MathCastle

USAMO

Day I and Day II

6 problems • 9 hours

Name _____

INSTRUCTIONS

1. Two days of 3 problems each, 4.5 hours per day.
2. Each problem is worth 7 points. Full credit needs a complete, rigorous proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Day I is handed in before Day II is opened.
5. Two sittings of 3 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

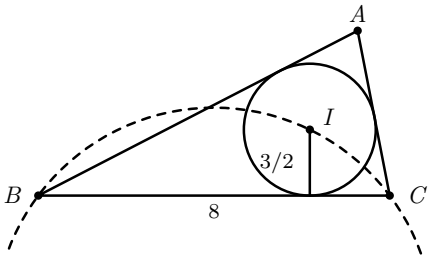
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Problem 1.

Positive real numbers x, y, z satisfy $x + y + z = 5$ and $xyz = 1$. Find the largest possible value of $xy + yz + zx$. Give your answer in the form $a + b\sqrt{c}$.

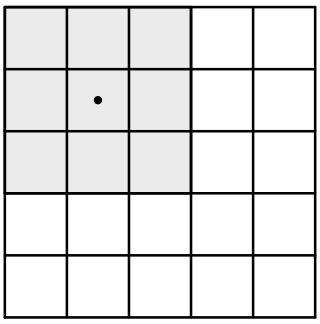
Problem 2.

Triangle ABC has $BC = 8$, and I is its incenter. The circle passing through B, I and C has radius 5, and the inradius of triangle ABC is $\frac{3}{2}$. Find $AB^2 + AC^2$.



Problem 3.

The 25 cells of a 5×5 board hold 25 different tiles, one per cell. A move chooses one of the nine 3×3 blocks of cells and turns the block through 180° about its centre cell: every tile of the block moves to the cell of the block symmetric to its old cell through the centre cell (the centre tile stays put, and tiles outside the block do not move). How many different arrangements of the tiles can be obtained from the starting arrangement by sequences of moves? (The starting arrangement counts.)



Problem 4.

Find the sum of $a + b$ over all ordered pairs (a, b) of positive integers for which both $a^2 + b$ and $a + b^2$ are powers of 3 (that is, numbers of the form 3^k with k a nonnegative integer).

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Problem 5.

For a positive integer n, let $f(n)$ be the number of ways to write n as a sum of powers of 2 (the numbers $1, 2, 4, 8, \ldots$), where the order of the terms does not matter and each power of 2 may be used at most four times. For example $f(5) = 3$, from $4 + 1$, $2 + 2 + 1$ and $2 + 1 + 1 + 1$. For how many integers n with $1 \leq n \leq 2026$ is $f(n)$ odd?

Problem 6.

For a set A of integers let $A + A = \{a + a' : a, a' \in A\}$, where $a = a'$ is allowed. How many 7-element subsets A of $\{1, 2, \ldots, 50\}$ satisfy $|A + A| = 14$, that is, have exactly 14 distinct sums $a + a'$?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — USAMO Practice Contest 2

Problem 1. $1 + 4\sqrt{2}$

Problem 2. $\frac{163}{2}$

Problem 3. 1128701952000

Problem 4. 14

Problem 5. 811

Problem 6. 308

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — USAMO Practice Contest 2

Problem 1. **Answer:** $1+4\sqrt{2}$. **The idea.** Freeze one variable. Once z is fixed, $x+y=5-z$ and $xy=1/z$ are fixed too, so $xy+yz+zx=xy+z(x+y)$ becomes a function of z alone. The only question is which values of z are allowed: x and y must be real (and then they are automatically positive), which is a discriminant condition on z. So the problem is a one-variable maximisation over an interval.

Work it out. Fix $z\in(0,5)$. Positive reals x,y with $x+y=5-z$ and $xy=1/z$ exist exactly when the quadratic $u^2-(5-z)u+1/z$ has real roots (both roots are then positive, since their sum and product are positive), i.e. when $(5-z)^2\geq 4/z$, i.e. $h(z)=z(5-z)^2-4\geq 0$. Now $h(z)=z^3-10z^2+25z-4=(z-4)(z^2-6z+1)$, with roots $3-2\sqrt{2}\approx 0.17$, 4 and $3+2\sqrt{2}\approx 5.83$. As $h(0)=-4<0$ and h is a cubic with positive leading coefficient, on $(0,5)$ we have $h\geq 0$ exactly for $3-2\sqrt{2}\leq z\leq 4$.

For such z the target is $g(z)=xy+z(x+y)=\frac{1}{z}+5z-z^2$. Then $g'(z)=-\frac{1}{z^2}+5-2z=-\frac{2z^3-5z^2+1}{z^2}=-\frac{(2z-1)(z^2-2z-1)}{z^2}$. On $[3-2\sqrt{2},4]$ the factor z^2-2z-1 is negative for $z<1+\sqrt{2}$ and positive after, so g decreases on $[3-2\sqrt{2},\frac{1}{2}]$, increases on $[\frac{1}{2},1+\sqrt{2}]$ and decreases on $[1+\sqrt{2},4]$. Its maximum is therefore at an endpoint of an increasing stretch: at $z=3-2\sqrt{2}$ or at $z=1+\sqrt{2}$.

At $z=1+\sqrt{2}$: $\frac{1}{z}=\sqrt{2}-1$ and $z(5-z)=(1+\sqrt{2})(4-\sqrt{2})=2+3\sqrt{2}$, so $g=1+4\sqrt{2}$. At $z=3-2\sqrt{2}$: $\frac{1}{z}=3+2\sqrt{2}$ and $z(5-z)=(3-2\sqrt{2})(2+2\sqrt{2})=-2+2\sqrt{2}$, so again $g=1+4\sqrt{2}$. (These are the same triple listed in different orders.) The maximum is $1+4\sqrt{2}\approx 6.657$, attained at $(x,y,z)=(1+\sqrt{2},\,1+\sqrt{2},\,3-2\sqrt{2})$: indeed the sum is 5 and the product is $(3+2\sqrt{2})(3-2\sqrt{2})=1$.

Check. The same answer from the uvw point of view: x,y,z are the roots of t^3-5t^2+qt-1 with $q=xy+yz+zx$, and the extreme values of q occur when two roots coincide. Putting $x=y=s$, $z=1/s^2$ gives $2s^3-5s^2+1=(2s-1)(s^2-2s-1)=0$, so $s=\frac{1}{2}$ (then $q=s^2+2/s=\frac{17}{4}$, the minimum) or $s=1+\sqrt{2}$ (then $q=3+2\sqrt{2}+2(\sqrt{2}-1)=1+4\sqrt{2}$, the maximum).

Problem 2. **Answer:** $\frac{163}{2}$. **The idea.** The circle through B, I, C is not a random circle: its centre is the midpoint M of arc BC of the circumcircle (the incenter–excenter lemma). That turns its radius into the half-angle $A/2$. With $A/2$ known, the inradius gives the tangent length from A, hence the semiperimeter, hence $AB+AC$ and the area, hence $AB\cdot AC$.

Work it out. Write $a=BC=8$, $b=CA$, $c=AB$, and A,B,C for the angles. Let the bisector AI meet the circumcircle again at M, the midpoint of the arc BC not containing A.

Lemma: $MB=MI=MC$. Indeed $\angle MBC=\angle MAC=A/2$ (same arc MC), so $\angle MBI=\angle MBC+\angle CBI=A/2+B/2$. And $\angle MIB$ is an exterior angle of triangle ABI, so $\angle MIB=\angle IAB+\angle IBA=A/2+B/2$. Triangle MBI has two equal angles, so $MB=MI$; in the same way $MC=MI$. Hence the circle through B, I, C has centre M and radius MB.

In the isosceles triangle MBC the base angles are $\angle MBC=\angle MCB=A/2$, so $BC=2\,MB\cos(A/2)$. With $BC=8$ and $MB=5$: $\cos(A/2)=\frac{4}{5}$, $\sin(A/2)=\frac{3}{5}$, $\tan(A/2)=\frac{3}{4}$, and $\sin A=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$.

Let the incircle touch AB at F. The tangent lengths from the vertices are $s-a$, $s-b$, $s-c$ (s the semiperimeter), so $AF = s-a$, and the right triangle AFI (right angle at F, angle $A/2$ at A) gives $r = (s-a)\tan(A/2)$. So $s-a = \frac{3/2}{3/4} = 2$ and $s = 10$. Hence $b+c = 2s-a = 12$, and the area is $rs = \frac{3}{2}\cdot 10 = 15$. Also area $= \frac{1}{2}bc\sin A$, so $bc = \frac{2\cdot 15}{24/25} = \frac{125}{4}$. Therefore $AB^2 + AC^2 = (b+c)^2 - 2bc = 144 - \frac{125}{2} = \frac{163}{2}$.

Such a triangle exists: b, c are the roots of $t^2 - 12t + \frac{125}{4}$, namely $6 \pm \frac{\sqrt{19}}{2}$, both positive, with $b+c = 12 > 8$ and $|b-c| = \sqrt{19} < 8$. By the law of cosines $\cos A = \frac{163/2-64}{125/2} = \frac{7}{25} = 2\left(\frac{4}{5}\right)^2 - 1$, so this triangle really has $\cos(A/2) = \frac{4}{5}$, $s = 10$, and hence circle radius $\frac{8}{2\cdot 4/5} = 5$ and inradius $2\cdot\frac{3}{4} = \frac{3}{2}$. All the steps above were forced, so the answer is unique.

Check. Heron with $s = 10$: the area is $\sqrt{10\cdot 2\cdot(10-b)(10-c)}$, and $(10-b)(10-c) = 100 - 10\cdot 12 + \frac{125}{4} = \frac{45}{4}$, so the area is $\sqrt{225} = 15 = rs$. Consistent.

Problem 3. **Answer:** 1128701952000**. The idea.** Every move preserves the parity of a tile's row and of its column, so tiles live in four classes of cells, of sizes 9, 6, 6, 4. A move changes the signs of the permutations of these classes in a rigid pattern, and this gives two independent parity invariants, cutting the count by a factor 4. The hard part is the converse: everything else is reachable. The engine is a clean trick: if two half-turns move exactly one cell in common, then doing them alternately twice is a single 3-cycle.

Work it out. Name the cells (r, c) with $1 \le r, c \le 5$ and let M_{ij} be the move whose block has centre (i, j), for $i, j \in \{2, 3, 4\}$; it sends (r, c) to $(2i - r, 2j - c)$ inside its block. Since the tiles are distinct, the reachable arrangements correspond one-to-one to the permutations of cells in the group G generated by the nine moves, so the answer is $|G|$. Products are composed right to left.

Classes. Call a cell's class the pair (parity of r, parity of c): OO (9 cells, odd row and odd column), OE (6), EO (6), EE (4). Since $2i - r \equiv r$ and $2j - c \equiv c \pmod 2$, every element of G maps each class to itself.

Two invariants. The move M_{ij} fixes its centre, swaps the two pairs of opposite corners (two transpositions inside the corners' class), swaps $(i-1, j)$ with $(i+1, j)$ and $(i, j-1)$ with $(i, j+1)$ (one transposition in each of two more classes). The corners' class differs from the centre's class in both parities, so centre and corners form one of the two opposite pairs $\{OO, EE\}$, $\{OE, EO\}$, and the two edge classes form the other. On the first pair the move acts evenly on each class; on the second it is odd on each class. So with sgn_X the sign of the permutation of class X, both $\epsilon_1 = \operatorname{sgn}_{OO}\operatorname{sgn}_{EE}$ and $\epsilon_2 = \operatorname{sgn}_{OE}\operatorname{sgn}_{EO}$ equal $+1$ for every move, hence for all of G. The class-preserving permutations form a group of order $9!\,6!\,6!\,4!$, and $\sigma \mapsto (\epsilon_1, \epsilon_2)$ maps it onto $\{\pm 1\}^2$ (a single transposition in OO, resp. OE, shows each sign occurs), so its kernel H has order $9!\,6!\,6!\,4!/4$ and $G \subseteq H$.

The 3*-cycle trick.* Let $u, v \in G$ be involutions (products of disjoint transpositions) whose sets of moved cells share exactly one cell x. Write $u = (x\ p)U$ and $v = (x\ q)V$, where U and V are the other transpositions. Then $p \ne q$, and the cells moved by U, by V and the set $\{x, p, q\}$ are pairwise disjoint, so $uv = [(x\ p)(x\ q)]\,UV$ with all factors commuting, and $(uv)^2 = [(x\ p)(x\ q)]^2U^2V^2 = [(x\ p)(x\ q)]^2$, a 3-cycle on $\{x, p, q\}$. Every move is an involution, and so is every conjugate $gM_{ij}g^{-1}$ (it moves exactly the images under g of the cells M_{ij} moves).

(a) OO: $u = M_{22}$ moves the 8 non-centre cells of rows 1–3, columns 1–3; $v = M_{44}$ moves

those of rows 3–5, columns 3–5. They share only $(3,3)$, which u pairs with $(1,1)$ and v with $(5,5)$. So G contains a 3-cycle on $\{(1,1),(3,3),(5,5)\}$.

(b) OE: let $u = M_{22}M_{23}M_{22}$. M_{23} moves the 8 non-centre cells of rows 1–3, columns 2–4, so u moves their images under M_{22}: $(1,1),(1,2),(1,4),(2,2),(2,4),(3,1),(3,2),(3,4)$. M_{23} pairs $(3,4)$ with $(1,2)$, and M_{22} fixes $(3,4)$ and sends $(1,2)$ to $(3,2)$, so u pairs $(3,4)$ with $(3,2)$. With $v = M_{44}$ the only common moved cell is $(3,4)$, which v pairs with $(5,4)$: G contains a 3-cycle on $\{(3,2),(3,4),(5,4)\}$.

(c) EE: the same u and $v = M_{34}$ (rows 2–4, columns 3–5, centre $(3,4)$ unmoved). Their only common moved cell is $(2,4)$; u pairs it with $(2,2)$ (as M_{23} does, M_{22} fixing both) and v with $(4,4)$. So G contains a 3-cycle on $\{(2,2),(2,4),(4,4)\}$.

(d) EO: reflecting the board in its main diagonal turns moves into moves, so (b) gives a 3-cycle on $\{(2,3),(4,3),(4,5)\}$.

Spreading them. If $t \in G$ is a 3-cycle on $\{x,y,z\}$ and $g \in G$, then $gtg^{-1} \in G$ is a 3-cycle on $\{g(x),g(y),g(z)\}$. In OO, conjugating (a) by $M_{23}, M_{24}, M_{32}, M_{34}, M_{42}, M_{43}$ (each fixes $(1,1)$ and $(5,5)$ and sends $(3,3)$ to $(1,3),(1,5),(3,1),(3,5),(5,1),(5,3)$ respectively) gives a 3-cycle on $\{(1,1),(5,5),z\}$ for every other $z \in OO$. In EE, conjugating (c) by M_{32} (fixes $(2,4),(4,4)$; sends $(2,2) \mapsto (4,2)$) gives both 3-cycles through $(2,4),(4,4)$. In OE, the pair $(3,4),(5,4)$: (b) gives $z = (3,2)$; conjugating by M_{22} gives $z = (1,2)$; by M_{42} gives $z = (5,2)$; conjugating by M_{24} (which sends $(3,4) \mapsto (1,4)$) and then by M_{33} (which sends $(3,2) \mapsto (3,4)$), fixing the other points each time, gives $z = (1,4)$. By the diagonal symmetry the same holds in EO for the pair $(4,3),(4,5)$.

Star lemma. If a group contains a 3-cycle on $\{x,y,z\}$ for every $z \neq x,y$ in a set X, it contains all even permutations of X. Proof: it contains each such 3-cycle and its inverse, so both orientations. For distinct p,q,a,b one checks $(p\ a\ b) = (p\ q\ b)(p\ q\ a)^{-1}$; with $p = x$, $q = y$ this gives every 3-cycle through x, and then $(a\ b\ c) = (a\ x\ c)(a\ x\ b)^{-1}$ gives every 3-cycle. Finally every even permutation is a product of pairs of transpositions, and $(a\ b)(b\ c)$ is a 3-cycle while $(a\ b)(c\ d) = (a\ b)(b\ c)\cdot(b\ c)(c\ d)$.

Our 3-cycles fix every cell outside their class, so G contains $K = \mathrm{Alt}(OO) \times \mathrm{Alt}(OE) \times \mathrm{Alt}(EO) \times \mathrm{Alt}(EE)$, of order $9!\,6!\,6!\,4!/16$. The cosets of K inside H are told apart by the sign vector $(\mathrm{sgn}_{OO}, \mathrm{sgn}_{OE}, \mathrm{sgn}_{EO}, \mathrm{sgn}_{EE})$, which in H is one of $(+,+,+,+)$, $(+,-,-,+)$, $(-,+,+,-)$, $(-,-,-,-)$. The identity, M_{22} (even on OO and EE, odd on OE and EO), M_{23} (odd on OO and EE, even on OE and EO) and $M_{22}M_{23}$ realise all four, so G meets every coset, and $G = H$. Hence the number of arrangements is $\frac{9!\cdot 6!\cdot 6!\cdot 4!}{4} = \frac{362880\cdot 720\cdot 720\cdot 24}{4} = 1128701952000$.

Check. A computer (Schreier–Sims) finds the group generated by the nine moves has exactly this order, with orbits of sizes $9,6,6,4$. Also each move is a product of four transpositions, an even permutation of all 25 cells, consistent with $\epsilon_1\epsilon_2 = +1$.

Problem 4. **Answer:** 14. **The idea.** Two powers of the same prime: the smaller divides the larger. Reducing the larger one modulo the smaller eliminates b and leaves a fixed polynomial in a, whose 3-adic valuation is tiny. That forces $a^2 + b$ to be small, and a with it.

Work it out. The conditions are symmetric under swapping a and b, so first find the pairs with $a \le b$. Then $(b^2 + a) - (a^2 + b) = (b - a)(b + a - 1) \ge 0$, so $N = a^2 + b$ is the smaller power of 3 and $N \mid b^2 + a$. Modulo N we have $b \equiv -a^2$, so $b^2 + a \equiv a^4 + a =$

$a(a+1)(a^2-a+1)$, and therefore $N \mid a(a+1)(a^2-a+1)$. Since N is a power of 3 and $N \geq 2$, N divides the 3-part of this product.

If $3 \mid a$: then $a+1 \equiv 1$ and $a^2-a+1 \equiv 1 \pmod 3$, so $N \mid a$; impossible as $N = a^2+b > a$.

If $a \equiv 1 \pmod 3$: then $a+1 \equiv 2$ and $a^2-a+1 \equiv 1$, so the product is prime to 3 and $N = 1$, impossible.

If $a \equiv 2 \pmod 3$: write $a^2-a+1 = (a+1)^2 - 3a$. Here $9 \mid (a+1)^2$ while $3 \parallel 3a$, so $3 \parallel a^2-a+1$. Hence $N \leq 3^{v_3(a+1)+1} \leq 3(a+1)$. But $N = a^2+b \geq a^2+a$, so $a^2+a \leq 3a+3$, i.e. $(a-3)(a+1) \leq 0$, $a \leq 3$. With $a \equiv 2 \pmod 3$ this gives $a = 2$. Then N divides the 3-part of $2 \cdot 3 \cdot 3$, so $N \leq 9$: $4+b \leq 9$ and $b \geq a = 2$ with $4+b$ a power of 3 give $b = 5$. Indeed $2^2+5 = 9$ and $2+5^2 = 27$.

So the pairs are $(2,5)$ and, by symmetry, $(5,2)$, and the requested sum is $7+7 = 14$.

Check. A direct search over all $a \leq 20000$ and all $b \leq 20000$ with a^2+b a power of 3 finds only $(2,5)$ and $(5,2)$. Small cases also behave: $a = b$ would need $a(a+1)$ to be a power of 3, impossible since one factor is even.

Problem 5. Answer: 811**. The idea.** Write the generating function as an infinite product and read it modulo 2. Over the integers mod 2 each factor $1+y+y^2+y^3+y^4$ equals $\frac{1+y^5}{1+y}$, and a product of $1+x^{m\cdot 2^k}$ over all k telescopes, by uniqueness of binary expansions, into $1+x^m+x^{2m}+\cdots$. The whole product collapses to $\frac{1+x}{1-x^5} \bmod 2$.

Work it out. Put $f(0) = 1$. Choosing how many copies (0 to 4) of each 2^k to use gives $F(x) = \sum_{n\geq 0} f(n)x^n = \prod_{k\geq 0}\left(1+y_k+y_k^2+y_k^3+y_k^4\right)$ with $y_k = x^{2^k}$. The product makes sense as a power series: the coefficient of x^n only involves the finitely many factors with $2^k \leq n$. We work in the ring of power series with coefficients mod 2, where $1+x$ is invertible with inverse $1+x+x^2+\cdots$.

Two identities. First, over any ring, $(1+y+y^2+y^3+y^4)(1-y) = 1-y^5$; mod 2 this reads $(1+y+\cdots+y^4)(1+y) = 1+y^5$. Second, for any $m \geq 1$, $\prod_{k\geq 0}(1+x^{m2^k}) = \sum_{j\geq 0} x^{mj}$ exactly, because expanding the product picks a finite set of distinct powers 2^k, and every $j \geq 0$ is a sum of distinct powers of 2 in exactly one way.

Multiply F by $P = \prod_k(1+x^{2^k}) = \sum_j x^j$. Factor by factor, mod 2, $F \cdot P = \prod_k(1+y_k+\cdots+y_k^4)(1+y_k) \equiv \prod_k(1+x^{5\cdot 2^k}) = \sum_j x^{5j}$. Since $P = (1-x)^{-1}$, multiplying by $1-x \equiv 1+x$ gives $F(x) \equiv (1+x)\sum_{j\geq 0} x^{5j} = \sum_{j\geq 0}\left(x^{5j}+x^{5j+1}\right) \pmod 2$. So $f(n)$ is odd exactly when $n \equiv 0$ or $1 \pmod 5$.

Count in $1 \leq n \leq 2026$: multiples of 5 are $5, 10, \ldots, 2025$, that is 405 numbers; $n \equiv 1$ are $1, 6, \ldots, 2026$, that is 406 numbers. Total $405+406 = 811$.

Check. Small values by hand: $f(1) = 1$, $f(2) = 2$ (2, 1+1), $f(3) = 2$, $f(4) = 4$ (4, 2+2, 2+1+1, 1+1+1+1), $f(5) = 3$, $f(6) = 5$ (4+2, 4+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1). Odd exactly at $n = 1, 5, 6$, as predicted. An exact dynamic-programming count of $f(n)$ for every $n \leq 2026$ confirms the pattern throughout.

Problem 6. Answer: 308**. The idea.** A k-element set always has at least $2k-1 = 13$ sums, with equality only for arithmetic progressions. One more sum than the minimum is still an extremely rigid condition: a Freiman-type lemma (proved in full below) says that after

normalising, the set must sit inside an interval of length k. Then the few candidate shapes can be listed and counted.

Work it out. *Normalising.* Write $2X = X + X$. Translating A or scaling it by a nonzero factor does not change $|2A|$. Given A, let $a_0 = \min A$ and $d = \gcd\{a - a_0 : a \in A\}$; then $A = a_0 + d \cdot N$ where N is in normal form: $\min N = 0$ and $\gcd(N) = 1$. The triple (a_0, d, N) is determined by A, and $|2A| = |2N|$.

Basic bound. For any $X = \{x_1 < \cdots < x_k\}$ the sums $x_1 + x_1 < x_1 + x_2 < x_2 + x_2 < x_2 + x_3 < \cdots < x_k + x_k$ are $2k - 1$ distinct elements of $2X$, so $|2X| \geq 2k - 1$.

Lemma. Let N be in normal form with $|N| = k \geq 4$ and $\max N = m \geq k + 1$. Then $|2N| \geq 2k + 1$.

Proof, by induction on k. *Base $k = 4$:* $N = \{0, a, b, m\}$, $0 < a < b < m$, $m \geq 5$. The seven sums $0 < a < b < m < m + a < m + b < 2m$ are distinct. The sums $2a < a + b < 2b$ lie strictly between a and $m + b$, so each is new unless $2a \in \{b, m\}$, $a + b = m$, or $2b \in \{m, m + a\}$. If $a + b \neq m$, then $a + b$ is new, and if neither $2a$ nor $2b$ were new: $2a = b$ forces $2b = 4a \in \{m, m + a\}$, so $m \in \{3a, 4a\}$ and $\gcd = a = 1$, $m \leq 4$; while $2a = m$ forces $2b = m + a = 3a$, so $(a, b, m) = (2t, 3t, 4t)$, $t = 1$, $m = 4$. If $a + b = m$, then $2a < m < 2b$, and both are new unless $2a = b$ (equivalently $2b = m + a$), which gives $N = \{0, a, 2a, 3a\}$, $a = 1$, $m = 3$. In every possible case at least two of them are new, so $|2N| \geq 9$.

Step, $k \geq 5$: let $B = N \setminus \{m\}$ and $m' = \max B$. The sums $m + m'$ and $2m$ exceed $2m' = \max 2B$, so they are new: $|2N| \geq |2B| + 2$. (i) If $g = \gcd(B) \geq 2$: $g \nmid m$ (as $\gcd N = 1$), so $2B \subseteq g\mathbb{Z}$ and $m + B \subseteq m + g\mathbb{Z}$ are disjoint, and $2m$ is larger than all of them. So $|2N| \geq (2(k - 1) - 1) + (k - 1) + 1 = 3k - 3 \geq 2k + 1$. (ii) If $\gcd(B) = 1$ and $m' \geq k$: B is in normal form with $k - 1 \geq 4$ elements and $m' \geq (k - 1) + 1$, so by induction $|2B| \geq 2k - 1$, and $|2N| \geq 2k + 1$. (iii) If $m' \leq k - 1$: B has $k - 1$ elements in $[0, m']$, so $m' = k - 2$ or $k - 1$. If $m' = k - 2$, $B = \{0, \ldots, k - 2\}$ and $2B = [0, 2k - 4]$ has $2k - 3$ elements, while $m + a > 2k - 4$ for the four elements $a = k - 4, k - 3, k - 2, m$ (as $m \geq k + 1$): $|2N| \geq 2k + 1$. If $m' = k - 1$, $B = [0, k - 1] \setminus \{j\}$ with $1 \leq j \leq k - 2$. Every $x \in [0, 2k - 2]$ other than 1 and $2k - 3$ is in $2B$: for $x \leq k - 1$ use $x + 0$, or $1 + (j - 1)$ when $x = j \geq 2$; for $x \geq k - 1$ use $(k - 1) + (x - k + 1)$, or $(k - 2) + (j + 1)$ when $x - k + 1 = j \leq k - 3$. Also $1 \in 2B$ iff $j \neq 1$, and $2k - 3 \in 2B$ iff $j \neq k - 2$, and j cannot be both (as $k \geq 5$). So $|2B| \geq 2k - 2$, with $2B \subseteq [0, 2k - 2]$. Besides $m + m'$ and $2m$ we need one more new sum. If $j \neq k - 2$: $m + (k - 2) \geq 2k - 1$ is new. If $j = k - 2$ and $m \geq k + 2$: $m + (k - 3) \geq 2k - 1$ is new. If $j = k - 2$ and $m = k + 1$: $m + (k - 4) = 2k - 3$, which is missing from $2B$, is new (here $k - 4 \geq 1$ and $k - 4 \neq j$, so $k - 4 \in B$). In all cases $|2N| \geq 2k + 1$. This proves the Lemma.

Finishing. Let $|2A| = 14$ and let N be the normal form, with $k = 7$ and $m = \max N$. By the Lemma $m \leq 7$, and $m \geq 6$ since N has 7 elements. If $m = 6$, $N = \{0, \ldots, 6\}$ has 13 sums. If $m = 7$, $N = \{0, \ldots, 7\} \setminus \{j\}$ with $1 \leq j \leq 6$ (both 0 and 7 are in N). The argument in (iii) (with $k - 1 = 7$) shows $2N = [0, 14]$ minus 1 if $j = 1$, minus 13 if $j = 6$, and all of $[0, 14]$ (15 sums) if $2 \leq j \leq 5$. So $N = \{0, 2, 3, 4, 5, 6, 7\}$ or $\{0, 1, 2, 3, 4, 5, 7\}$, each with exactly 14 sums (and each has $\gcd 1$).

Counting. So $A = a_0 + d \cdot N$ for one of these two N, with $a_0 \geq 1$ and $a_0 + 7d \leq 50$, and different (a_0, d, N) give different sets. For each d from 1 to 7 there are $50 - 7d$ choices of a_0, so the total is $2 \sum_{d=1}^{7} (50 - 7d) = 2(350 - 196) = 308$.

Check. A pruned exhaustive search over all $\binom{50}{7}$ subsets finds exactly 308 with 14 sums. Small example: $A = \{1, 3, 4, 5, 6, 7, 8\}$ gives sums 2 and 4 through 16, fourteen of them, while $\{1, 2, 4, 5, 6, 7, 8\}$ (a middle gap) gives all fifteen of $2, \ldots, 16$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.