MathCastle

USAMO

Day I and Day II

6 problems • 9 hours

Name _____

INSTRUCTIONS

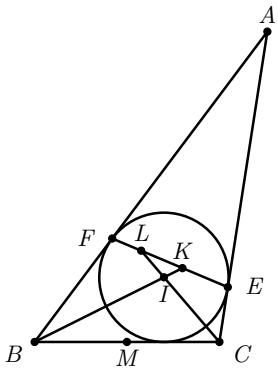
1. Two days of 3 problems each, 4.5 hours per day.
2. Each problem is worth 7 points. Full credit needs a complete, rigorous proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Day I is handed in before Day II is opened.
5. Two sittings of 3 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

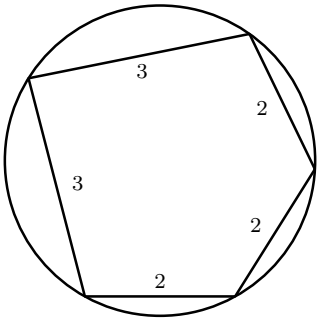
Problem 1.

Triangle ABC has $BC = 10$, $CA = 17$ and $AB = 21$. Its incircle, with centre I, touches side CA at E and side AB at F. The line EF meets line BI at K and line CI at L, and M is the midpoint of BC. Find the area of triangle MKL. Give your answer as a common fraction.



Problem 2.

A pentagon is inscribed in a circle, and its five side lengths, in some order around the pentagon, are $2, 2, 2, 3, 3$. Find the area of the pentagon. Give your answer in simplest radical form.



Problem 3.

Each cell of an $m \times n$ board of unit squares (m rows, n columns) is coloured black or white. The colouring is called balanced if every cell shares a side with exactly two cells of its own colour. (Cells on the edge or in a corner of the board have fewer neighbours, but they too must have exactly two neighbours of their own colour.) Colourings that differ by a rotation or reflection of the board count as different. Let $N(m, n)$ be the number of balanced colourings of the $m \times n$ board. Find $\sum_{m=1}^{10} \sum_{n=1}^{10} N(m, n)$.

Problem 4.

For integers a, b, c let $f(x) = x^3 + ax^2 + bx + c$. How many ordered triples (a, b, c) with $a, b, c \in \{0, 1, \ldots, 10\}$ have the property that the eleven numbers $f(0), f(1), \ldots, f(10)$ leave eleven different remainders when divided by 11?

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Problem 5.

How many monic polynomials P of degree 10 with complex coefficients satisfy $P(x)\cdot P(-x) = P(x^2)$ for every complex number x?

Problem 6.

Call a positive integer n centred if there is an integer t with $1 \le t < n$ such that both $n - t$ and $n + t$ divide $n^2 + 1$. For example, 3 is centred: with $t = 2$, both 1 and 5 divide $3^2 + 1 = 10$. Find the sum of all centred integers n with $n < 10000$.

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Answer Key — USAMO Practice Contest 1

Problem 1. $\frac{100}{17}$

Problem 2. $\frac{5\sqrt{15}}{2}$

Problem 3. 106

Problem 4. 121

Problem 5. 1024

Problem 6. 4059

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Solutions — USAMO Practice Contest 1

Problem 1. **Answer:** $\frac{100}{17}$. **The idea.** The chord EF of the incircle has a hidden right angle: the point K where it meets the bisector BI is exactly the foot of the perpendicular from C to BI, so $\angle BKC = 90°$. In the same way $\angle BLC = 90°$. So K and L both lie on the circle with diameter BC, whose centre is M: the triangle MKL is isosceles with legs 5, and inscribed angles give its apex angle.

Work it out. First the angles. $\cos A = \frac{17^2+21^2-10^2}{2\cdot 17\cdot 21} = \frac{630}{714} = \frac{15}{17}$, so $\sin A = \frac{8}{17}$, and $\cos B = \frac{10^2+21^2-17^2}{2\cdot 10\cdot 21} = \frac{3}{5}$. The semiperimeter is $s = 24$, so $AE = AF = s - a = 14$, $BF = s - b = 7$ and $CE = s - c = 3$.

Why the right angle appears: $\angle BIC = 90° + \frac{A}{2}$ is obtuse, so the foot K' of the perpendicular from C to line BI lies on ray BI beyond I, and $\angle ICK' = 90° - \angle CIK' = 90° - (180° - \angle BIC) = \frac{A}{2}$. Both K' and E see CI at a right angle, so they lie on the circle with diameter CI, where $\angle IEK' = \angle ICK' = \frac{A}{2}$. On the other hand triangle AEF is isosceles with apex angle A, so $\angle AEF = 90° - \frac{A}{2}$ and $\angle IEF = 90° - \angle AEF = \frac{A}{2}$ (as $IE \perp AC$). Hence EK' and EF are the same line and $K' = K$. To make every position claim airtight, here is the same computation in coordinates.

Put $B = (0,0)$, $C = (10,0)$, $A = (\frac{63}{5}, \frac{84}{5})$ (indeed $AB^2 = \frac{63^2+84^2}{25} = 441$ and $AC^2 = \frac{13^2+84^2}{25} = 289$). Then $I = \frac{10A+17B+21C}{48} = (7, \frac{7}{2})$, $F = B + \frac{7}{21}(A - B) = (\frac{21}{5}, \frac{28}{5})$ and $E = C + \frac{3}{17}(A - C) = (\frac{889}{85}, \frac{252}{85})$. The direction of EF is proportional to $(-19, 8)$, so its points are $(\frac{21}{5} - 19t, \frac{28}{5} + 8t)$. Line BI is $y = \frac{x}{2}$: this gives $t = -\frac{1}{5}$ and $K = (8,4)$. Line CI is $y = -\frac{7}{6}(x - 10)$: this gives $t = -\frac{7}{85}$ and $L = (\frac{98}{17}, \frac{84}{17})$.

Now $\overrightarrow{CK} = (-2,4)$ is perpendicular to the direction $(2,1)$ of BI, as the lemma says. With $M = (5,0)$ we get $\overrightarrow{MK} = (3,4)$ and $\overrightarrow{ML} = (\frac{13}{17}, \frac{84}{17})$, both of length 5 (since $13^2 + 84^2 = 85^2$). Inscribed angles explain the apex: $\angle KMC = 2\angle KBC = B$ and $\angle LMB = 2\angle LCB = C$, so $\angle KML = 180° - B - C = A$. The area is $\frac{1}{2} \cdot 5 \cdot 5 \cdot \sin A = \frac{25}{2} \cdot \frac{8}{17} = \frac{100}{17}$, and the cross product agrees: $\frac{1}{2}\left|3 \cdot \frac{84}{17} - 4 \cdot \frac{13}{17}\right| = \frac{1}{2} \cdot \frac{200}{17} = \frac{100}{17}$.

Check. The formula area $= \frac{a^2 \sin A}{8}$ depends only on a and A: here $\frac{100 \cdot 8/17}{8} = \frac{100}{17}$. Also $\overrightarrow{MK} = (3,4)$ makes the angle $\arccos \frac{3}{5} = B$ with $\overrightarrow{MC}$, exactly as the inscribed-angle step predicts.

Problem 2. **Answer:** $\frac{5\sqrt{15}}{2}$. **The idea.** Join the centre O to the five vertices. If O is inside the pentagon, the pentagon splits into five isosceles triangles with apex O, one per side, and their apex angles add up to 360°. Rearranging the sides only permutes these triangles, so neither the circle nor the area depends on the order. The tie for the longest side forces O inside, and then one trigonometric equation pins down everything.

Work it out. Let R be the radius. A chord of length ℓ that cuts off an arc (free of other vertices) of angle φ has $\ell = 2R \sin \frac{\varphi}{2}$.

O is inside. Suppose O is not inside the pentagon (or lies on a side). Then some side XY separates O from the other vertices (or passes through O), so the arc of XY free of other vertices is at least a semicircle, say $\varphi_0 \geq 180°$, and the other four arcs $\varphi_1, \ldots, \varphi_4$ add up to $\Sigma = 360° - \varphi_0 \leq 180°$. Then $XY = 2R \sin \frac{\varphi_0}{2} = 2R \sin \frac{\Sigma}{2}$, while every other side is $2R \sin \frac{\varphi_i}{2} < 2R \sin \frac{\Sigma}{2}$ because $\varphi_i < \Sigma \leq 180°$. So XY would be strictly longer

than every other side. But the longest length 3 occurs twice, so O is strictly inside, all five arcs are less than $180°$, and they add up to $360°$.

The equation. Let each side 2 subtend the central angle 4θ and each side 3 the angle β. Then $12\theta + 2\beta = 360°$, so $\frac{\beta}{2} = 90° - 3\theta$, and $2 = 2R\sin 2\theta$, $3 = 2R\cos 3\theta$. Dividing, $3\sin 2\theta = 2\cos 3\theta$, i.e. $6\sin\theta\cos\theta = 2(4\cos^3\theta - 3\cos\theta)$. Here $0 < 4\theta < 120°$, so $\cos\theta \neq 0$ and we may divide: $3\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$, so $4\sin^2\theta + 3\sin\theta - 1 = (4\sin\theta - 1)(\sin\theta + 1) = 0$ and $\sin\theta = \frac{1}{4}$. This is the only possibility, and it does give a pentagon: $\theta = \arcsin\frac{1}{4} < 30°$ makes $\beta = 180° - 6\theta$ positive, and all arcs are below $180°$.

The area. $\cos\theta = \frac{\sqrt{15}}{4}$, so $\sin 2\theta = \frac{\sqrt{15}}{8}$, $\cos 2\theta = \frac{7}{8}$ and $R^2 = \frac{1}{\sin^2 2\theta} = \frac{64}{15}$. Next $\sin 4\theta = 2\cdot\frac{\sqrt{15}}{8}\cdot\frac{7}{8} = \frac{7\sqrt{15}}{32}$ and $\sin\beta = \sin 6\theta = 3\sin 2\theta - 4\sin^3 2\theta = \frac{3\sqrt{15}}{8} - \frac{60\sqrt{15}}{512} = \frac{33\sqrt{15}}{128}$. Each triangle O-side has area $\frac{1}{2}R^2\sin(\text{its central angle})$, so the area is $\frac{R^2}{2}(3\sin 4\theta + 2\sin 6\theta) = \frac{32}{15}\left(\frac{84\sqrt{15}}{128} + \frac{66\sqrt{15}}{128}\right) = \frac{32}{15}\cdot\frac{150\sqrt{15}}{128} = \frac{5\sqrt{15}}{2}$.

Check. Numerically $R \approx 2.0656$ and the area is ≈ 9.68246. Building the pentagon in each of the 10 distinct orders of the sides and applying the shoelace formula gives this same value every time, and the five arcs add to exactly $360°$.

Problem 3. **Answer:** 106. **The idea.** Call a side of a cell closed if it lies on the border of the board or separates cells of different colours. A cell with d same-colour neighbours has $4 - d$ closed sides, so balanced means: every cell has exactly two closed sides. For a cell C and one of its corners v, let $k(C, v) \in \{0, 1, 2\}$ be the number of closed sides of C through v. These numbers pass their parity along diagonals, and a chain argument shows that at every lattice point with odd coordinate sum the walls run straight through. That makes colours alternate along diagonals, so the top row determines the board and its colours travel down like tokens bouncing between the side walls. Counting token orbits gives $N(m, n) = 2^{\gcd(m,n)/2}$ when m and n are both even, and 0 otherwise.

Work it out. If $m = 1$ or $n = 1$, an end cell has one neighbour, so $N = 0$. From now on $m, n \geq 2$. Number rows $0, \ldots, m-1$ from the top and columns $0, \ldots, n-1$ from the left; cell (i, j) has corners at the lattice points $(i, j), (i, j+1), (i+1, j), (i+1, j+1)$, where $0 \leq x \leq m$, $0 \leq y \leq n$ for a lattice point (x, y). Its level is $x + y$. The points $(0, 0), (0, n), (m, 0), (m, n)$ are board corners, the other points on the border are border points, the rest are interior points. An interior point $v = (x, y)$ is a corner of four cells: NW $= (x-1, y-1)$, NE $= (x-1, y)$, SW $= (x, y-1)$, SE $= (x, y)$.

Step 1: local facts. (F1) A cell's two closed sides are either opposite (its k-values are $1, 1, 1, 1$) or adjacent (going round, $2, 1, 0, 1$, with 2 where they meet). Either way opposite corners of a cell carry k-values of the same parity, and if $k = 1$ at one corner, then $k = 1$ at the opposite corner. (F2) Going round an interior point v, the colour changes an even number of times, so an even number w of the four segments at v are walls. Then $k(\text{NW}) + k(\text{SE}) = w$ and $k(\text{NE}) + k(\text{SW}) = w$: opposite cells at v have k-values of the same parity, and the four k-values add up to $2w \in \{0, 4, 8\}$. (F3) At a border point both cells touching it have the same $k \geq 1$ (the border segment, plus 1 if the segment between them is a wall). (F4) A corner cell of the board has its two border sides as its closed sides, so $k = 2$ at the board corner. Call a non-corner point good if every cell touching it has $k = 1$ there.

Step 2: every non-corner point of odd level is good. Induction on the odd level ℓ. For $\ell = 1$: the points are $(0, 1)$ and $(1, 0)$; by (F4) the side between cells $(0, 0)$ and $(0, 1)$

is open, so both have $k = 1$ at $(0, 1)$; likewise at $(1, 0)$. Now let $\ell \geq 3$ and assume level $\ell - 2$. (*) If the top-left corner of a cell D has level $\ell - 2$, it is not a board corner, so it is good and $k = 1$ there; by (F1), $k(D, \text{bottom-right corner}) = 1$.

(a) Interior $v = (x, y)$ of level ℓ. By (*) the NW cell has $k = 1$ at v. By (F2), $k(\text{SE}) = w-1$ is odd and at most 2, so $k(\text{SE}) = 1$ and $w = 2$. $k(\text{NE}) \equiv k(\text{SW})$. If both are odd, all four are 1: v is good. Otherwise both are even with $k(\text{NE}) + k(\text{SW}) = w = 2$, so one of them is 2 and the other 0. Case A: $k(\text{NE}) = 2$, $k(\text{SW}) = 0$. The NE cell P has its left and bottom sides closed, so its top and right sides are open and $k(P, w) = 0$ at its top-right corner $w = (x - 1, y + 1)$, which has level ℓ again. By (F3) and (F4), w is not on the border, so it is interior with SW cell P. The NW cell of w has top-left corner $(x - 2, y)$ of level $\ell - 2$, so it has $k = 1$ at w by (*), and the reasoning above gives Case A at w. Repeating, Case A holds at $(x-2, y+2), (x-3, y+3), \ldots$, all interior. That is impossible. Case B ($k(\text{SW}) = 2$, $k(\text{NE}) = 0$) is Case A on the transposed board. (Swapping rows and columns keeps levels, NW and SE cells and top-left corners, and swaps NE with SW.) So v is good.

(b) A border point on the bottom edge, $v = (m, y)$: the cell $(m - 1, y - 1)$ has top-left corner of level $\ell - 2$, so $k = 1$ at v by (*), and by (F3) the other cell has $k = 1$ too. The right edge is the same, using the cell $(x - 1, n - 1)$.

(c) A point on the top edge, $v = (0, y)$ with $y = \ell \geq 3$. If it is not good, then by (F3) the cell $L = (0, y - 1)$ has $k = 2$ at v: its closed sides are its top and right sides, so $k = 0$ at its bottom-left corner $z = (1, y - 1)$, an interior point of level ℓ with NE cell L. The NW cell $(0, y - 2)$ of z has $k = 1$ at z by (*), so as in (a) the SW cell of z has $k = 2$: Case B at z, which (a) showed impossible. The left edge follows by transposing.

Step 3: m and n are even. If n is odd, the corner cell $(0, n - 1)$ has closed top and right sides (F4), so $k = 0$ at its corner $(1, n - 1)$. That is an interior point of odd level n, so it is not good, contradicting Step 2. Likewise for m. So $N(m, n) = 0$ unless m and n are both even. Assume that from now on.

Step 4: the colour rules. At a good interior point all four k-values are 1: two walls, no cell with both, so the walls are opposite and the four cells form two dominoes of different colours. In particular both diagonal pairs have different colours. At a good border point the two cells have the same colour. Writing $c(i, j)$ for colours, Step 2 therefore gives: (R1) $c(i, j) \neq c(i+1, j+1)$ whenever $i+j$ is odd; (R2) $c(i, j+1) \neq c(i+1, j)$ whenever $i+j$ is odd; (R3) $c(0, 2t) = c(0, 2t+1)$, $c(m-1, 2t) = c(m-1, 2t+1)$, $c(2t, 0) = c(2t+1, 0)$ and $c(2t, n-1) = c(2t+1, n-1)$. Conversely (R1)–(R3) imply balance. Take an interior cell $X = (a, b)$. If $a+b$ is even, (R1) gives $c(a-1, b) \neq c(a, b+1)$ and $c(a, b-1) \neq c(a+1, b)$, so exactly one of the upper and right neighbours and exactly one of the left and lower neighbours match X. If $a+b$ is odd, (R2) does the same with the pairs (upper, left) and (right, lower). A top-edge cell $(0, b)$ has one neighbour matching by (R3) (right if b is even, left if b is odd), and (R1) or (R2) makes exactly one of the other two match. The corners $(0, 0)$ and $(0, n - 1)$ match both of their neighbours by (R3). The other edges and corners follow by rotating the board by 180° or transposing it: for even m, n these map the rules (R1)–(R3) to themselves. So balanced colourings are exactly the solutions of (R1)–(R3).

Step 5: the top row decides. Fix $0 \leq i \leq m - 2$. For each column j, exactly one rule ties $(i + 1, j)$ to row i. If $i + j$ is even it is $c(i + 1, j) \neq c(i, j - 1)$ (R1), or for $j = 0$ (then i is even) $c(i + 1, 0) = c(i, 0)$. If $i + j$ is odd it is $c(i + 1, j) \neq c(i, j + 1)$ (R2), or

for $j = n-1$ (then i is even) $c(i+1, n-1) = c(i, n-1)$. Every rule except the top and bottom rows of (R3) is one of these ties. Picture it as a token carrying the colour of (i, p). It moves to $(i+1, p+1)$ if $i+p$ is odd and to $(i+1, p-1)$ if $i+p$ is even, flipping colour. If that would leave the board, it drops straight down and keeps its colour. Give a token moving right at column p the label p, and a token moving left at column p the label $2n-1-p$, modulo $2n$. Every move raises the label by 1. (A right-mover stays a right-mover, a left-mover stays a left-mover, and the drops at column $n-1$ and column 0 turn label $n-1$ into n and label $2n-1$ into 0.) In row 0 the column p moves right if and only if p is odd, so the n tokens carry exactly the odd labels. Call the token with row-0 label r token r: in row i its label is $r+i$. The top pair $(0, 2t), (0, 2t+1)$ has labels $-(2t+1)$ and $2t+1$. In row $m-1$ (with m even) column p moves right if and only if p is even, so the bottom pair $(m-1, 2t), (m-1, 2t+1)$ has labels $2t$ and $-2t-2$, which are tokens r and $-r-2m$ with $r = 2t-m+1$.

Step 6: counting. Let v_r be the colour of token r in row 0; its colour in the bottom row is $v_r + f_r \pmod 2$, where the number of flips f_r does not depend on the colouring. Balanced colourings correspond exactly to vectors (v_r) with $v_r = v_{-r}$ and $v_r + f_r = v_{-r-2m} + f_{-r-2m}$ for all odd r. This system has a solution: the colouring $c(i, j) \equiv \lfloor i/2 \rfloor + \lfloor j/2 \rfloor \pmod 2$ (a chessboard of 2×2 blocks) is balanced. So the number of solutions equals the number of solutions of $v_r = v_{-r}$, $v_r = v_{-r-2m}$, which is 2 to the number of orbits of the odd residues mod $2n$ under the maps $r \mapsto -r$ and $r \mapsto -r-2m$. Composing them gives $r \mapsto r-2m$, and the multiples of $2m$ modulo $2n$ are the multiples of $2g$, $g = \gcd(m, n)$. So the orbits are the sets $\{\pm r + 2g\mathbb{Z}\}$. The odd residues mod $2g$ are g classes, and $r \mapsto -r$ pairs them up with no fixed class ($r \equiv -r \pmod{2g}$ would need $g \mid r$, impossible for odd r as g is even). Hence there are $g/2$ orbits and $N(m, n) = 2^{\gcd(m,n)/2}$.

The sum. Only $m = 2a$, $n = 2b$ with $a, b \le 5$ contribute $2^{\gcd(a,b)}$. The diagonal gives $2+4+8+16+32 = 62$; the pairs $(2, 4), (4, 2)$ give $4+4$; the other 18 ordered pairs have $\gcd 1$ and give 36. Total $62 + 8 + 36 = 106$.

Check. On the 2×2 board all four cells must match: $2 = 2^1$ colourings. The 4×4 board has $4 = 2^2$: the two 2×2-block chessboards and the two ring colourings (outer ring one colour, central 2×2 the other). The 4×6 board has only $2 = 2^{\gcd(4,6)/2}$, the block chessboards. A transfer-matrix count over rows reproduces $N(m, n)$ for every $m, n \le 10$, and the sum 106.

Problem 4. **Answer:** 121**. The idea.** Since 3 is invertible modulo 11, a shift $x = y - a/3$ removes the x^2 term and leaves $y^3 + b'y + c'$. The pure cube y^3 permutes the residues because $\gcd(3, 10) = 1$. Any nonzero b' spoils injectivity, and we can write down the collision explicitly. So the good cubics are exactly $(x+d)^3 + e$.

Work it out. Work in $\mathbb{Z}_{11}$. Let $h = 3^{-1} = 4$ and put $x = y - ah$. Since $x \mapsto x + ah$ is a bijection, f permutes $\mathbb{Z}_{11}$ if and only if $g(y) = f(y - ah)$ does. Expanding, $g(y) = y^3 + b'y + c'$ with $b' = b - a^2 h$ and some constant c'.

If $b' = 0$: $y^3 = z^3$ with $y \ne z$ is impossible. If one of them is 0 so is the other; otherwise $(yz^{-1})^3 = 1$, but the order of yz^{-1} divides 10 and 3, so it is 1 and $y = z$. So g is a bijection.

If $b' \ne 0$: $g(y) - g(z) = (y-z)(y^2 + yz + z^2 + b')$, so it suffices to find $y \ne z$ with $y^2 + yz + z^2 = -b'$. The nonzero squares mod 11 are $1, 3, 4, 5, 9$. If $-b'$ is a square,

$-b' = s^2$ with $s \neq 0$, take $y = s$, $z = -s$: then $y^2 + yz + z^2 = s^2 = -b'$ and $y \neq z$. If $-b'$ is a non-square, take $z = 2y$: we need $7y^2 = -b'$, i.e. $y^2 = -b' \cdot 7^{-1}$, and 7 is a non-square, so the right side is a nonzero square. So a $y \neq 0$ exists and $z = 2y \neq y$. Either way $g(y) = g(z)$ and f is not a permutation.

So f permutes the residues exactly when $b \equiv 4a^2 \pmod{11}$. For each of the 11 values of a and 11 values of c there is exactly one b, giving $11^2 = 121$ triples.

Check. Equivalently the good cubics are $(x + d)^3 + e$: expanding gives $a = 3d$, $b = 3d^2 = a^2/3$, $c = d^3 + e$, again 121 choices of (d, e). For example $a = 1$ forces $b = 4$: $x^3 + x^2 + 4x = (x+4)^3 - 64$, a shifted cube. A direct test of all 1331 triples finds exactly 121.

Problem 5. **Answer:** 1024. **The idea.** For even degree the identity says exactly that squaring maps the multiset of roots onto itself. Then squaring permutes the distinct roots, which must be 0 or roots of unity of odd order, and multiplicities are constant on each squaring cycle. So the count is the coefficient of t^{10} in a product over cycles. Taking logarithms, the product collapses to $\frac{1}{1-2t}$ because exactly 2^L points satisfy $z^{2^L} = z$.

Work it out. Write $P(x) = \prod_{i=1}^{10}(x - r_i)$. As the degree is even, $P(-x) = \prod(-x - r_i) = \prod(x + r_i)$, so $P(x)P(-x) = \prod(x^2 - r_i^2)$, while $P(x^2) = \prod(x^2 - r_i)$. Two polynomials in x^2 agree for all x exactly when they agree as polynomials in $y = x^2$. So the identity holds if and only if $\prod(y - r_i^2) = \prod(y - r_i)$, that is, the multiset $\{r_i^2\}$ equals the multiset $\{r_i\}$.

Structure. Let S be the set of distinct roots. Every r^2 with $r \in S$ is in S, and every element of S is some r_i^2. So squaring maps the finite set S onto itself and is a permutation of S. Each $z \in S$ is then periodic: $z^{2^k} = z$ for some $k \geq 1$, so $z = 0$ or $z^{2^k-1} = 1$. Let $\mu(w)$ be the multiplicity of w. The multiplicity of w in $\{r_i^2\}$ is the sum of $\mu(z)$ over roots z with $z^2 = w$, and exactly one such z is in S. So $\mu(w) = \mu(z)$: multiplicities are constant along each cycle of the permutation. Conversely, any multiset that is a union of whole squaring cycles, each with a constant multiplicity, satisfies $\{r_i^2\} = \{r_i\}$.

Let U be the set of all complex z with $z^{2^k} = z$ for some $k \geq 1$. Squaring permutes U (on the $(2^k - 1)$-th roots of unity it is a bijection because $2^k - 1$ is odd), so U splits into cycles. Our polynomials correspond exactly to choices of a multiplicity $e_C \geq 0$ for every cycle C with $\sum e_C|C| = 10$. So the answer is the coefficient of t^{10} in $G(t) = \prod_C \frac{1}{1-t^{|C|}}$.

The generating function. Let a_d be the number of cycles of length d. The points with $z^{2^L} = z$ are 0 and the $2^L - 1$ roots of $z^{2^L-1} = 1$: 2^L points. They are exactly the points on cycles whose length divides L, so $\sum_{d|L} d\, a_d = 2^L$. Now, as formal power series, $\log G(t) = \sum_d a_d \sum_{k\geq 1} \frac{t^{dk}}{k} = \sum_{L\geq 1} \frac{t^L}{L} \sum_{d|L} d\, a_d = \sum_{L\geq 1} \frac{(2t)^L}{L} = -\log(1 - 2t)$. Since log is injective on power series with constant term 1, $G(t) = \frac{1}{1-2t}$. The coefficient of t^{10} is $2^{10} = 1024$.

Check. In degree 2 the formula predicts 4. The cycles of length at most 2 are $\{0\}$, $\{1\}$ and $\{\omega, \omega^2\}$ ($\omega = e^{2\pi i/3}$), giving x^2, $x(x - 1)$, $(x - 1)^2$ and $x^2 + x + 1$; indeed $(x^2 + x + 1)(x^2 - x + 1) = x^4 + x^2 + 1$. Solving the coefficient equations directly in degree 4 gives $16 = 2^4$ polynomials. For degree 10, listing the squaring cycles of length up to 10 explicitly and counting the multisets also gives 1024.

Problem 6. **Answer:** 4059. **The idea.** The two divisors $n-t$ and $n+t$ are always coprime, so their product n^2-t^2 divides n^2+1. Writing the quotient as $k+1$ gives the equation $kn^2-(k+1)t^2=1$. The substitution $y=n-t$, $x=t-ky$ turns it into the negative Pell equation $x^2-k(k+1)y^2=-1$, and a descent shows that this has no solution unless $k=1$. For $k=1$ we are left with $n^2-2t^2=1$, whose solutions all come from $(3,2)$ by one fixed linear map.

Work it out. Step 1: the two divisors are coprime. Let $d=n-t$, $e=n+t$ and $g=\gcd(d,e)$. Then g divides $d+e=2n$ and divides n^2+1. An odd prime dividing g would divide both n and n^2+1, which is impossible, and $4\nmid n^2+1$ because squares are 0 or 1 modulo 4. So $g=1$ or $g=2$. Suppose $g=2$. Then n^2+1 is even, so n is odd and $n^2+1\equiv 2 \pmod 4$. The even divisors d and e of n^2+1 are therefore both $\equiv 2 \pmod 4$, so $2n=d+e\equiv 0 \pmod 4$ and n is even, a contradiction. Hence $g=1$.

Step 2: the equation. Since d and e are coprime divisors of n^2+1, their product n^2-t^2 divides n^2+1. As $0<n^2-t^2<n^2+1$, the quotient is an integer at least 2; call it $k+1$ with $k\geq 1$. Then $(k+1)(n^2-t^2)=n^2+1$, that is, $kn^2-(k+1)t^2=1$. Call this equation $(\star)$. Conversely, if $(\star)$ holds with $k\geq 1$ and $1\leq t<n$, then $(k+1)(n-t)(n+t)=n^2+1$, so both $n-t$ and $n+t$ divide n^2+1 and n is centred.

Step 3: a Pell equation. Put $y=n-t\geq 1$ and $x=t-ky$, so that $t=x+ky$ and $n=x+(k+1)y$. Expanding, $kn^2-(k+1)t^2=k\,(x+(k+1)y)^2-(k+1)(x+ky)^2=k(k+1)y^2-x^2$, because the terms in xy cancel and $k(k+1)^2-(k+1)k^2=k(k+1)$. So $(\star)$ says exactly $x^2-k(k+1)y^2=-1$.

Step 4: no solution when $k\geq 2$. Let $k\geq 2$ and $D=k(k+1)$, and suppose that $x^2-Dy^2=-1$ has a solution in integers with $y\geq 1$. Replacing x by $|x|$ we may take $x\geq 0$; choose such a solution with y as small as possible. Then $x^2=k^2y^2+(ky^2-1)>k^2y^2$ because $ky^2\geq 2$, so $x>ky$. Also $x^2<Dy^2<\left(k+\frac{1}{2}\right)^2y^2$, so $2x<(2k+1)y$. Now set $x'=(2k+1)x-2Dy$ and $y'=(2k+1)y-2x$. Expanding shows $x'^2-Dy'^2=\left((2k+1)^2-4D\right)(x^2-Dy^2)$, and $(2k+1)^2-4k(k+1)=1$, so $x'^2-Dy'^2=-1$. Moreover $y'>0$ because $2x<(2k+1)y$, and $y'<y$ because $2ky<2x$. So $(|x'|,y')$ is a solution with a smaller positive y, a contradiction. Hence $k=1$.

Step 5: solving $n^2-2t^2=1$. With $k=1$, $(\star)$ reads $n^2-2t^2=1$, and by Step 2 every solution with $t\geq 1$ gives a centred n (then $t<n$ automatically, since $n^2=2t^2+1>t^2$). Take such a solution. Here $t\neq 1$ (that would need $n^2=3$), so $t\geq 2$. Put $n'=3n-4t$ and $t'=3t-2n$. The same expansion as in Step 4 gives $n'^2-2t'^2=(9-8)(n^2-2t^2)=1$. Next, $t'\geq 0$ because $9t^2\geq 8t^2+4=4n^2$ (as $t^2\geq 4$); $t'<t$ because $t<n$; and $n'>0$ because $9n^2=18t^2+9>16t^2$. Repeating this step, t strictly decreases while staying non-negative, so we reach $t=0$, $n=1$. The step is inverted by $(n,t)\mapsto(3n+4t,2n+3t)$, so every solution lies on the chain $(1,0)\to(3,2)\to(17,12)\to(99,70)\to(577,408)\to(3363,2378)\to(19601,13860)\to\cdots$, along which n strictly increases.

So the centred numbers are exactly $3,17,99,577,3363,19601,\ldots$, and those below 10000 have sum $3+17+99+577+3363=4059$.

Check. For $n=17$, $t=12$: 5 and 29 both divide $17^2+1=290=2\cdot 5\cdot 29$. For $n=99$, $t=70$: 29 and 169 both divide $99^2+1=9802=2\cdot 13^2\cdot 29$. In each case $n^2+1=2(n-t)(n+t)$, as $k=1$ predicts. A direct search over every $n<10000$ and every divisor $d<n$ of n^2+1 (with $t=n-d$, testing whether $2n-d$ also divides n^2+1) finds exactly $3,17,99,577,3363$.

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