MathCastle

CHAPTER COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A car brochure printed in Europe says the car uses 6.5 litres of fuel for every 100 kilometres it drives. An American buyer wants to compare it with cars rated in miles per gallon. Using 1 mile = 1.609 km and 1 gallon = 3.785 litres, what is the car's fuel economy in miles per gallon? Express your answer to the nearest whole number.

1. _____

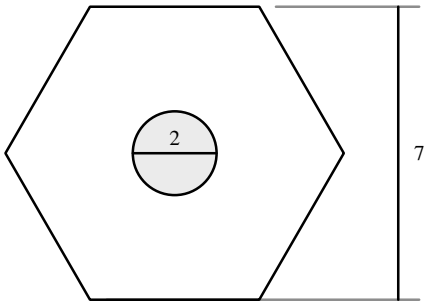
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2. A family needs a moving van for one day. Company *A* charges \$135 for the day, which includes the first 300 km driven; every kilometre beyond 300 costs \$0.30. Company *B* charges \$90 for the day plus \$0.20 for every kilometre driven, with no free kilometres. The family will drive a whole number of kilometres. For how many different whole numbers of kilometres is Company *A* strictly cheaper than Company *B*?

2. _____

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3. An unsharpened pencil is a right prism 175 mm long whose cross-section is a regular hexagon. Measured with calipers across two opposite flat faces, the pencil is 7 mm wide. A cylindrical graphite core 2 mm in diameter runs along the whole length of the pencil, through its centre. The rest of the pencil is wood. What is the volume of the wood, in cubic millimetres? Express your answer to the nearest whole number.



3. _____

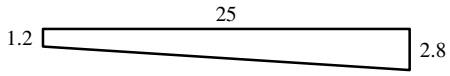
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4. Three lighthouses stand along a coast. Lighthouse *A* flashes exactly at midnight and then every 12 seconds. Lighthouse *B* first flashes 6 seconds after midnight and then every 18 seconds. Lighthouse *C* first flashes 12 seconds after midnight and then every 30 seconds. Each flash is instantaneous. How many seconds after midnight do all three lighthouses flash at the same instant for the 10th time?

4. _____

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5. A rectangular swimming pool is 25 m long and 10 m wide. Its floor slopes evenly along the length, so the pool is 1.2 m deep at the shallow end and 2.8 m deep at the deep end. The empty pool is filled with a hose delivering 400 litres per minute until the water surface is 0.2 m below the top edge of the pool. How many hours does the filling take? (1 cubic metre = 1000 litres.) Express your answer as a decimal.



5. _____

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6. An online poll has two choices, Yes and No, and each voter picks exactly one. The results page shows each choice's share of the votes as a percentage rounded to the nearest tenth of a percent, with halves rounded up (so a share of exactly 66.65% would be shown as 66.7%). At the moment the page shows Yes at 66.6%. What is the smallest possible number of votes cast so far?

6. _____

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7. Ana changes \$500 into euros at an exchange booth. The rate is 0.92 euro per dollar, and the booth keeps a fee of 2% of the money she hands over. On her way home she changes all of her euros back into dollars at a rate of 1.10 dollars per euro, again paying a 2% fee on the money she hands over. Do not round any amount until the end. How many dollars has Ana lost in all? Express your answer to the nearest cent.

7. _____

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8. A lottery draws 6 different numbers at random from the integers 1 through 49; every set of 6 is equally likely. Call a draw lucky if exactly one pair of the drawn numbers consists of two consecutive integers. For example, $\{7, 8, 20, 31, 33, 40\}$ is lucky; $\{7, 8, 9, 20, 31, 40\}$ is not (it has the two pairs 7, 8 and 8, 9), and neither is $\{7, 8, 20, 21, 33, 40\}$ or $\{3, 11, 20, 31, 33, 40\}$. What is the probability that a draw is lucky? Express your answer as a decimal to the nearest thousandth.

8. _____

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Answer Key — MathCounts Chapter Target Practice Contest 4

1. 36

2. 224

3. 6876

4. 1752

5. 18.75

6. 287

7. 14.04

8. 0.388

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Solutions — MathCounts Chapter Target Practice Contest 4

1. **Answer:** 36**. The idea.** "Litres per 100 km" is fuel per distance, while "miles per gallon" is distance per fuel, so the European figure has to be turned upside down before the units are converted. Dividing 6.5 by anything, or multiplying 6.5 by the conversion factors, gives nonsense like 2.4 or 15.

   Work it out. Flip the rate: the car goes 100 km on 6.5 L, which is $\frac{100}{6.5} \approx 15.385$ km per litre. A gallon is 3.785 litres, so on one gallon it goes $15.385 \times 3.785 \approx 58.23$ km. A mile is 1.609 km, so that is $58.23 \div 1.609 \approx 36.19$ miles. The fuel economy is about 36.19 mpg, which rounds to 36.

   Check. In one line: $\frac{100 \times 3.785}{6.5 \times 1.609} = \frac{378.5}{10.4585} \approx 36.19$. Sanity check: a thirsty car at 10 L/100 km is known to be about 23.5 mpg, and 6.5 L/100 km uses 0.65 as much fuel, so it should get about $23.5/0.65 \approx 36$ mpg.

2. **Answer:** 224**. The idea.** Company A's price is not one straight line: it is flat at \$135 up to 300 km and then climbs at \$0.30 per km, faster than Company B's \$0.20 per km. So B is cheaper for short trips, A wins in a middle stretch, and B wins again for long trips. Compare the prices separately on each side of 300 km.

   Work it out. Let d be the distance in km; B costs $90 + 0.2d$. For $d \le 300$, A costs 135, and $135 < 90 + 0.2d$ means $0.2d > 45$, so $d > 225$. For $d > 300$, A costs $135 + 0.3(d - 300) = 45 + 0.3d$, and $45 + 0.3d < 90 + 0.2d$ means $0.1d < 45$, so $d < 450$. Together, A is strictly cheaper exactly when $225 < d < 450$, that is for $d = 226, 227, \ldots, 449$: $449 - 226 + 1 = 224$ whole numbers.

   Check. At 225 km both cost \$135 (a tie, so not counted), at 226 km A costs \$135 and B costs \$135.20; at 449 km A costs $45 + 134.70 =$ \$179.70 and B costs $90 + 89.80 =$ \$179.80; at 450 km both cost \$180. Ignoring the free kilometres (treating A as $135 + 0.3d$) would make A never cheaper, a sign that the allowance is the whole point.

3. **Answer:** 6876**. The idea.** The 7 mm is measured across the flats, not corner to corner, so it is twice the apothem (the distance from the centre to a side), and the side of the hexagon is not 3.5 mm. Split the hexagon into six equilateral triangles whose height is the apothem.

   Work it out. The apothem is 3.5 mm, and it is the height of an equilateral triangle of side s, so $\frac{\sqrt{3}}{2}s = 3.5$ and $s = \frac{7}{\sqrt{3}}$. Each triangle has area $\frac{1}{2} \cdot s \cdot 3.5 = \frac{12.25}{\sqrt{3}}$, so the hexagon has area $6 \cdot \frac{12.25}{\sqrt{3}} = \frac{73.5}{\sqrt{3}} = 24.5\sqrt{3} \approx 42.435$ square mm. The core's cross-section is a circle of radius 1, area $\pi \approx 3.1416$. The wood's cross-section is $24.5\sqrt{3} - \pi \approx 39.294$ square mm, and its volume is $175(24.5\sqrt{3} - \pi) \approx 6876.39$, so about 6876 cubic mm.

   Check. The hexagon lies between its inscribed circle (radius 3.5, area ≈ 38.5) and its circumscribed circle (radius $\frac{7}{\sqrt{3}} \approx 4.04$, area ≈ 51.3), and 42.4 lies between them, closer to the inner circle as it should for a hexagon. Mistaking 7 mm for the corner-to-corner width would give side 3.5 and area ≈ 31.8, far too small since it is smaller than the inscribed circle.

4. **Answer:** 1752**. The idea.** The lighthouses do not start together, so the first triple flash is not at midnight. Find the first common flash by listing, and then use the fact that the whole pattern repeats every $\mathrm{lcm}(12, 18, 30) = 180$ seconds.

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Work it out. A flashes at the multiples of 12. B flashes at $6, 24, 42, 60, \ldots$; the first of these that is a multiple of 12 is 24, and after that A and B agree every $\text{lcm}(12, 18) = 36$ seconds: at $24, 60, 96, 132, 168, \ldots$. C flashes at $12, 42, 72, 102, 132, \ldots$, i.e. at times that are 12 more than a multiple of 30. Checking the A-and-B list: $24, 60, 96$ leave remainders $24, 0, 6$ on division by 30, but $132 = 4 \cdot 30 + 12$ works. So the first triple flash is at 132 seconds, and they repeat every 180 seconds. The 10th is at $132 + 9 \cdot 180 = 1752$ seconds (that is 29 minutes 12 seconds after midnight).

Check. $1752 = 146 \cdot 12$, $1752 - 6 = 1746 = 97 \cdot 18$ and $1752 - 12 = 1740 = 58 \cdot 30$, so all three do flash at 1752. The gaps between triple flashes cannot be shorter than 180 seconds, because the difference of two triple-flash times is a multiple of 12, 18 and 30.

5. Answer: 18.75**. The idea.** Seen from the side, the water is a trapezoid: its depth is 0.2 m less than the pool's depth at every point, so it is 1.0 m deep at the shallow end and 2.6 m deep at the deep end. The water is a prism with that trapezoid as its cross-section and the 10-m width as its length.

Work it out. The trapezoid has parallel sides 1.0 and 2.6 and length 25, so its area is $\frac{1.0+2.6}{2} \cdot 25 = 45$ square metres. The water's volume is $45 \times 10 = 450$ cubic metres = 450,000 litres. At 400 litres per minute that takes $450{,}000/400 = 1125$ minutes = 18.75 hours.

Check. The average water depth is the average of the two end depths, 1.8 m, and $25 \times 10 \times 1.8 = 450$ cubic metres. A common slip is to use the pool's full depths (1.2 and 2.8, average 2.0), which gives 500 cubic metres and 20.83 hours, or to subtract 0.2 only at one end.

6. Answer: 287**. The idea.** It is tempting to say $66.6\% \approx \frac{2}{3}$, so 3 votes. But $\frac{2}{3} = 66.666\ldots\%$ is shown as 66.7%, so the true share must be a little below $\frac{2}{3}$. Measure how far below: if there are n votes and y of them are Yes, the whole number $k = 2n - 3y$ is at least 1, and $\frac{y}{n} = \frac{2}{3} - \frac{k}{3n}$. A small gap $\frac{k}{3n}$ forces n to be large.

Work it out. The page shows 66.6% exactly when $0.6655 \le \frac{y}{n} < 0.6665$. Since $\frac{2}{3} - 0.6655 = \frac{7}{6000}$ and $\frac{2}{3} - 0.6665 = \frac{1}{6000}$, this says $\frac{1}{6000} < \frac{k}{3n} \le \frac{7}{6000}$. The right-hand inequality gives $n \ge \frac{2000k}{7}$. With $k = 1$ this is $n \ge 285.7$, so $n \ge 286$; with $k \ge 2$ it is $n \ge 571.4$, which is worse. For $k = 1$ we need $3y = 2n - 1$, so $2n - 1$ must be a multiple of 3, which means n leaves remainder 2 on division by 3. The first such n from 286 on is 287, with $y = \frac{2 \cdot 287 - 1}{3} = 191$. The left-hand inequality, $n < 2000k$, holds easily. So the smallest possible number of votes is 287.

Check. $\frac{191}{287} = 0.665505\ldots$, which is 66.55% and a bit, so it rounds to 66.6%. With 286 votes, $\frac{190}{286} \approx 66.43\%$ and $\frac{191}{286} \approx 66.78\%$ show as 66.4% and 66.8%, so 286 fails. The same size bound $n \ge 286$ covers every smaller n.

7. Answer: 14.04**. The idea.** Each step multiplies the money by a factor: a 2% fee multiplies by 0.98, and each conversion multiplies by its rate. Because multiplication can be done in any order, it does not matter whether the fee is taken before or after converting. The overall factor is $0.98 \times 0.92 \times 0.98 \times 1.10$.

Work it out. Going out: $500 \times 0.98 = 490$ dollars are converted, giving $490 \times 0.92 = 450.80$ euros. Coming back: $450.80 \times 0.98 = 441.784$ euros are converted, giving $441.784 \times 1.10 = 485.9624$ dollars. Her loss is $500 - 485.9624 = 14.0376$, which is \$14.04 to the nearest cent.

Check. The overall factor is $0.98^2 \times 0.92 \times 1.10 = 0.9604 \times 1.012 = 0.9719248$, and $500 \times (1 -$

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$0.9719248) = 14.0376$. Notice that the rates alone would have earned her money ($0.92 \times 1.10 = 1.012 > 1$, a 1.2% gain); the two fees, about 3.96% together, more than cancel it.

8. Answer: 0.388**. The idea.** A lucky draw is made of five blocks: one "domino" of two consecutive numbers and four single numbers, with at least one undrawn number between any two neighbouring blocks. Squeeze the blocks together: remove one undrawn number from each of the 4 gaps between neighbouring blocks, and shrink the domino to a single number. What is left is 5 different numbers from a shorter range, plus the information of which of the 5 was the domino, and the squeezing can be undone uniquely. So lucky draws can be counted with a single binomial coefficient.

Work it out. Write the blocks' starting numbers in increasing order as $b_1 < b_2 < \cdots < b_5$, and say block number j is the domino. Squeezing replaces b_i by $c_i = b_i - (i-1)$ if block i comes before the domino or is the domino, and by $c_i = b_i - (i-1) - 1$ if it comes after the domino. The rule "at least one gap between blocks" becomes exactly "$c_1 < c_2 < \cdots < c_5$", and the largest drawn number ≤ 49 becomes $c_5 \le 49 - 4 - 1 = 44$. Conversely any 5 numbers $c_1 < \cdots < c_5$ from 1 to 44 and any choice of j from 1 to 5 rebuild exactly one lucky draw. Hence there are $5\binom{44}{5} = 5 \times 1{,}086{,}008 = 5{,}430{,}040$ lucky draws out of $\binom{49}{6} = 13{,}983{,}816$ draws in all, and the probability is $\frac{5{,}430{,}040}{13{,}983{,}816} \approx 0.38831$, or 0.388.

Check. A small case: from $\{1, \ldots, 5\}$ choose 3. The same argument gives $2\binom{3}{2} = 6$ sets with exactly one consecutive pair, and listing confirms it: $\{1,2,4\}, \{1,2,5\}, \{2,3,5\}, \{1,3,4\}, \{1,4,5\}, \{2,4,5\}$. For the lottery, the same squeezing with no domino shows that $\binom{44}{6} = 7{,}059{,}052$ draws (about 50.5%) have no consecutive pair at all, so the draws with two or more consecutive pairs make up the remaining $1 - 0.5048 - 0.3883 \approx 0.107$, a reasonable split.

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