MathCastle

CHAPTER COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The hybrid version of a car costs \$2,400 more than the regular model. The hybrid travels 32 miles per gallon of gas, the regular model travels 24 miles per gallon, and gas costs \$3.60 per gallon. After how many miles of driving will the money the hybrid saves on gas exactly equal the extra \$2,400 it cost?

1. _____

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2. A small theatre has 240 seats. At a ticket price of \$20 it has 200 buyers for a show, and every \$1 cut in the price brings 15 more buyers (so at \$18 there would be 230 buyers). The theatre cannot sell more tickets than it has seats, so if there are more buyers than seats, only 240 tickets are sold. The price must be a whole number of dollars, at most \$20. What is the greatest possible ticket revenue, in dollars, for one show?

2. _____

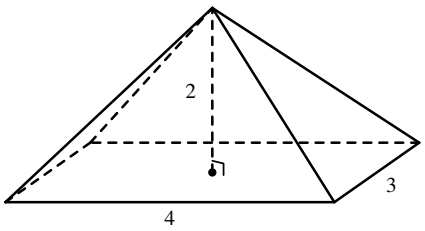
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3. Four different roads join Ana's home to the park, three different roads join the park to her school, and two more roads run directly from her home to the school; there are no other roads. Ana walks from home to school, either through the park or directly, and later walks from school back home, again either through the park or directly. During the whole round trip she never walks along the same road twice (in either direction), and on neither walk does she pass through any place twice. How many different round trips are possible?

3. _____

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4. A tent is shaped like a pyramid. Its floor is a rectangle 4 m long and 3 m wide, and its peak is 2 m directly above the centre of the floor. The four sloping triangular sides are made of fabric; the floor is not. How many square metres of fabric are needed for the four sides? Express your answer in simplest radical form.



4. _____

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5. A steel washer is a flat ring 0.3 cm thick. Its outer diameter is 5 cm and the hole in its centre has diameter 2 cm. Steel has a density of 7.85 grams per cubic centimetre. A shipping box may hold at most 10 kg of washers (ignore the mass of the box). What is the greatest number of whole washers that can be put in the box?

5. _____

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6. A game-show wheel has 20 equal sectors labelled $5, 10, 15, \ldots, 100$. A contestant spins the wheel twice; each spin lands on each sector with equal chance, independently of the other spin. If the sum of her two numbers is 100 or less, her score is that sum; if the sum is more than 100, she busts and her score is 0. What is her expected (average) score? Express your answer as a decimal.

6. _____

7. A ribbon is 996 cm long. Starting from its left end, Lia makes a mark every 12 cm ($12, 24, 36, \ldots$ cm from the left end). Starting from its right end, Ben makes a mark every 18 cm ($18, 36, 54, \ldots$ cm from the right end). Neither of them marks the ends of the ribbon themselves. Some of Ben's marks land exactly on Lia's marks. The ribbon is then cut at every marked point. Into how many pieces is it cut?

7. _____

8. A digital display shows a six-digit whole number (whose first digit is not 0). When the display is turned upside down (rotated half a turn), each digit is turned upside down and the order of the digits is reversed. Upside down, the digits $0, 1, 2, 5$ and 8 each read as themselves, 6 reads as 9 and 9 reads as 6, and $3, 4$ and 7 do not read as any digit. For example, 168891 turned upside down reads 168891 again. How many six-digit numbers read as the same number after the display is turned upside down and are also divisible by 6?

8. _____

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Answer Key — MathCounts Chapter Target Practice Contest 3

1. 64000

2. 4140

3. 122

4. $10 + 6\sqrt{2}$

5. 257

6. 33.25

7. 111

8. 45

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Solutions — MathCounts Chapter Target Practice Contest 3

1. Answer: 64000**. The idea.** Do not compare gallons or tanks; compare the *cost of one mile* in each car. The savings grow by the same amount every mile, so the break-even distance is the extra price divided by the savings per mile.

Work it out. The regular model spends $\frac{3.60}{24} = \$0.15$ per mile on gas. The hybrid spends $\frac{3.60}{32} = \$0.1125$ per mile. Every mile the hybrid saves $0.15 - 0.1125 = \$0.0375$. To save \$2,400 takes $\frac{2400}{0.0375} = 64{,}000$ miles.

Check. Over 64,000 miles the regular model burns $64000/24 \approx 2666.67$ gallons and the hybrid burns $64000/32 = 2000$ gallons; the difference is $666.\overline{6}$ gallons, which at \$3.60 each costs exactly $666.\overline{6} \times 3.60 = \$2{,}400$.

2. Answer: 4140**. The idea.** Without the seat limit, revenue is a downward parabola in the price cut, and its peak is the usual answer. But the peak here needs more seats than the theatre has, so the real maximum sits where the seat limit starts to bite, and the neighbouring prices must be compared by hand.

Work it out. Cut the price by x dollars. Buyers: $200+15x$; price: $20-x$. Ignoring seats, revenue is $(20-x)(200+15x) = 4000+100x-15x^2$, whose vertex is at $x = \frac{100}{30} = 3\frac{1}{3}$, suggesting \$17 (245 buyers, \$4,165). But 245 buyers do not fit in 240 seats. Buyers fit only when $200 + 15x \le 240$, i.e. $x \le 2$. For $x \le 2$ the revenue $4000 + 100x - 15x^2$ is still rising (we are left of the vertex), so the best of these is $x = 2$: price \$18, 230 tickets, $18 \times 230 = \$4{,}140$. For $x \ge 3$ the house is full, so revenue is $240(20 - x)$, largest at $x = 3$: $240 \times 17 = \$4{,}080$. Compare: $4140 > 4080$.

Check. A short table: \$20 gives 4000, \$19 gives $19 \cdot 215 = 4085$, \$18 gives 4140, \$17 gives $17 \cdot 240 = 4080$, \$16 gives 3840. The largest is \$4,140. (The trap answer \$4,165 sells tickets for seats that do not exist.)

3. Answer: 122**. The idea.** The trip back depends on which roads the trip there used up, so split into cases by the route there and count the routes back that remain in each case.

Work it out. *Case 1: through the park.* There are $4 \times 3 = 12$ ways to go. That uses one home–park road and one park–school road, leaving 3 and 2 of them, so $3 \times 2 = 6$ ways back through the park, plus the 2 direct roads: 8 ways back. This case gives $12 \times 8 = 96$ trips.

Case 2: directly. There are 2 ways to go. Back, all 12 park routes are still free, plus the 1 unused direct road: 13 ways back. This case gives $2 \times 13 = 26$ trips.

Total: $96 + 26 = 122$.

Check. Without the no-reuse rule there would be $(12 + 2)^2 = 196$ trips. The forbidden ones reuse a road: going through the park and returning through the park on at least one same road gives $12 \times (12 - 6) = 72$; going direct and returning on the same direct road gives 2. And $196 - 72 - 2 = 122$.

4. Answer: $10+6\sqrt{2}$**. The idea.** The area of each triangular side is $\frac{1}{2} \times$base$\times$slant height, where the slant height runs from the peak straight down the face to the midpoint of the base edge. Because the floor is not square, the two pairs of faces have *different* slant heights, and each one comes from a right triangle through the tent's centre.

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Work it out. Put the peak 2 m above the centre. The midpoint of a 4-m edge is 1.5 m from the centre (half the width), so the slant height of those faces is $\sqrt{2^2 + 1.5^2} = \sqrt{6.25} = 2.5$ m. The midpoint of a 3-m edge is 2 m from the centre (half the length), so the slant height of those faces is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$ m. Fabric: $2 \cdot \frac{1}{2} \cdot 4 \cdot 2.5 + 2 \cdot \frac{1}{2} \cdot 3 \cdot 2\sqrt{2} = 10 + 6\sqrt{2}$ square metres.

Check. Numerically $10 + 6\sqrt{2} \approx 18.49$ m^2. A common slip pairs each edge with the wrong half-dimension, giving $4 \cdot 2\sqrt{2} + 3 \cdot 2.5 \approx 18.81$; the rule is that the face over a long edge leans across the short half-width. Also, the total is more than the floor's 12 m^2, as a tent roof must be.

5. Answer: 257**. The idea.** A washer is a cylinder with a smaller cylinder removed, so its volume is (ring area) × (thickness). Mass is density times volume, and the count must be rounded *down*, since the limit may not be exceeded.

Work it out. Radii are 2.5 cm and 1 cm, so the ring's area is $\pi(2.5^2 - 1^2) = 5.25\pi$ cm^2, and the volume is $5.25\pi \times 0.3 = 1.575\pi \approx 4.948$ cm^3. Mass: $7.85 \times 1.575\pi \approx 38.842$ g. Then $\frac{10000}{38.842} \approx 257.45$, so at most 257 washers fit the limit.

Check. $257 \times 38.842 \approx 9982$ g, under 10 kg, while $258 \times 38.842 \approx 10021$ g is over. (Forgetting the hole gives $6.25\pi \cdot 0.3 \cdot 7.85 \approx 46.24$ g per washer and only 216 washers.)

6. Answer: 33.25**. The idea.** Work in units of 5: the spins are $5i$ and $5j$ with i, j from 1 to 20, and she keeps the total exactly when $i + j \le 20$. Expected score is (sum of all scores) ÷ (number of equally likely outcomes), and the scores are easiest to total by grouping the outcomes by their sum $s = i + j$.

Work it out. There are $20 \times 20 = 400$ equally likely ordered outcomes. For $2 \le s \le 20$, the sum $i + j = s$ happens in $s - 1$ ways (with $i = 1, \ldots, s - 1$), each worth $5s$ points. So the total of all scores is $5 \sum_{s=2}^{20} s(s-1) = 5\left(\sum_{s=1}^{20} s^2 - \sum_{s=1}^{20} s\right) = 5(2870 - 210) = 5 \cdot 2660 = 13{,}300$. Busting outcomes add 0. Expected score: $\frac{13300}{400} = 33.25$.

Check. She keeps her total in $\sum_{s=2}^{20}(s-1) = 190$ of the 400 outcomes, a bit under half; the kept totals average $\frac{13300}{190} = 70$ points, and $\frac{190}{400} \times 70 = 33.25$. Without the bust rule the average would be $2 \times 52.5 = 105$, so busting costs a lot, as it should when most totals exceed 100.

7. Answer: 111**. The idea.** Pieces = (number of distinct cut points) +1. Count Lia's marks and Ben's marks, then subtract the places where a mark of each coincides. Measuring everything from the left end turns "coincide" into a linear equation in whole numbers.

Work it out. Lia's marks are at $12a$ cm for $a = 1, \ldots, 82$ ($12 \cdot 83 = 996$ is the end): 82 marks. Ben's marks are at $996 - 18b$ cm from the left for $b = 1, \ldots, 55$ ($996 - 18 \cdot 55 = 6 > 0$, while $b = 56$ is past the left end): 55 marks. They coincide when $12a = 996 - 18b$, i.e. $2a + 3b = 166$. Then $3b$ is even, so b is even, and $a = \frac{166 - 3b}{2}$. The even values $b = 2, 4, \ldots, 54$ give $a = 80, 77, \ldots, 2$, all between 1 and 82; $b = 56$ is too big. That is 27 coincidences. Distinct cut points: $82 + 55 - 27 = 110$, so there are 111 pieces.

Check. Since $996 = 12 \cdot 83$, Ben's marks sit at $6 + 18k$ cm from the left for $k = 0, \ldots, 54$, and such a point is a multiple of 12 exactly when $6 + 18k \equiv 0 \pmod{12}$, i.e. $6k \equiv 6 \pmod{12}$, i.e. k odd: $k = 1, 3, \ldots, 53$, which is 27 values, matching. (Marking both from the left end instead would give coincidences at every multiple of 36, a different and wrong count.)

8. Answer: 45**. The idea.** Write the number as $d_1 d_2 d_3 d_4 d_5 d_6$ and let f be the upside-down rule.

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Turned over, the display reads $f(d_6)f(d_5)\ldots f(d_1)$, so the number reads the same exactly when $d_6 = f(d_1)$, $d_5 = f(d_2)$, $d_4 = f(d_3)$ (then automatically $d_1 = f(d_6)$, and so on, since f undoes itself). So the first three digits decide everything. Divisibility by 6 means even (a condition on $d_6 = f(d_1)$) and digit sum divisible by 3, and the digit sum is the sum of the three pair sums $d + f(d)$.

Work it out. Every digit must be one of $0, 1, 2, 5, 6, 8, 9$. The pair sums $d + f(d)$ are: $0 \to 0$, $1 \to 2$, $2 \to 4$, $5 \to 10$, $8 \to 16$, $6 \to 15$, $9 \to 15$. Modulo 3 these are: residue 0 for $\{0, 6, 9\}$, residue 1 for $\{2, 5, 8\}$, residue 2 for $\{1\}$.

Even. The last digit $f(d_1)$ must be even, and $d_1 \neq 0$: $d_1 = 2$ ($f = 2$), $d_1 = 8$ ($f = 8$) or $d_1 = 9$ ($f = 6$). Here $d_1 = 9$ has pair-sum residue 0, and $d_1 = 2, 8$ have residue 1.

Middle digits. d_2, d_3 are free among the 7 digits; the residue counts are 3, 3, 1 (for residues 0, 1, 2). The pairs (d_2, d_3) with total residue 0: $3 \cdot 3 + (3 \cdot 1 + 1 \cdot 3) = 15$; residue 2: $3 \cdot 1 + 1 \cdot 3 + 3 \cdot 3 = 15$; residue 1: the remaining $49 - 30 = 19$.

Divisible by 3. $d_1 = 9$ needs (d_2, d_3) of residue 0: 15 numbers. $d_1 = 2$ or 8 needs residue 2: $2 \times 15 = 30$ numbers. Total: $15 + 30 = 45$.

Check. Example: $d_1 d_2 d_3 = 900$ gives 900006; turned over it reads $f(6)f(0)f(0)f(0)f(0)f(9) = 900006$, and $900006 = 6 \times 150001$. As a count check, without the divisibility condition there are $6 \cdot 7 \cdot 7 = 294$ such six-digit numbers; with only "even" there are $3 \cdot 49 = 147$, and 45 of those have digit sum divisible by 3, a little under a third, as the uneven residue counts suggest.

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