MathCastle

CHAPTER COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Halfway through the season, a basketball player has made exactly 55% of her first 140 free throws. From now on she makes every free throw she attempts. What is the least number of additional free throws she must attempt so that her free-throw percentage for the whole season is at least 65%?

1. _____

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2. A basketball team has 12 players: a pair of twins, 4 seniors (neither twin is a senior), and 6 other players. The coach chooses 5 of the players to start and then names one of the 5 starters as captain. The twins refuse to start together, and the captain must be a senior. In how many ways can the coach choose the set of starters and the captain?

2. _____

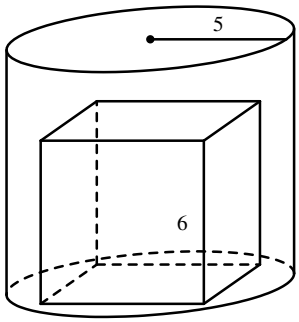
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3. For how many positive integers n does the product $n(n+1)(n+2)$ of three consecutive integers have exactly 7 digits?

3. _____

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4. A cylindrical tank stands upright. Its inside radius is 5 cm, and it holds water to a depth of 3 cm. A solid metal cube with edges 6 cm long is lowered into the tank until it rests flat on the bottom. No water spills out. How deep is the water now, in centimetres? Express your answer as a decimal to the nearest hundredth.



4. _____

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5. A running track has two straightaways, each 84.39 m long, joined by two semicircular curves. Each lane is 1.2 m wide. The inner edge of lane 1 follows semicircles of radius 36.8 m, and each lane's inner edge is 1.2 m farther out than the one before it. Each runner runs along the inner edge of her own lane. For one full lap, how many more metres does the runner in lane 4 run than the runner in lane 1? Express your answer as a decimal to the nearest hundredth.

5. _____

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6. A bank PIN is a string of four digits, each from 0 to 9, and it may begin with 0 (so there are 10,000 PINs, from 0000 to 9999). How many PINs use four different digits and, read as an ordinary number, are divisible by 4? (For example, 0724 counts, since $724 = 4 \cdot 181$.)

6. _____

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7. A drawer holds 10 red socks, 8 blue socks and 6 green socks, all loose and mixed together. In the dark, Tomas pulls out socks one at a time without looking. He wants three matching pairs: six of his socks that split into three pairs, each pair of a single colour (pairs may share a colour, so two or even all three pairs can be the same colour). What is the fewest socks he must pull out to be certain of having three matching pairs?

7. _____

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8. A snail starts at the bottom of an 85 m tower and climbs straight up its side. On day 1 it climbs 2 m. Each day after that, it climbs exactly 98% as far as it climbed the day before. Every night, starting after day 1, it slides straight down 5 cm. On which day does the snail first reach the top of the tower? (Answer with the day number; day 1 is the first day.)

8. _____

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Answer Key — MathCounts Chapter Target Practice Contest 1

1. 40

2. 1176

3. 115

4. 5.54

5. 22.62

6. 1232

7. 8

8. 119

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Solutions — MathCounts Chapter Target Practice Contest 1

1. Answer: 40**. The idea.** Every new attempt is a make, so both the number made and the number attempted go up by the same amount k. Write the season percentage as a fraction in k and solve the inequality; do not just "add 10% of 140".

Work it out. She has made $0.55 \times 140 = 77$ free throws. After k more (all made) her percentage is $\frac{77+k}{140+k}$. We need $\frac{77+k}{140+k} \geq 0.65$, i.e. $77 + k \geq 91 + 0.65k$, so $0.35k \geq 14$ and $k \geq 40$. The least number is 40.

Check. With $k = 40$: $\frac{117}{180} = 0.65$ exactly. With $k = 39$: $\frac{116}{179} \approx 0.648$, just short. So the answer is 40.

2. Answer: 1176**. The idea.** Choose the captain first. Picking the captain before the rest of the line-up turns "the captain must be a senior" into a simple first step, and the twin rule then only affects the other four starters.

Work it out. The captain is one of the 4 seniors. The other 4 starters come from the remaining 11 players, which include both twins. There are $\binom{11}{4} = 330$ ways to pick them, and $\binom{9}{2} = 36$ of these include both twins (both twins, plus 2 of the other 9). That leaves $330 - 36 = 294$ allowed choices. Every (line-up, captain) pair is counted exactly once, so the total is $4 \times 294 = 1176$.

Check. Count line-ups first instead. Line-ups without both twins: $\binom{12}{5} - \binom{10}{3} = 792 - 120 = 672$. Each has between 0 and 4 seniors, and the number of ways to pick the captain is the number of seniors in the line-up. By symmetry each of the 4 seniors is in $\binom{11}{4} - \binom{9}{2} = 294$ of these line-ups, so the sum over all line-ups is again $4 \times 294 = 1176$.

3. Answer: 115**. The idea.** Write the product around its middle factor $m = n+1$: $n(n+1)(n+2) = (m-1)m(m+1) = m^3 - m$. So the product is just a little LESS than m^3. Estimate with cube roots, but check the boundary cases exactly, because "a little less" matters right at 10^6.

Work it out. The product grows as n grows, so the good values of n form one unbroken block. Lower end: $\sqrt[3]{10^6} = 100$, but $m = 100$ gives $100^3 - 100 = 999{,}900$, which has only 6 digits. So the smallest good middle factor is $m = 101$ (product 1,030,200). Upper end: $\sqrt[3]{10^7} \approx 215.4$. $m = 215$ gives $9{,}938{,}375 - 215 = 9{,}938{,}160$ (7 digits), and $m = 216$ gives $10{,}077{,}696 - 216 = 10{,}077{,}480$ (8 digits). So m runs from 101 to 215, that is n from 100 to 214: $214 - 100 + 1 = 115$ values.

Check. Directly: $99 \cdot 100 \cdot 101 = 999{,}900$ (too small), $100 \cdot 101 \cdot 102 = 1{,}030{,}200$ (first good one), $214 \cdot 215 \cdot 216 = 9{,}938{,}160$ (last good one), $215 \cdot 216 \cdot 217 = 10{,}077{,}480$ (too big). The count is 115. The tempting answer 116 comes from starting at $n = 99$.

4. Answer: 5.54**. The idea.** First decide whether the cube ends up completely under water. If you assume it does and add the whole 216 cm^3 of the cube, you get a depth of about 5.75 cm. That is less than 6, so the top of the cube sticks out. Then the water only fills the space around the cube, and the answer is smaller.

Work it out. The water's volume is $\pi \cdot 5^2 \cdot 3 = 75\pi$ cm^3. (The cube fits, since its face diagonal $6\sqrt{2} \approx 8.49$ is less than the diameter 10.) If the cube were covered, the depth would be $3 + \frac{216}{25\pi} \approx 5.75 < 6$, which is impossible. So the water reaches some depth $h < 6$, and at every height below h the water fills a cross-section of area $25\pi - 36$ (the circle minus the cube's 6×6 square). So

$h(25\pi - 36) = 75\pi$, which gives $h = \frac{75\pi}{25\pi - 36} \approx \frac{235.62}{42.54} \approx 5.539$. To the nearest hundredth, the depth is 5.54 cm.

Check. The value $5.54 < 6$ agrees with our assumption that the cube sticks out. Water around the cube: $5.539 \times (78.54 - 36) \approx 235.6 \approx 75\pi$. That matches.

5. Answer: 22.62**. The idea.** The straightaways are the same length in every lane, so only the curves differ. The two semicircles together make one full circle, and the circumference $2\pi r$ grows by 2π for every metre added to the radius. So the straight length and the radius 36.8 are not needed. Only the extra radius matters.

Work it out. Lane 4's inner edge is $3 \times 1.2 = 3.6$ m farther out than lane 1's. The two curves form one full circle, so the difference is $2\pi(r + 3.6) - 2\pi r = 7.2\pi \approx 22.619$ m. To the nearest hundredth, that is 22.62 m.

Check. Compute both laps in full. Lane 1: $2(84.39) + 2\pi(36.8) \approx 168.78 + 231.22 = 400.00$ m. Lane 4: $168.78 + 2\pi(40.4) \approx 168.78 + 253.84 = 422.62$ m. The difference is 22.62 m.

6. Answer: 1232**. The idea.** A number is divisible by 4 exactly when the number formed by its last two digits is. So choose the last two digits first, then fill the first two places with digits not used yet. Leading zeros cause no trouble: 0724 read as a number is 724, whose last two digits are still 24.

Work it out. The endings $00, 04, 08, \ldots, 96$ are the 25 multiples of 4 from 0 to 99 written with two digits. Three of them repeat a digit: 00, 44 and 88. That leaves 22 endings with two different digits. For each, the first digit can be any of the 8 digits not in the ending, and the second digit any of the 7 digits left. So the count is $22 \times 8 \times 7 = 1232$.

Check. Split by the ending instead. Endings containing 0: $04, 08, 20, 40, 60, 80$, which is 6. Endings without 0: $22 - 6 = 16$. Each ending leaves exactly 8 unused digits whatever they are, so both kinds give $8 \cdot 7 = 56$ completions and the total is $(6 + 16) \cdot 56 = 1232$. A common slip is to forbid a leading 0; that would count four-digit numbers, not PINs, and give a smaller answer.

7. Answer: 8**. The idea.** Find the largest handful that still does NOT contain three pairs. Each colour gives $\lfloor c/2 \rfloor$ pairs and can leave at most one sock unpaired. So a handful with only two pairs has at most $2 \cdot 2 = 4$ paired socks plus one leftover sock of each of the 3 colours.

Work it out. Seven socks can fail: 3 red, 3 blue and 1 green give only one red pair and one blue pair. Now take any 8 socks. At most 3 of them (one per colour) are left unpaired, so at least 5 are in pairs. Paired socks come in twos, so at least 6 are paired, which makes 3 pairs. The answer is 8.

Check. Other bad handfuls of 7 also exist, e.g. 5 red, 1 blue, 1 green (two red pairs). Adding any eighth sock to 3 red, 3 blue, 1 green completes a third pair, whichever colour it is. So 7 is not enough and 8 is.

8. Answer: 119**. The idea.** Stepping through the days one by one would take well over a hundred steps. Instead, get a formula for the height at the end of day n: the climbs form a geometric series and the slides add up linearly. Then find the first n that works by trying values on a calculator. The snail only reaches a new height during a day's climb, so it first reaches the top on the first day n whose end-of-day height is at least 85 m.

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Work it out. Day k's climb is $2(0.98)^{k-1}$ m. By the end of day n the snail has climbed $2(1 + 0.98 + \cdots + 0.98^{n-1}) = \frac{2(1-0.98^n)}{0.02} = 100(1 - 0.98^n)$ m, and it has slid on $n - 1$ nights, losing $0.05(n - 1)$ m. So its height is $H(n) = 100(1 - 0.98^n) - 0.05(n - 1)$. Each day's climb is more than 5 cm until $2(0.98)^{k-1}$ drops below 0.05, around day 183, so $H(n)$ keeps rising well past the values we need, and the first n with $H(n) \geq 85$ is the answer. Try values: $H(110) \approx 100(1 - 0.10836) - 5.45 \approx 83.71$ and $H(120) \approx 100(1 - 0.08854) - 5.95 \approx 85.20$. Closer in: $H(118) \approx 100(1-0.092189)-5.85 \approx 84.93$, which is short, and $H(119) \approx 100(1-0.090345)-5.90 \approx 85.07$, which is enough. So the snail first reaches the top on day 119.

Check. At the end of day 118 the snail is at about 84.93 m, its highest point so far, so it has not reached the top. That night it slides to about 84.88 m, and on day 119 it climbs $2(0.98)^{118} \approx 0.184$ m, passing 85 m. Do not forget the slides. With no sliding, the snail would reach 85 m once $100(1 - 0.98^n) \geq 85$, that is $0.98^n \leq 0.15$, on day 94. The 5 cm nights cost it 25 extra days.

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