors of n, and let $\varphi(n)$ be the number of integers k with $1 \le k \le n$ and $\gcd(k, n) = 1$. Find the sum of all positive integers n such that the product $\varphi(n) \cdot d(n)$ divides $\sigma(n)$.

Problem 3.

Each cell of a rectangular board with 23 rows and 35 columns is filled with a 0 or a 1. Two cells are neighbours if they share a side. A filling is called *balanced* if every cell of the board (whatever it contains itself) has an even number of neighbours that contain a 1. How many balanced fillings are there? (The all-0 filling counts.)

Problem 4.

Find the sum of $x + y + z$ over all triples (x, y, z) of positive integers such that $2^x \cdot 3^y + 1 = 7^z$.

Problem 5.

Let U be the set of the 18 complex numbers z with $z^{18} = 1$. How many subsets of U (the empty set included) have the property that the sum of their elements is 0?

Problem 6.

Some queens are placed on a 2025×2025 chessboard, at most one queen on each cell. Two queens *see* each other if they lie in the same row, in the same column, or on the same diagonal (a line of cells parallel to one of the two long diagonals of the board), and no other queen stands on a cell strictly between them on that line. What is the largest possible number of queens if every queen sees exactly two other queens?

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Answer Key — USAMO Practice Contest 3

Problem 1. $\frac{2\sqrt{3}}{3}$

Problem 2. 60

Problem 3. 2048

Problem 4. 10

Problem 5. 1000

Problem 6. 4048

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — USAMO Practice Contest 3

Problem 1. **Answer:** $\frac{2\sqrt{3}}{3}$. **The idea.** Guess the point first. Because $BD = CE$, there should be a rotation taking B to C and D to E; its centre is equidistant from B and C, so it should be the midpoint M of the arc BAC of the circumcircle. We prove directly that M lies on circle ADE, so $P = M$ (and P does not depend on the common length $BD = CE$ at all). Then Ptolemy's theorem finds AM.

Work it out. Let M be the midpoint of the arc BC of the circumcircle that contains A, so $MB = MC$. Since $AB < AC$, the point A lies on this arc between B and M: going round the circle the order is B, A, M, C.

Step 1: M is on circle ADE. The inscribed angles $\angle MBA$ and $\angle MCA$ both stand on the arc AM that contains neither B nor C, so $\angle MBA = \angle MCA$. As D is on ray BA and E on ray CA, this says $\angle MBD = \angle MCE$. Together with $MB = MC$ and $BD = CE$, triangles MBD and MCE are congruent (SAS), so $\angle MDB = \angle MEC$, and taking supplements, $\angle MDA = \angle MEA$. Finally D and E are on the same side of line AM: with the order B, A, M, C on the circle, the chord AM leaves B and C on the same side, and D (on segment AB) and E (on segment AC) go with them. Equal angles $\angle ADM = \angle AEM$ over the segment AM, on the same side, mean that A, D, E, M are concyclic.

Step 2: $P = M$. The two circles are different (D is inside the circumcircle), so they share at most two points: A and M, where $M \neq A$ because $AB \neq AC$. Hence $P = M$.

Step 3: the length. In the isosceles triangle BMC the apex angle is $\angle BMC = \angle BAC = 120^\circ$ (M and A are on the same arc), so $MB = MC = \dfrac{BC}{2\sin 60^\circ} = \dfrac{7}{\sqrt{3}}$. Ptolemy's theorem for the cyclic quadrilateral $BAMC$ (in this order) gives $BA \cdot MC + AM \cdot CB = BM \cdot AC$, i.e. $3 \cdot MB + 7 \cdot AM = 5 \cdot MB$. So $AM = \dfrac{2\,MB}{7} = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$.

Check. In coordinates $A = (0,0)$, $B = (3,0)$, $C = \left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$, $D = (2,0)$, $E = (-2, 2\sqrt{3})$, the circles are $x^2 + y^2 - 2x - \frac{10}{\sqrt{3}}y = 0$ (through A, D, E) and $x^2 + y^2 - 3x - \frac{13}{\sqrt{3}}y = 0$ (through A, B, C). Subtracting gives the common chord $x + \sqrt{3}\,y = 0$; putting $x = -\sqrt{3}\,y$ into the first circle gives $4y^2 - \frac{4}{\sqrt{3}}y = 0$, so $P = \left(-1, \frac{\sqrt{3}}{3}\right)$ and $AP = \sqrt{1 + \frac{1}{3}} = \frac{2\sqrt{3}}{3}$. As predicted, $PB^2 = 16 + \frac{1}{3} = \frac{49}{3} = \frac{9}{4} + \frac{169}{12} = PC^2$.

Problem 2. **Answer:** 60. **The idea.** All three functions are multiplicative, so the quotient $R(n) = \dfrac{\sigma(n)}{\varphi(n)\,d(n)}$ is a product of one factor for each prime power exactly dividing n. Almost every such factor is at most 1, and only 2 and 4 give factors above 1 (namely $\frac{3}{2}$ and $\frac{7}{6}$). So $R(n) \le \frac{3}{2} < 2$, which forces $R(n) = 1$, and then the sizes of the factors allow at most one "defective" prime power, which must exactly cancel the factor from the power of 2.

Work it out. *Step 1 (local factors).* If $\gcd(a,b) = 1$ then $\sigma(ab) = \sigma(a)\sigma(b)$, $d(ab) = d(a)d(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$, so for $n = \prod p^e$ we get $R(n) = \prod r(p,e)$ over the prime powers p^e exactly dividing n (and $R(1) = 1$), where

$$r(p,e)=\frac{\sigma(p^e)}{\varphi(p^e)(e+1)}=\frac{p^{e+1}-1}{(p-1)^2\,p^{e-1}\,(e+1)}<\frac{p^2}{(p-1)^2(e+1)}.$$

For $p=2$: $r(2,1)=\frac{3}{2}$, $r(2,2)=\frac{7}{6}$, and for $e\ge 3$, $r(2,e)<\frac{4}{e+1}\le 1$.

For odd p and $e=1$: $r(p,1)=\frac{p+1}{2(p-1)}$, which equals 1 for $p=3$, $\frac{3}{4}$ for $p=5$, and is at most $\frac{8}{12}=\frac{2}{3}$ for $p\ge 7$ (it decreases in p). It equals $\frac{2}{3}$ exactly when $3p+3=4p-4$, i.e. $p=7$.

For odd p and $e\ge 2$: $r(p,e)<\frac{p^2}{3(p-1)^2}\le\frac{9}{12}=\frac{3}{4}$. Moreover $r(3,2)=\frac{13}{18}$, $r(3,e)<\frac{9}{4(e+1)}\le\frac{9}{16}$ for $e\ge 3$, and $r(p,e)<\frac{25}{48}$ for $p\ge 5$; so $r(p,e)\ne\frac{2}{3}$ whenever $e\ge 2$.

Call an odd prime power *defective* if it is not exactly 3^1. We have shown: the factor of 3^1 is 1, the factor of every defective prime power is at most $\frac{3}{4}$, and it equals $\frac{2}{3}$ only for 7^1.

Step 2 ($R(n)=1$). Every factor is at most 1 except $r(2,1)=\frac{3}{2}$ and $r(2,2)=\frac{7}{6}$, and at most one power of 2 occurs, so $0<R(n)\le\frac{3}{2}$. If $\varphi(n)d(n)\mid\sigma(n)$ then $R(n)$ is a positive integer below 2, so $R(n)=1$; conversely $R(n)=1$ certainly gives divisibility.

Step 3 (solve $R(n)=1$). Write $n=2^a m$ with m odd, and let Q be the product of the factors of the odd prime powers, so $Q\le 1$.

If $a=0$ or $a\ge 3$, the factor from 2^a is at most 1, so $R(n)=1$ forces it to be 1 (so $a=0$) and $Q=1$, i.e. no defective prime power: $m\in\{1,3\}$, giving $n=1,3$.

If $a=2$, we need $Q=\frac{6}{7}$. With no defective prime power $Q=1$; with one or more, $Q\le\frac{3}{4}<\frac{6}{7}$. No solutions.

If $a=1$, we need $Q=\frac{2}{3}$. With no defective prime power $Q=1$; with two or more, $Q\le\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}<\frac{2}{3}$. So there is exactly one defective prime power and its factor is $\frac{2}{3}$, i.e. it is 7^1. Together with the optional neutral factor 3^1, $m\in\{7,21\}$ and $n=14$ or 42.

The solutions are $n=1,3,14,42$, with sum $1+3+14+42=60$.

Check. $n=14$: $\varphi=6$, $d=4$, $\sigma=1+2+7+14=24=6\cdot 4$. $n=42$: $\varphi=12$, $d=8$, $\sigma=3\cdot 4\cdot 8=96=12\cdot 8$. $n=3$: $\varphi=2$, $d=2$, $\sigma=4$. A near miss: $n=28$ has $\varphi\cdot d=12\cdot 6=72$ and $\sigma=56$, matching $R(28)=\frac{7}{6}\cdot\frac{2}{3}=\frac{7}{9}$. A sieve over all $n\le 3\cdot 10^5$ finds exactly $1,3,14,42$.

Problem 3. **Answer:** 2048. **The idea.** Work modulo 2 and read the board row by row. The condition at a cell of row i says (row $i-1$) + (row $i+1$) + (row i shifted left and right) $=0$, so the first row determines every later row, and a first row is allowed exactly when the imaginary row $m+1$ comes out as 0. This turns the count into the size of the kernel of a polynomial in the $n\times n$ "path" matrix, and that kernel is governed by a gcd of Fibonacci polynomials. For an $m\times n$ board the answer is $2^{\gcd(m+1,\,n+1)-1}$.

Work it out. Let $m=23$, $n=35$, and do all arithmetic mod 2. Write row i as a row vector $r_i\in\mathbb{F}_2^n$, with $r_0=r_{m+1}=0$ (outside the board). Let T be the $n\times n$ matrix with 1s in positions $(j,j\pm 1)$ and 0 elsewhere; then the j-th entry of r_iT is the sum of the left and right neighbours of cell (i,j). So the filling is balanced exactly when $r_{i-1}+r_{i+1}+r_iT=0$ for $i=1,\dots,m$.

Step 1 (propagation). Define polynomials over $\mathbb{F}_2$ by $f_0=0$, $f_1=1$, $f_{k+1}=tf_k+f_{k-1}$, so f_k has degree $k-1$ and leading coefficient 1. The equations for $i=1,\dots,m-1$ say

$r_{i+1} = r_i T + r_{i-1}$, so by induction $r_i = r_1 f_i(T)$ for $1 \le i \le m$, and the last equation says $r_1 f_m(T)T + r_1 f_{m-1}(T) = 0$, i.e. $r_1 f_{m+1}(T) = 0$. Conversely any r_1 with $r_1 f_{m+1}(T) = 0$ gives a balanced filling. Hence the answer is the number of $v \in \mathbb{F}_2^n$ with $v f_{m+1}(T) = 0$.

Step 2 (a model for $\mathbb{F}_2^n$). Let $e_1, \dots, e_n$ be the standard basis, so $e_1 T = e_2$ and $e_k T = e_{k-1} + e_{k+1}$ (with $e_{n+1} = 0$). By induction $e_1 f_k(T) = e_k$ for $1 \le k \le n$, and then $e_1 f_{n+1}(T) = e_n T + e_{n-1} = 0$. So the linear map $g \mapsto e_1 g(T)$ from polynomials onto $\mathbb{F}_2^n$ is onto (it hits every e_k), and its kernel is an ideal (if $e_1 g(T) = 0$ then $e_1(tg)(T) = e_1 g(T)\, T = 0$) containing f_{n+1}, of codimension $n = \deg f_{n+1}$. Hence the kernel is exactly the multiples of f_{n+1}, and every v is $e_1 g(T)$ for exactly one g of degree less than n.

Step 3 (count). For $v = e_1 g(T)$ we have $v f_{m+1}(T) = e_1(g f_{m+1})(T)$, which is 0 iff $f_{n+1} \mid g f_{m+1}$. With $d = \gcd(f_{m+1}, f_{n+1})$ this means $\frac{f_{n+1}}{d} \mid g$. The multiples of a fixed polynomial of degree $n - \deg d$ with degree below n form a space of dimension $\deg d$, so there are $2^{\deg d}$ balanced fillings.

Step 4 (the gcd). With $Q = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}$ one checks by induction $Q^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$; comparing corners in $Q^{a+b} = Q^a Q^b$ gives $f_{a+b} = f_{a+1} f_b + f_a f_{b-1}$. Consecutive f's are coprime (a common factor of f_{k+1} and f_k divides f_{k-1}, and so on down to $f_1 = 1$). Therefore $\gcd(f_{a+b}, f_b) = \gcd(f_a f_{b-1}, f_b) = \gcd(f_a, f_b)$, and running Euclid's algorithm on the indices gives $\gcd(f_a, f_b) = f_{\gcd(a,b)}$. Here $\gcd(24, 36) = 12$, so $d = f_{12}$ has degree 11 and the number of balanced fillings is $2^{11} = 2048$.

Check. Small boards by hand: a 1×1 board has no neighbours at all, so both fillings are balanced, and $2^{\gcd(2,2)-1} = 2$. On a 1×3 board abc, the conditions are $b = 0$, $a + c = 0$, $b = 0$, giving 2 fillings, and $2^{\gcd(2,4)-1} = 2$. On a 1×2 board only the zero filling works, and $\gcd(2, 3) = 1$. A computer solves the full 805-variable linear system mod 2 for the 23×35 board and finds 2^{11} solutions.

Problem 4. **Answer:** 10. **The idea.** $7^z - 1$ can contain only a little of the primes 2 and 3: its 2-part and 3-part grow only like the 2-part and 3-part of z (lifting the exponent). So $7^z - 1 = 2^x 3^y$ would force $7^z - 1 \le 24z$, which fails for $z \ge 3$.

Work it out. Write $z = 2^s u = 3^{s'} w$ with u odd and $3 \nmid w$.

The power of 3. $7^w - 1 = 6(7^{w-1} + \cdots + 1)$ and the bracket is $\equiv w \not\equiv 0 \pmod 3$, so $3 \parallel 7^w - 1$. If $b \equiv 1 \pmod 3$ then $b^3 - 1 = (b-1)(b^2 + b + 1)$ with $b^2 + b + 1 \equiv 3 \pmod 9$, so cubing adds exactly one factor 3. Hence $3^y \parallel 7^z - 1$ with $y = 1 + s'$, and $3^{s'} \mid z$.

The power of 2. $7^u - 1 = 6(7^{u-1} + \cdots + 1)$ with an odd number of odd terms in the bracket, so $2 \parallel 7^u - 1$. If $s = 0$ we get $x = 1$. If $s \ge 1$, $7^z - 1 = (7^u - 1)(7^u + 1)(7^{2u} + 1) \cdots (7^{2^{s-1}u} + 1)$. Here $7^u \equiv 7 \pmod{16}$ (u odd), so $2^3 \parallel 7^u + 1$, and each $7^{2^i u} + 1 \equiv 2 \pmod 4$ contributes exactly one 2. So $x = 1 + 3 + (s - 1) = s + 3$. In all cases $2^x \le 8 \cdot 2^s$.

The bound. Since 2^s and $3^{s'}$ both divide z, $2^s 3^{s'} \le z$, and so $7^z - 1 = 2^x 3^y \le 8 \cdot 2^s \cdot 3 \cdot 3^{s'} \le 24z$. For $z = 3$ the left side is $342 > 72$, and when z grows by 1 the left side is multiplied by more than 7 while $24z$ grows by a factor at most $\frac{4}{3}$; so $7^z - 1 > 24z$ for all $z \ge 3$. Hence $z \in \{1, 2\}$.

$z = 1$: $6 = 2 \cdot 3$, so $(x, y, z) = (1, 1, 1)$. $z = 2$: $48 = 2^4 \cdot 3$, so $(4, 1, 2)$. The requested sum is $(1 + 1 + 1) + (4 + 1 + 2) = 10$.

Check. The formulas predict $x = 1$ for odd z and $x = 3 + v_2(z)$ for even z: $7^2 - 1 = 48 = 2^4 \cdot 3$ ($s = 1$, $x = 4$) and $7^4 - 1 = 2400 = 2^5 \cdot 3 \cdot 5^2$ ($s = 2$, $x = 5$), matching. A computer factorisation of $7^z - 1$ for every $z < 400$ finds only the two triples above.

Problem 5. **Answer:** 1000. **The idea.** Encode a subset by its generating polynomial $A(x) = \sum a_k x^k$ ($a_k \in \{0, 1\}$ says whether ζ^k is chosen, $\zeta = e^{2\pi i/18}$). The subset sums to zero exactly when $A(\zeta) = 0$, i.e. when the minimal polynomial $\Phi_{18}(x) = x^6 - x^3 + 1$ divides A. Using $\zeta^9 = -1$ first folds opposite points together, and then divisibility by Φ_{18} becomes a rigid coefficient pattern.

Work it out. Since $\zeta^{k+9} = -\zeta^k$, the sum is $\sum_{k=0}^{8} b_k \zeta^k$ with $b_k = a_k - a_{k+9} \in \{-1, 0, 1\}$. Put $B(x) = \sum_{k=0}^{8} b_k x^k$. The minimal polynomial of the primitive 18th root of unity ζ over $\mathbb{Q}$ is the cyclotomic polynomial $\Phi_{18}(x) = x^6 - x^3 + 1$ (degree $\varphi(18) = 6$), so $B(\zeta) = 0$ iff $\Phi_{18} \mid B$. Because Φ_{18} is monic with integer coefficients and $\deg B \le 8$, this means $B = (c_0 + c_1 x + c_2 x^2)(x^6 - x^3 + 1)$ with integers c_j. Expanding, the coefficient of x^j, x^{j+3}, x^{j+6} is c_j, $-c_j$, c_j respectively ($j = 0, 1, 2$), and these nine terms do not overlap. So the condition is: for each $j \in \{0, 1, 2\}$ there is c_j with $b_j = c_j$, $b_{j+3} = -c_j$, $b_{j+6} = c_j$; in particular $c_j \in \{-1, 0, 1\}$.

For a fixed j this involves the three antipodal pairs $\{\zeta^j, \zeta^{j+9}\}$, $\{\zeta^{j+3}, \zeta^{j+12}\}$, $\{\zeta^{j+6}, \zeta^{j+15}\}$ and nothing else. If $c_j = 0$, each pair is taken whole or not at all: $2^3 = 8$ ways. If $c_j = 1$, the choice is forced (ζ^j, ζ^{j+12}, ζ^{j+6} in; their partners out), and likewise for $c_j = -1$. That is $8 + 1 + 1 = 10$ ways for each j, independently, so the number of zero-sum subsets is $10^3 = 1000$.

Check. The case $c_0 = 1$, $c_1 = c_2 = 0$ with no pairs chosen is $\{1, \zeta^6, \zeta^{12}\}$, the vertices of an equilateral triangle, which indeed sum to 0; the case "all $c_j = 0$" gives the $2^9 = 512$ unions of antipodal pairs. A computer check over all 2^{18} subsets, with exact arithmetic in $\mathbb{Z}[\zeta]$, also finds 1000.

Problem 6. **Answer:** 4048. **The idea.** Count the pairs of queens that see each other line by line. A line holding $k \ge 1$ queens contributes exactly $k - 1$ such pairs, so with Q queens and D lines that contain at least one queen there are $4Q - D$ pairs; if every queen sees exactly two others there are also exactly Q pairs. Hence $D = 3Q$, and since the board has only $6n - 2$ lines, $Q \le 2n - 1$. Equality would leave just one line empty, which forces all four corners into play and then breaks down near a corner. A closed "loop" of queens reaches $2n - 2$. For $n = 2025$ the answer is $2n - 2 = 4048$.

Work it out. Let $n = 2025$, number the rows and columns 1 to n, and write (r, c) for the cell in row r and column c. The lines are the n rows, the n columns, the $2n - 1$ *falling* diagonals ($r - c$ constant) and the $2n - 1$ *rising* diagonals ($r + c$ constant): $6n - 2$ lines in four families, and every cell lies on exactly one line of each family.

Step 1 (the count $D = 3Q$). Two queens see each other along the unique line through them, exactly when they are consecutive among the queens of that line. So a line with $k \ge 1$ queens gives $k - 1$ seeing pairs, and summing over the D occupied lines gives $\sum(k - 1) = 4Q - D$ pairs, because each queen is on four lines. If every queen sees exactly two others, the number of pairs is $\frac{2Q}{2} = Q$. So $4Q - D = Q$, i.e. $D = 3Q$. As $D \le 6n - 2$, we get $3Q \le 6n - 2$, so $Q \le 2n - 1$.

Step 2 ($Q = 2n - 1$ is impossible). Suppose $Q = 2n - 1$. Then $D = 6n - 3$: exactly one line is empty. Each family of diagonals has $2n - 1 = Q$ lines; if all of them are occupied,

each holds exactly one queen, since every queen is on only one of them. Note the corner cells: $(n,1)$ and $(1,n)$ are each a whole falling diagonal, and $(1,1)$ and (n,n) are each a whole rising diagonal.

Case A: the empty line is a row or a column. Then every diagonal of both families holds exactly one queen. The four one-cell diagonals are occupied, so all four corners hold queens. But $(1,1)$ and (n,n) both lie on the falling diagonal $r-c=0$, which holds only one queen. Contradiction.

Case B: the empty line is a diagonal. Reflecting the board in its vertical axis swaps the two diagonal families, so we may assume it is a rising diagonal. Then every falling diagonal holds exactly one queen (so no two queens see each other along a falling diagonal), every row and column is occupied, and the $2n-1$ queens occupy $2n-2$ rising diagonals, so exactly one rising diagonal holds two queens and every other holds at most one. The one-cell falling diagonals give queens on $(n,1)$ and $(1,n)$. Both are on the rising diagonal $r+c=n+1$, so this is the rising diagonal with two queens; these two queens see each other, and no other pair sees each other along a rising diagonal. At most one of $(1,1)$ and (n,n) is occupied (they share the falling diagonal $r-c=0$), so one of the one-cell rising diagonals $\{(1,1)\}$, $\{(n,n)\}$ is empty. It is the only empty line, so exactly one of these corners holds a queen. The half-turn about the centre maps each family of lines to itself and fixes the set $\{(1,n),(n,1)\}$, so we may assume $(1,1)$ holds a queen and the only empty line is $\{(n,n)\}$. Now the queen on $(1,n)$ sees the queen on $(n,1)$. Row 1 also holds the queen on $(1,1)$, so the queen on $(1,n)$ sees some queen in row 1 too. That is two already, so column n holds no other queen (otherwise it would see the nearest one below it). In the same way the queen on $(n,1)$ sees $(1,n)$ and a queen above it in column 1 (which holds $(1,1)$), so row n holds no other queen. Now look at the rising diagonal $r+c=2n-1$, which consists of the two cells $(n-1,n)$ and $(n,n-1)$. The first is in column n and the second in row n, and neither is a corner, so both are empty. But this diagonal is not $\{(n,n)\}$, so it must be occupied. Contradiction.

So $Q \le 2n-2 = 4048$.

Step 3 (a loop of $2n-2$ queens). Write $n=2m+1$ with $m=1012$ and place queens on $(1,1)$; $(r,2)$ for $2 \le r \le m$, and $(n,2)$; $(n-1,c)$ for $3 \le c \le m+1$; (r,n) for $m+1 \le r \le n-1$; $(1,c)$ for $m+2 \le c \le n$.

That is $1+m+(m-1)+m+m = 4m = 2n-2$ queens. Lines with at least two queens: row 1 (the queen on $(1,1)$ and those on $(1,m+2),\dots,(1,n)$), row $n-1$ ($(n-1,3),\dots,(n-1,m+1)$ and $(n-1,n)$), column 2 ($(2,2),\dots,(m,2)$ and $(n,2)$) and column n ($(1,n)$ and $(m+1,n),\dots,(n-1,n)$). Every other row and column holds at most one queen. For the falling diagonals, the values of $r-c$ on the five groups above are 0; $0,1,\dots,m-2$ and $2m-1$; $m-1,\dots,2m-3$; $-m,\dots,-1$; $-2m,\dots,-m-1$. These are all different except for the value 0 taken by $(1,1)$ and $(2,2)$. For the rising diagonals the values of $r+c$ are 2; $4,\dots,m+2$ and $2m+3$; $2m+3,\dots,3m+1$; $3m+2,\dots,4m+1$; $m+3,\dots,2m+2$. These are all different except for the value $2m+3$ taken by $(n,2)$ and $(n-1,3)$.

A queen sees two queens on a line where it is between others, one where it is at an end, and none where it is alone. Hence: $(1,1)$ sees $(1,m+2)$ and $(2,2)$; each $(1,c)$ with $m+2 \le c \le n-1$ sees its two neighbours in row 1; $(1,n)$ sees $(1,n-1)$ and $(m+1,n)$; each (r,n) with $m+1 \le r \le n-2$ sees its two neighbours in column n; $(n-1,n)$ sees $(n-2,n)$ and $(n-1,m+1)$; each $(n-1,c)$ with $4 \le c \le m+1$ sees its two neighbours in row $n-1$; $(n-1,3)$ sees $(n-1,4)$ and $(n,2)$; $(n,2)$ sees $(n-1,3)$ and $(m,2)$; each

$(r, 2)$ with $3 \le r \le m$ sees its two neighbours in column 2; and $(2, 2)$ sees $(3, 2)$ and $(1, 1)$. Every queen sees exactly two others: the queens form one closed loop. So 4048 queens are possible, and the answer is 4048.

Check. The count $D = 3Q$ is tight in the construction: its $Q = 4m$ queens leave exactly 4 lines empty, the falling diagonals $r - c = 2m - 2$ and $r - c = 2m$ (the corner $(n, 1)$) and the rising diagonals $r + c = 3$ and $r + c = 2n$ (the corner (n, n)), and indeed $D = 6n - 2 - 4 = 12m = 3Q$. Small boards agree with the formula $2n - 2$: a computer search over all subsets finds the maxima 4, 6, 8 on the 3×3, 4×4 and 5×5 boards, and a direct simulation confirms that the loop above works on every odd board from 5×5 up to 41×41 and on the 2025×2025 board.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.