MathCastle

CHAPTER COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A racing bicycle has a front chainring with 48 teeth, and the chain runs to a rear cog with 16 teeth. The rear wheel is 70 cm in diameter (including the tyre). The rider pedals at a steady 80 full turns of the pedals per minute, and the wheel rolls without slipping. How fast is the bicycle moving, in kilometres per hour? Express your answer as a decimal to the nearest tenth.

1. _____

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2. A country taxes income in brackets: the first \$12,000 of a person's income is taxed at 0%, the part of the income between \$12,000 and \$42,000 is taxed at 10%, and every dollar above \$42,000 is taxed at 22%. Rosa earned a positive whole number of dollars, and her total tax came to exactly 14% of her income. What was her income, in dollars?

2. _____

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3. A sculptor first makes a small model of a statue out of solid bronze. The model is 25 cm tall, contains 1.2 kg of bronze, and its entire outer surface, 600 cm^2 in all, is covered in gold leaf. The finished statue will be an exact enlargement of the model (every length multiplied by the same factor), 2 m tall, also made of solid bronze, and its entire outer surface will also be covered in gold leaf. Bronze costs \$12.50 per kilogram and gold leaf costs \$0.05 per square centimetre. What is the total cost, in dollars, of the bronze and the gold leaf for the finished statue?

3. _____

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4. An 8×8 chessboard is made of 64 unit squares, drawn with 9 equally spaced horizontal grid lines and 9 equally spaced vertical grid lines. Consider every rectangle, of any size (squares included), whose four sides lie along these grid lines. How many of these rectangles have an area, in unit squares, that is an odd number?

4. _____

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5. A theatre has its seats in straight rows. The front row has 20 seats, and every row has exactly 2 more seats than the row just in front of it. There are no partial rows, and the theatre has exactly 1100 seats in all. How many seats are in the back row?

5. _____

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6. A 10-digit book number $d_1 d_2 d_3 \ldots d_{10}$ (each d_i a digit from 0 to 9) is valid when $10d_1 + 9d_2 + 8d_3 + 7d_4 + 6d_5 + 5d_6 + 4d_7 + 3d_8 + 2d_9 + 1d_{10}$ is a multiple of 11. A librarian typed a valid book number but accidentally swapped two neighbouring digits that were different from each other; every other digit is in its correct place. What she typed was 1695488306. What is the correct book number? Give all ten digits as a single number.

6. _____

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7. A restaurant's dinner special must include exactly one of its 6 main courses. The diner may also add one of its 4 appetizers, or skip the appetizer. Finally, the diner may choose no dessert, one dessert, or two different desserts from the dessert menu (the order in which desserts are chosen does not matter). The owner wants at least 1000 different dinners to be possible. What is the smallest number of desserts the dessert menu must have?

7. _____

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8. A long water trough has the same cross-section all along its length: an isosceles trapezoid with its horizontal bottom 40 cm wide and its open top 80 cm wide, the two sloping sides being the equal legs. When the water in the trough is 20 cm deep, the trough is exactly half full (by volume). How deep is the trough, in centimetres, from the bottom to the top? Express your answer in simplest radical form.

8. _____

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Answer Key — MathCounts Chapter Target Practice Contest 2

1. 31.7

2. 78000

3. 9600

4. 400

5. 68

6. 1695848306

7. 8

8. $\frac{40+20\sqrt{10}}{3}$

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Solutions — MathCounts Chapter Target Practice Contest 2

1. Answer: 31.7**. The idea.** The chain moves the same number of teeth past the front chainring and the rear cog. One turn of the pedals moves 48 teeth of chain, which turns the 16-tooth cog (and the wheel fixed to it) $48/16 = 3$ times. Each wheel turn moves the bike one circumference.

Work it out. Pedals: 80 turns per minute, so the wheel makes $80 \cdot 3 = 240$ turns per minute. One turn covers $\pi \cdot 0.70 = 0.7\pi$ m, so the bike covers $240 \cdot 0.7\pi = 168\pi$ m per minute, which is $168\pi \cdot 60 = 10080\pi$ m per hour $= 10.08\pi$ km/h ≈ 31.667 km/h. To the nearest tenth: 31.7.

Check. In metres per second: $168\pi/60 = 2.8\pi \approx 8.80$ m/s, and $8.80 \times 3.6 \approx 31.7$ km/h, a sensible speed for a road cyclist.

2. Answer: 78000**. The idea.** The tax is a different formula in each bracket, so first find which bracket Rosa is in. An overall rate of 14% is higher than any rate the first two brackets can produce, so she must be in the 22% bracket.

Work it out. Below \$12,000 the tax is 0, not 14%. For an income I between \$12,000 and \$42,000 the tax is $0.10(I - 12000)$, which is less than $0.10I$, so it is never 14% of I. At exactly \$42,000 the tax is $0.10 \cdot 30000 =$ \$3000, only about 7%. So $I > 42000$ and

$3000 + 0.22(I - 42000) = 0.14I$, so $3000 + 0.22I - 9240 = 0.14I$, so $0.08I = 6240$ and $I = 78000$.

Check. Tax on \$78,000: $0 + 3000 + 0.22 \cdot 36000 = 3000 + 7920 =$ \$10,920, and $0.14 \cdot 78000 = 10920$. Rosa earned \$78,000.

3. Answer: 9600**. The idea.** When a solid is enlarged by a length factor k, every area is multiplied by k^2 and every volume (hence the mass of bronze) by k^3. The trap is multiplying everything by k.

Work it out. The length factor is $k = 200/25 = 8$. Bronze: $1.2 \cdot 8^3 = 1.2 \cdot 512 = 614.4$ kg, costing $614.4 \cdot 12.50 = 7680$ dollars. Gold leaf: $600 \cdot 8^2 = 38{,}400$ cm^2, costing $38{,}400 \cdot 0.05 = 1920$ dollars. The total is $7680 + 1920 = 9600$ dollars.

Check. Think of the model as built from small cubes. Enlarging by 8 turns each cube into an 8×8×8 block of 512 cubes, and each exposed square face into an 8×8 patch of 64 faces, confirming the factors 512 and 64. The model's own materials cost $1.2 \cdot 12.50 + 600 \cdot 0.05 = 15 + 30 = 45$ dollars, and $15 \cdot 512 + 30 \cdot 64 = 7680 + 1920 = 9600$.

4. Answer: 400**. The idea.** A rectangle is chosen by picking two of the vertical lines (its width) and two of the horizontal lines (its height), independently. Its area is width × height, and a product is odd exactly when both factors are odd. So count the ways to get an odd width, and square.

Work it out. Number the vertical lines $0, 1, \ldots, 8$. A width of w can be placed in $9 - w$ ways (the left line can be $0, \ldots, 8 - w$). The odd widths $1, 3, 5, 7$ give $8 + 6 + 4 + 2 = 20$ choices. The heights work the same way: 20 choices. So the number of odd-area rectangles is $20 \cdot 20 = 400$.

Check. By parity: two lines give an odd width exactly when one has an even number and one an odd number. Lines 0–8 contain 5 even and 4 odd numbers, so $5 \cdot 4 = 20$ odd widths, agreeing with the count above. (All rectangles: $\binom{9}{2}^2 = 1296$, and 400 is a bit under a third of them.)

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5. Answer: 68**. The idea.** The row sizes form an arithmetic sequence, so the total is (number of rows) × (average of the first and last row). Find the number of rows n first, then the back row.

Work it out. Row n has $20 + 2(n-1) = 2n + 18$ seats. The total is $n \cdot \frac{20+(2n+18)}{2} = n(n+19)$. So $n(n+19) = 1100$, i.e. $n^2+19n-1100 = 0$, which factors as $(n-25)(n+44) = 0$ (or: two numbers 19 apart with product 1100 are 25 and 44). So $n = 25$ rows, and the back row has $2 \cdot 25 + 18 = 68$ seats.

Check. The average row is $(20 + 68)/2 = 44$ seats, and $25 \cdot 44 = 1100$. The back row has 68 seats.

6. Answer: 1695848306**. The idea.** See what a swap does to the weighted sum. If neighbours a (weight $w+1$) and b (weight w) change places, the sum changes from $(w+1)a+wb$ to $(w+1)b+wa$, a change of exactly $b - a$. So undoing the typo changes the typed sum by the difference of two neighbouring typed digits (right minus left), and we need that change to make the sum a multiple of 11.

Work it out. The typed sum is $10{\cdot}1+9{\cdot}6+8{\cdot}9+7{\cdot}5+6{\cdot}4+5{\cdot}8+4{\cdot}8+3{\cdot}3+2{\cdot}0+1{\cdot}6 = 282 = 11{\cdot}25+7$. So swapping positions i and $i+1$ must change the sum by an amount $\equiv -7 \equiv 4 \pmod{11}$, that is, by 4 or -7 (a difference of digits lies between -9 and 9). The nine neighbour differences (right minus left) of $1, 6, 9, 5, 4, 8, 8, 3, 0, 6$ are $5, 3, -4, -1, 4, 0, -5, -3, 6$. Only the fifth, $8 - 4 = 4$, works (note -4 is the wrong sign). So digits 5 and 6 were swapped, and the correct number is 1695848306.

Check. $10 \cdot 1 + 9 \cdot 6 + 8 \cdot 9 + 7 \cdot 5 + 6 \cdot 8 + 5 \cdot 4 + 4 \cdot 8 + 3 \cdot 3 + 2 \cdot 0 + 1 \cdot 6 = 286 = 11 \cdot 26$, so 1695848306 is valid, and it is the only neighbour swap that is.

7. Answer: 8**. The idea.** Each part of the meal is an independent choice, so the number of dinners is a product. Count it as a function of the number of desserts d, then find the smallest d that reaches 1000. The trap is forgetting that "none" is itself an option, both for the appetizer and for dessert.

Work it out. Main course: 6 ways. Appetizer: 4 appetizers or none, 5 ways. Desserts: none (1 way), one (d ways) or two different ones ($\binom{d}{2} = \frac{d(d-1)}{2}$ ways). So the number of dinners is $30\left(1 + d + \frac{d(d-1)}{2}\right)$. We need $1 + d + \frac{d(d-1)}{2} \geq \frac{1000}{30} \approx 33.3$, so at least 34. For $d = 7$: $1 + 7 + 21 = 29$, too few. For $d = 8$: $1 + 8 + 28 = 37$, enough. So the menu needs 8 desserts.

Check. With 7 desserts there are $30 \cdot 29 = 870 < 1000$ dinners; with 8 there are $30 \cdot 37 = 1110 \geq 1000$. The answer is 8.

8. Answer: $\frac{40+20\sqrt{10}}{3}$**. The idea.** Because the cross-section is the same all along the trough, "half the volume" means "half the cross-sectional area". The depth H of the trough is the unknown, so describe the wet part of the cross-section as a function of the water depth h and of H. The trap is to think half full means half the depth: the trough widens upward, so the top half of the depth holds more water than the bottom half, and the half-full level is above the halfway mark.

Work it out. Going from the bottom to the top, the width grows from 40 to 80, steadily. So at height h the width is $40+40\frac{h}{H}$ (similar triangles along each sloping side). The water is a trapezoid with parallel sides 40 and $40 + 40h/H$ and height h, so its area is $\frac{h}{2}\left(80 + \frac{40h}{H}\right) = 40h + \frac{20h^2}{H}$. The whole cross-section has area $\frac{H}{2}(40 + 80) = 60H$.

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Half full at $h = 20$: $800 + \frac{8000}{H} = 30H$. Multiply by H: $30H^2 - 800H - 8000 = 0$, i.e. $3H^2 - 80H - 800 = 0$. By the quadratic formula $H = \frac{80 \pm \sqrt{6400+9600}}{6} = \frac{80 \pm 40\sqrt{10}}{6}$. The depth is positive, so $H = \frac{40+20\sqrt{10}}{3} \approx 34.42$ cm.

Check. With $H \approx 34.42$: the full area is $60H \approx 2065.2$ cm^2. At 20 cm the water surface is $40 + 40 \cdot 20/34.42 \approx 63.24$ cm wide, so the water's area is $\frac{20}{2}(40 + 63.24) \approx 1032.4$, half of 2065.2. Also the half-full level 20 is above the halfway mark $H/2 \approx 17.2$, as the widening shape predicts.

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