MathCastle

Putnam

Session A and Session B

12 problems • 6 hours

Name _____

INSTRUCTIONS

1. Two sessions, A and B, of 6 problems each. Each session lasts 3 hours.
2. Each problem is worth 10 points. Full credit needs a complete proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Session A is handed in before Session B is opened.
5. Two sittings of 6 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

A1

The number 2027 is prime, and for every integer k the number k^2+k+1 is not divisible by 2027. For each $k = 0, 1, 2, \ldots, 2026$ let r_k be the unique integer with $1 \le r_k \le 2026$ such that $r_k\,(k^2+k+1)-1$ is divisible by 2027.
Find the remainder when $r_0 + r_1 + r_2 + \cdots + r_{2026}$ is divided by 2027.

A2

A solid unit cube has a solid ball inscribed in it (radius $\frac{1}{2}$, same centre O). A random line is produced as follows. First a unit vector u is chosen uniformly at random on the unit sphere. Then a point Q is chosen uniformly at random from the disk of radius 1 centred at O in the plane through O perpendicular to u, and the line is the line through Q in direction u.
Given that the line meets the cube, what is the probability that it also meets the ball? Give an exact answer.

A3

For each positive integer n let
$I_n = \displaystyle\int_0^1 \left(1 + x^n + x^{2n}\right)^{1/n}\,dx.$
Then $I_n \to 1$ as $n \to \infty$. Find the exact value of $\displaystyle\lim_{n\to\infty} n^2\,(I_n - 1)$.

A4

For each 3-element subset $\{a, b, c\}$ of $\{1, 2, \ldots, 30\}$ with $a < b < c$, form the product $(b-a)(c-b)(c-a)$. Find the sum of the squares of these products over all $\binom{30}{3} = 4060$ such subsets.

A5

For $n \ge 0$ let
$r_n = \ln 2 - \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$
be the tails of the alternating harmonic series, so $r_0 = \ln 2$, $r_1 = \ln 2 - 1$, $r_2 = \ln 2 - \frac{1}{2}$, and so on.
Evaluate
$\displaystyle\sum_{n=0}^{\infty} r_n^3$
in closed form.

A6

A plane passes through the centre of a solid unit cube, and its unit normal vector is chosen uniformly at random on the unit sphere. The plane cuts the cube in a polygon (a parallelogram or a hexagon, depending on the plane). Find the exact expected area of this cross-section.

B1

Exactly 2026 points are placed independently and uniformly at random on a circle of circumference 1. Distance between two of the points is measured along the circle (the length of the shorter arc joining them). Two of the points form a *close pair* if each of them is the other's nearest neighbour among the 2026 points.

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Find the expected number of close pairs. (With probability 1 no two distances are equal, so nearest neighbours are well defined.)

B2

Real numbers $x_1, x_2, \ldots, x_{2026}$ are written around a circle, so that x_{2026} and x_1 are also neighbours. Any two neighbours differ by at most 1, that is, $|x_{i+1} - x_i| \le 1$ for $1 \le i \le 2025$ and $|x_1 - x_{2026}| \le 1$, and the numbers add up to 0.
Find the largest possible value of $x_1^2 + x_2^2 + \cdots + x_{2026}^2$. Give your answer as a fraction in lowest terms.

B3

Evaluate the infinite sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\operatorname{lcm}(k,\, k+1,\, k+2,\, k+3)} = \frac{1}{12} + \frac{1}{60} + \frac{1}{60} + \frac{1}{420} + \frac{1}{840} + \frac{1}{504} + \cdots$,
where lcm denotes the least common multiple. Give the exact value.

B4

Consider 4×4 matrices A whose entries are 0 or 1, with all arithmetic done modulo 2 (that is, matrices over the field $\mathbb{F}_2$). There are 2^{16} such matrices.
How many of them satisfy $A^4 = A$?

B5

A permutation σ of $\{1, 2, \ldots, 18\}$ is chosen uniformly at random. Write σ as a product of disjoint cycles (fixed points count as cycles of length 1), and let C be the number of cycles whose length is a multiple of 3.
Find the probability that C is a multiple of 3 (including $C = 0$). Give your answer as a fraction in lowest terms.

B6

A stick of length 1 is broken at a point chosen uniformly at random along it; the shorter piece is thrown away and the longer piece is kept. The kept piece is then broken at a uniformly random point of it, again keeping the longer piece, and so on, all the break points being chosen independently.
For $0 < t < 1$ let $N(t)$ be the number of breaks made until, for the first time, the kept piece is shorter than t. Find the exact value of
$\displaystyle\lim_{t \to 0^+} \left(E[N(t)] - \frac{\ln(1/t)}{1 - \ln 2} \right)$.

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Answer Key — Putnam Practice Contest 4

A1 1352

A2 $\frac{\pi}{6}$

A3 $\frac{\pi^2}{9}$

A4 9051851200

A5 $\frac{3}{2}\ln^2 2 - \frac{\pi^2}{24}$

A6 $\frac{3}{2}\ln\left(2+\sqrt{3}\right) - \frac{\pi}{4}$

B1 $\frac{2026}{3}$

B2 $\frac{346503741}{2}$

B3 $\frac{4}{3} - \frac{2\sqrt{3}\,\pi}{9}$

B4 9874

B5 $\frac{6211}{14580}$

B6 $\frac{2-2\ln 2-\ln^2 2}{2\,(1-\ln 2)^2}$

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Solutions — Putnam Practice Contest 4

A1 **Answer:** 1352. **The idea.** Work in the field $\mathbb{F}_p$ with $p = 2027$. There r_k is simply $(k^2+k+1)^{-1}$, so the question asks for $S = \sum_{k\in\mathbb{F}_p} \frac{1}{k^2+k+1}$. Split the fraction over the two roots of x^2+x+1 (they live in the field with p^2 elements), and sum each simple fraction $\frac{1}{x-k}$ over all k at once, using $\prod_{k\in\mathbb{F}_p}(x-k) = x^p - x$.

Work it out. *No zero denominators.* If $p \mid k^2+k+1$, then $k^3-1 = (k-1)(k^2+k+1) \equiv 0$, and $k \not\equiv 1$ (because $1+1+1 = 3 \not\equiv 0$), so k has multiplicative order exactly 3 modulo p. The order divides $p-1 = 2026 = 2 \cdot 1013$, which is not a multiple of 3; contradiction. So every k^2+k+1 is invertible, and $\sum r_k \equiv S \pmod p$.

The roots. Let $\mathbb{K}$ be the field with p^2 elements, which contains $\mathbb{F}_p$. Its multiplicative group is cyclic of order p^2-1, a multiple of 3 (as $p \equiv 2 \pmod 3$), so $\mathbb{K}$ has an element ω with $\omega^2+\omega+1 = 0$. By the previous paragraph $\omega \notin \mathbb{F}_p$; also $\omega^3 = 1$ and $p \equiv 2 \pmod 3$ give $\omega^p = \omega^2$ and $(\omega^2)^p = \omega^4 = \omega$. Since $x^2+x+1 = (x-\omega)(x-\omega^2)$,

$$\frac{1}{k^2+k+1} = \frac{1}{\omega-\omega^2}\left(\frac{1}{k-\omega} - \frac{1}{k-\omega^2}\right).$$

Summing simple fractions. In $\mathbb{F}_p[x]$ we have $\prod_{k\in\mathbb{F}_p}(x-k) = x^p - x$ (both sides are monic of degree p and vanish at every element of $\mathbb{F}_p$). The logarithmic derivative of a product is the sum of the logarithmic derivatives, and $\frac{d}{dx}(x^p-x) = px^{p-1} - 1 = -1$ in characteristic p. So, as rational functions and hence at every $x \in \mathbb{K} \setminus \mathbb{F}_p$,

$$\sum_{k\in\mathbb{F}_p} \frac{1}{x-k} = \frac{-1}{x^p-x}.$$

At $x = \omega$ this gives $\sum_k \frac{1}{\omega-k} = \frac{-1}{\omega^2-\omega}$, that is, $\sum_k \frac{1}{k-\omega} = \frac{1}{\omega^2-\omega}$. At $x = \omega^2$ it gives $\sum_k \frac{1}{\omega^2-k} = \frac{-1}{\omega-\omega^2}$, that is, $\sum_k \frac{1}{k-\omega^2} = \frac{1}{\omega-\omega^2}$. Therefore

$$S = \frac{1}{\omega-\omega^2}\left(\frac{1}{\omega^2-\omega} - \frac{1}{\omega-\omega^2}\right) = \frac{-2}{(\omega-\omega^2)^2}.$$

Finally $(\omega-\omega^2)^2 = \omega^2 - 2\omega^3 + \omega^4 = \omega^2+\omega-2 = -3$, so $S = \frac{2}{3}$ in $\mathbb{F}_p$. Since $3 \cdot 1352 = 4056 = 2 \cdot 2027 + 2$, we have $\frac{2}{3} \equiv 1352$, and the remainder is 1352.

Check. The argument works for every prime $p \equiv 2 \pmod 3$, so test it on $p = 5$: the values of k^2+k+1 for $k = 0, \dots, 4$ are $1, 3, 2, 3, 1$ modulo 5, with inverses $1, 2, 3, 2, 1$, whose sum is $9 \equiv 4$; and indeed $3 \cdot 4 = 12 \equiv 2$, so $\frac{2}{3} \equiv 4 \pmod 5$. For $p = 11$ the inverses sum to $8 \equiv \frac{2}{3}$ as well. A direct computer sum of the 2027 inverses modulo 2027 gives 1352.

A2 **Answer:** $\frac{\pi}{6}$. **The idea.** For a fixed direction u, the line meets a convex body exactly when Q lands in the body's shadow (orthogonal projection) on the plane $u^\perp$. So everything reduces to shadow areas, and the average shadow of a cube is computable because a cube's shadow area is a simple function of u.

Work it out. Both shadows lie in the disk: the cube's shadow lies within distance $\frac{\sqrt{3}}{2} < 1$ of O. So for fixed u,

$$P(\text{meets ball} \mid u) = \frac{\pi/4}{\pi} = \frac{1}{4}, \qquad P(\text{meets cube} \mid u) = \frac{S(u)}{\pi},$$

where $S(u)$ is the area of the cube's shadow. Every line meeting the ball meets the cube, so the requested probability is $\dfrac{E[1/4]}{E[S(u)/\pi]} = \dfrac{\pi/4}{E[S(u)]}$.

Shadow of the cube: the projection of a convex polyhedron is covered exactly twice by the projections of its faces (once by the faces facing u, once by those facing away), and a face of area A with unit normal n projects to area $A\,|n \cdot u|$. The six faces have area 1 and normals $\pm e_1, \pm e_2, \pm e_3$, so $S(u) = \frac{1}{2} \cdot 2\,(|u_1| + |u_2| + |u_3|) = |u_1| + |u_2| + |u_3|$.

By Archimedes' hat-box theorem, a coordinate u_1 of a uniform point on the sphere is uniform on $[-1, 1]$, so $E|u_1| = \frac{1}{2}$ and $E[S(u)] = \frac{3}{2}$. The answer is $\dfrac{\pi/4}{3/2} = \dfrac{\pi}{6}$.

Check. Cauchy's formula says the mean shadow of a convex body is one quarter of its surface area: $6/4 = \frac{3}{2}$ for the cube and $\pi/4$ for the ball, as used above. So the answer is also the ratio of surface areas, $\dfrac{\pi}{6} \approx 0.5236$ (for these two bodies it happens to equal the volume ratio too). Sanity check in one direction: for $u = e_3$ the shadows are the unit square and a disk of area $\pi/4$, giving $\pi/4$; tilted directions make the cube's shadow larger (up to $\sqrt{3}$ along a diagonal), so the average must be below $\pi/4 \approx 0.785$ and above $\frac{\pi/4}{\sqrt{3}} \approx 0.453$.

A3 Answer: $\frac{\pi^2}{9}$**. The idea.** All the excess over 1 comes from x within about $1/n$ of 1, so zoom in with $x = u^{1/n}$. After the substitution, the extra factor n is absorbed by $(\,\cdot\,)^{1/n} - 1 \approx \frac{1}{n}\ln(\cdot)$, and the limit becomes a fixed integral, $\int_0^1 \frac{\ln(1+u+u^2)}{u}\,du$, which the factorisation $1 + u + u^2 = \frac{1-u^3}{1-u}$ evaluates.

Work it out. Put $x = u^{1/n}$, $dx = \frac{1}{n}u^{1/n-1}du$, and use $\int_0^1 dx = 1$:

$$n^2(I_n - 1) = \int_0^1 n\left[(1 + u + u^2)^{1/n} - 1\right] u^{1/n-1}\,du.$$

Let $L(u) = \ln(1 + u + u^2)$, so $0 \le L \le \ln 3$ and $0 \le L(u) \le u + u^2 \le 2u$. Since $e^t - 1 \to t$ with $t \le e^t - 1 \le te^t$ for $t \ge 0$, we get $n[(1+u+u^2)^{1/n} - 1] = n(e^{L/n} - 1) \to L(u)$, with $0 \le n(e^{L/n} - 1) \le 3L(u)$. Also $u^{1/n-1} \to u^{-1}$ and $u^{1/n-1} \le u^{-1}$. So the integrand converges pointwise to $L(u)/u$ and is dominated by $3L(u)/u \le 6$. By dominated convergence,

$$\lim_{n\to\infty} n^2(I_n - 1) = \int_0^1 \frac{\ln(1 + u + u^2)}{u}\,du.$$

Now $\ln(1 + u + u^2) = \ln(1 - u^3) - \ln(1 - u)$. Using $\int_0^1 \frac{-\ln(1-v)}{v}dv = \sum_{m\ge1} \frac{1}{m^2} = \frac{\pi^2}{6}$ and, with $v = u^3$, $\int_0^1 \frac{\ln(1-u^3)}{u}du = \frac{1}{3}\int_0^1 \frac{\ln(1-v)}{v}dv = -\frac{\pi^2}{18}$, the limit is

$$-\frac{\pi^2}{18} + \frac{\pi^2}{6} = \frac{\pi^2}{9}.$$

Check. Expanding $\ln(1+u+u^2)/u = \sum_m c_m u^{m-1}$ with $c_m = \frac{1}{m}$ (1 if $3 \nmid m$, -2 if $3 \mid m$) gives $\sum c_m/m = \sum 1/m^2 - 3\sum_{3|m} 1/m^2 = \frac{\pi^2}{6}(1 - \frac{3}{9}) = \frac{\pi^2}{9}$. Numerically $\pi^2/9 \approx 1.0966$, and a high-precision evaluation of $n^2(I_n - 1)$ for $n = 32, \ldots, 1024$, extrapolated in powers of $1/n$, reproduces this value.

A4 Answer: 9051851200**. The idea.** $(b-a)(c-b)(c-a)$ is the Vandermonde determinant of the rows $(1, a, a^2)$, $(1, b, b^2)$, $(1, c, c^2)$. So the sum is the sum of the squares of all 3×3

minors of the 30×3 matrix V whose k-th row is $(1, k, k^2)$, and by the Cauchy–Binet formula that equals the single 3×3 determinant $\det(V^{\mathsf{T}}V)$. Column operations that do not change the minors can make the columns of V orthogonal, and then $V^{\mathsf{T}}V$ is diagonal.

Work it out. *Cauchy–Binet.* For an $N \times 3$ matrix V, $\det(V^{\mathsf{T}}V) = \sum_S \det(V_S)^2$, the sum over all 3-element row sets S, where V_S is the 3×3 submatrix on those rows. With rows $(1, k, k^2)$, $\det V_S = (b-a)(c-a)(c-b)$ for $S = \{a, b, c\}$, $a < b < c$. So the requested sum is $\det(V^{\mathsf{T}}V)$.

Change columns. If U is a 3×3 matrix, then $(VU)_S = V_S U$, so every minor gets multiplied by $\det U$. Put $s = 2k - 31$, so s runs over the odd integers $-29, -27, \dots, 29$. Then $k = \frac{s+31}{2}$ and $k^2 = \frac{s^2+62s+961}{4}$, so the columns $(1, k, k^2)$ are obtained from $(1, \frac{s}{2}, \frac{s^2}{4})$ by adding multiples of earlier columns ($\det = 1$). Next replace s^2 by $s^2 - \mu$, where $\mu = \frac{1}{30}\sum s^2$ (again $\det = 1$). Every minor of V is therefore $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ times the corresponding minor of the matrix W with rows $(1,\ s,\ s^2 - \mu)$, and the sum equals $\frac{1}{64}\det(W^{\mathsf{T}}W)$.

The columns of W are orthogonal. The values of s are symmetric about 0, so $\sum s = \sum s(s^2 - \mu) = 0$; and $\sum(s^2 - \mu) = 0$ by the choice of μ. Hence

$$\det(W^{\mathsf{T}}W) = 30 \cdot \sum s^2 \cdot \sum (s^2 - \mu)^2, \qquad \sum (s^2 - \mu)^2 = \sum s^4 - 30\mu^2.$$

Power sums of the odd numbers. $\sum_s s^2 = 2(1^2 + 3^2 + \cdots + 29^2) = 2\big(\sum_{j=1}^{30} j^2 - 4\sum_{i=1}^{15} i^2\big) = 2(9455 - 4960) = 8990$, so $\mu = \frac{899}{3}$. Similarly $1^4 + 3^4 + \cdots + 29^4 = \sum_{j=1}^{30} j^4 - 16\sum_{i=1}^{15} i^4 = 5273999 - 16 \cdot 178312 = 2421007$, so $\sum s^4 = 4842014$. Then

$$\sum (s^2 - \mu)^2 = 4842014 - 30 \cdot \frac{808201}{9} = \frac{14526042 - 8082010}{3} = \frac{6444032}{3}.$$

Therefore the sum is

$$\frac{1}{64} \cdot 30 \cdot 8990 \cdot \frac{6444032}{3} = \frac{89900 \cdot 6444032}{64} = 89900 \cdot 100688 = 9051851200.$$

Check. The same computation for $\{1, \dots, n\}$ gives $\dfrac{n^3(n^2-1)^2(n^2-4)}{2160}$. For $n = 3$ this is 4, and indeed the only triple $\{1, 2, 3\}$ gives $(1 \cdot 1 \cdot 2)^2 = 4$. For $n = 4$ it is 80, and by hand the triples $\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$ give $4 + 4 + 36 + 36 = 80$. With $n = 30$ the formula gives $27000 \cdot 899^2 \cdot 896/2160 = 9051851200$, and a direct computer sum over all 4060 triples agrees.

A5 Answer: $\frac{3}{2}\ln^2 2 - \frac{\pi^2}{24}$**. The idea.** Write the tails as integrals, $r_n = (-1)^n a_n$ with $a_n = \int_0^1 \frac{x^n}{1+x}dx$. These satisfy the two-term relation $a_n + a_{n+1} = \frac{1}{n+1}$, and the identity $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ converts the cubic sum into a telescoping sum plus the *quadratic* sum $\sum(-1)^n a_n a_{n+1}$, which is a double integral we can do.

Work it out. *Step 1.* $\frac{1}{1+x} = \sum_{k=0}^{n-1}(-x)^k + \frac{(-x)^n}{1+x}$ integrates to $\ln 2 = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} + (-1)^n a_n$, so $r_n = (-1)^n a_n$ with $0 < a_n \le \frac{1}{n+1}$, and $a_n + a_{n+1} = \int_0^1 x^n\,dx = \frac{1}{n+1}$. The target is $T = \sum_{n \ge 0}(-1)^n a_n^3$, absolutely convergent.

Step 2. With $a = a_n$, $b = a_{n+1}$, $a + b = \frac{1}{n+1}$: $(n+1)(a_n^3 + a_{n+1}^3) = \frac{1}{(n+1)^2} - 3a_n a_{n+1}$. Multiply by $(-1)^n$ and sum over $n \ge 0$. The series $\sum(-1)^n(n+1)a_n^3$ and $\sum(-1)^n(n+1)a_{n+1}^3$ converge absolutely (terms are $O(1/n^2)$), so we may split and reindex the second ($m = n + 1$):

$$\sum_{n\ge0}(-1)^n(n+1)a_n^3-\sum_{m\ge1}(-1)^m m\,a_m^3=a_0^3+\sum_{m\ge1}(-1)^m a_m^3=T.$$

On the right, $\sum(-1)^n/(n+1)^2=\frac{\pi^2}{12}$. So $T=\frac{\pi^2}{12}-3U$ with $U=\sum_{n\ge0}(-1)^n a_n a_{n+1}$.

Step 3. $a_n a_{n+1}=\iint_{[0,1]^2}\frac{x^n y^{n+1}}{(1+x)(1+y)}dx\,dy$, and the partial sums of $\sum(-1)^n(xy)^n$ are bounded by 1, so dominated convergence gives

$$U=\iint_{[0,1]^2}\frac{y\,dx\,dy}{(1+xy)(1+x)(1+y)}.$$

Since $\frac{y}{(1+xy)(1+y)}=\frac{1}{1-x}\left[\frac{1}{1+xy}-\frac{1}{1+y}\right]$, integrating in y gives

$$U=\int_0^1\frac{\frac{\ln(1+x)}{x}-\ln 2}{(1-x)(1+x)}\,dx.$$

Step 4. Partial fractions, $\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1/2}{1-x}-\frac{1/2}{1+x}$ and $\frac{1}{(1-x)(1+x)}=\frac{1/2}{1-x}+\frac{1/2}{1+x}$, split the integrand into

$$\frac{\ln(1+x)}{x}+\frac12\cdot\frac{\ln(1+x)-\ln 2}{1-x}-\frac12\cdot\frac{\ln(1+x)}{1+x}-\frac12\cdot\frac{\ln 2}{1+x},$$

each piece integrable on $[0,1]$. The four integrals are: $\int_0^1\frac{\ln(1+x)}{x}dx=\sum\frac{(-1)^{m-1}}{m^2}=\frac{\pi^2}{12}$; with $t=1-x$, $\int_0^1\frac{\ln(1+x)-\ln 2}{1-x}dx=\int_0^1\frac{\ln(1-t/2)}{t}dt=-\operatorname{Li}_2(\frac12)=\frac{\ln^2 2}{2}-\frac{\pi^2}{12}$ (Euler's reflection $\operatorname{Li}_2(z)+\operatorname{Li}_2(1-z)=\frac{\pi^2}{6}-\ln z\ln(1-z)$ at $z=\frac12$); $\int_0^1\frac{\ln(1+x)}{1+x}dx=\frac{\ln^2 2}{2}$; and $\int_0^1\frac{\ln 2}{1+x}dx=\ln^2 2$. Hence

$$U=\frac{\pi^2}{12}+\frac12\left(\frac{\ln^2 2}{2}-\frac{\pi^2}{12}\right)-\frac{\ln^2 2}{4}-\frac{\ln^2 2}{2}=\frac{\pi^2}{24}-\frac{\ln^2 2}{2},$$

and

$$T=\frac{\pi^2}{12}-\frac{\pi^2}{8}+\frac32\ln^2 2=\frac32\ln^2 2-\frac{\pi^2}{24}\approx 0.309446.$$

Check. The first terms are $r_0^3\approx 0.3330$, $r_1^3\approx -0.0289$, $r_2^3\approx 0.0072$, $r_3^3\approx -0.0028$, $r_4^3\approx 0.0013$; the partial sums 0.3086 (four terms) and 0.3099 (five terms) bracket 0.309446, and the alternating terms shrink like $\frac{1}{8n^3}$. An accelerated high-precision summation of $\sum r_n^3$ (with a_n computed from the digamma function) agrees to 60 digits.

A6 Answer: $\frac32\ln\left(2+\sqrt3\right)-\frac{\pi}{4}$. **The idea.** Averaging an area over all central planes is an integral over planes, and integral geometry turns it into a single volume integral: a point x lies in the plane $u^\perp$ exactly for the u on a great circle, and the weight that point picks up is proportional to $1/|x|$. So the expected area is a constant times $\int_{\text{cube}}\frac{dx}{|x|}$, a Newtonian-potential-type integral we can evaluate exactly.

Work it out. *The identity.* For integrable $f\ge0$ on $\mathbb{R}^3$, using polar coordinates in each plane $u^\perp$,

$$\int_{S^2}\int_{u^\perp}f\,dA\,du=\int_{S^2}\int_{v\in S^2,\,v\perp u}\int_0^\infty f(rv)\,r\,dr\,dv\,du.$$

The set of orthonormal pairs (u,v) can be integrated in either order, and for fixed v the admissible u form a great circle of length 2π; so the right side equals $2\pi\int_{S^2}\int_0^\infty f(rv)\,r\,dr\,dv=2\pi\int_{\mathbb{R}^3}\frac{f(x)}{|x|}dx$ (because $dx=r^2dr\,dv$). As a test, for the unit ball both sides equal $4\pi^2$.

Apply it. With f the indicator of the cube $C=[-\frac12,\frac12]^3$ and du normalised by 4π,

$$E[\text{area}] = \frac{2\pi}{4\pi}\int_C \frac{dx}{|x|} = \frac{1}{2}\cdot 8\int_{[0,\frac12]^3}\frac{dx}{|x|} = 4\cdot\frac14\int_{[0,1]^3}\frac{dx}{|x|} = J,$$

using the eight octants and the scaling $x \mapsto x/2$ (volume factor $\frac18$, $\frac{1}{|x|}$ factor 2).

Evaluate J. Split $[0,1]^3$ into three congruent pyramids with apex 0 according to which coordinate is largest. On the pyramid where x is largest, write $y = xs$, $z = xt$ with $(s,t)\in[0,1]^2$; then $dy\,dz = x^2 ds\,dt$ and $|\mathbf{x}| = x\sqrt{1+s^2+t^2}$, so its contribution is

$$\int_0^1 x\,dx \iint_{[0,1]^2}\frac{ds\,dt}{\sqrt{1+s^2+t^2}} = \frac12 K, \qquad J = \frac32 K.$$

For K, integrate in s: $\int_0^1 \frac{ds}{\sqrt{a^2+s^2}} = \operatorname{arsinh}\frac{1}{a}$ with $a = \sqrt{1+t^2}$. So $K = \int_0^1 \operatorname{arsinh}\frac{1}{\sqrt{1+t^2}}\,dt$. Integrate by parts: $\frac{d}{dt}\operatorname{arsinh}\frac{1}{\sqrt{1+t^2}} = -\frac{t}{(1+t^2)\sqrt{2+t^2}}$, so

$$K = \operatorname{arsinh}\tfrac{1}{\sqrt2} + \int_0^1 \frac{t^2\,dt}{(1+t^2)\sqrt{2+t^2}} = \ln\frac{1+\sqrt3}{\sqrt2} + \int_0^1\frac{dt}{\sqrt{2+t^2}} - \int_0^1\frac{dt}{(1+t^2)\sqrt{2+t^2}}.$$

Here $\int_0^1\frac{dt}{\sqrt{2+t^2}} = \operatorname{arsinh}\frac{1}{\sqrt2} = \ln\frac{1+\sqrt3}{\sqrt2}$, and with $w = \frac{t}{\sqrt{2+t^2}}$ one gets $\int_0^1\frac{dt}{(1+t^2)\sqrt{2+t^2}} = \int_0^{1/\sqrt3}\frac{dw}{1+w^2} = \arctan\frac{1}{\sqrt3} = \frac{\pi}{6}$. (Indeed $t^2 = \frac{2w^2}{1-w^2}$, $1+t^2 = \frac{1+w^2}{1-w^2}$, $dt = \frac{\sqrt2\,dw}{(1-w^2)^{3/2}}$, $\sqrt{2+t^2} = \frac{\sqrt2}{\sqrt{1-w^2}}$, and the integrand becomes $\frac{dw}{1+w^2}$.) Since $2\ln\frac{1+\sqrt3}{\sqrt2} = \ln\frac{4+2\sqrt3}{2} = \ln(2+\sqrt3)$,

$$K = \ln(2+\sqrt3) - \frac{\pi}{6}, \qquad E[\text{area}] = J = \frac32\ln(2+\sqrt3) - \frac{\pi}{4} \approx 1.19004.$$

Check. Every central section has area between 1 (a face-parallel square) and $\sqrt2$ (the diagonal rectangle), and 1.19004 lies in between. Independently, writing $\frac{1}{|x|} = \frac{2}{\sqrt\pi}\int_0^\infty e^{-\lambda^2|x|^2}d\lambda$ factorises $J = \frac{2}{\sqrt\pi}\int_0^\infty\left(\frac{\sqrt\pi\,\operatorname{erf}\lambda}{2\lambda}\right)^3 d\lambda$, and this one-dimensional integral agrees with the closed form to 60 digits; averaging the exact polygon areas directly over the sphere (a fine Gauss quadrature in $\cos\theta$ and ϕ) gives 1.1900387, in agreement.

B1 Answer: $\frac{2026}{3}$. The idea. A point's nearest neighbour is always one of the two points next to it on the circle, across the shorter of its two adjacent gaps. So a close pair is two adjacent points whose gap is shorter than both gaps beside it. The gaps are exchangeable, so this happens with probability $\frac13$ for each of the 2026 gaps.

Work it out. Let $n = 2026$. The points cut the circle into n arcs (gaps) $g_1,\dots,g_n$ in cyclic order, where g_i runs from point P_i to point P_{i+1} (indices mod n). With probability 1 all gaps are positive and all distances distinct.

Nearest neighbours. Take a point P_i, with gap g_{i-1} behind it and g_i ahead of it. The distance from P_i to any other point Q is the shorter of the clockwise and counterclockwise arcs. The clockwise arc to Q is at least g_i, and the counterclockwise arc is at least g_{i-1}, so the distance is at least $\min(g_{i-1}, g_i)$. Suppose, say, g_i is the smaller gap. The distance from P_i to P_{i+1} is $\min(g_i, 1-g_i) \le g_i$, and by the bound above it is also at least g_i, so it equals g_i. Every other point is strictly farther, since distances are distinct. So the nearest neighbour of P_i is the adjacent point across the smaller of g_{i-1}, g_i. (Since $n \ge 3$, the two adjacent points are different points.)

Close pairs. Two points that are not adjacent can never be a close pair, because each point's nearest neighbour is adjacent to it. The adjacent points P_i, P_{i+1} form a close pair exactly when $g_i < g_{i-1}$ (so P_i chooses P_{i+1}) and $g_i < g_{i+1}$ (so P_{i+1} chooses P_i). Since

$n \geq 3$, the gaps g_{i-1}, g_i, g_{i+1} are three different gaps. So the number of close pairs is the number of indices i with $g_i < \min(g_{i-1}, g_{i+1})$.

Probability. Cut the circle at P_1. The other $n-1$ points are independent uniform points on an interval of length 1, so $(g_1, \ldots, g_n)$ is uniformly distributed on the simplex $g_j > 0$, $\sum g_j = 1$. That law is symmetric under every permutation of the coordinates. So the three gaps g_{i-1}, g_i, g_{i+1} are exchangeable, and each of their $3! = 6$ orderings has probability $\frac{1}{6}$ (ties have probability 0). Exactly two of those orderings put g_i first, so $P(g_i < \min(g_{i-1}, g_{i+1})) = \frac{1}{3}$.

By linearity of expectation, the expected number of close pairs is $n \cdot \frac{1}{3} = \frac{2026}{3}$.

Check. For $n = 3$ the argument predicts exactly 1 close pair on average, and indeed there is always exactly one: the two endpoints of the smallest of the three gaps. For $n = 4, 5, 6, 7$, integrating the Dirichlet density $(n-1)(n-2)(n-3)(1-g_1-g_2-g_3)^{n-4}$ of three consecutive gaps over $\{g_2 < g_1, g_2 < g_3\}$ gives exactly $\frac{1}{3}$ each time.

B2 Answer: $\frac{346503741}{2}$**. The idea.** Describe the circle by its steps $s_i = x_{i+1} - x_i$. The sum of squares is a convex function of the steps, so it is largest when every step is ± 1. Then the numbers form a closed walk on consecutive integers, which must pass through every level between its lowest and highest value twice, once going up and once coming down; that forced repetition is exactly what caps the spread.

Work it out. Let $n = 2026 = 2m$ with $m = 1013$, indices taken modulo n. Because $\sum x_i = 0$, $Q := \sum x_i^2 = \sum (x_i - \bar{x})^2$ with $\bar{x} = 0$, and Q does not change if a constant is added to all x_i.

Step 1: all steps are ± 1 at a maximum. The steps satisfy $s_i \in [-1, 1]$ and $\sum s_i = 0$; conversely such steps determine the x_i up to an additive constant, which the condition $\sum x_i = 0$ fixes. Each $x_i - \bar{x}$ is a linear function of $(s_1, \ldots, s_n)$, so Q is a sum of squares of linear functions, a convex function of the steps. The feasible set is compact, so a maximum exists. Take a maximiser. If two steps s_i, s_j $(i \neq j)$ lie strictly between -1 and 1, replace them by $s_i + \tau$, $s_j - \tau$: the sum stays 0, the allowed τ form a closed interval with 0 inside it, and Q is convex in τ, so at one of the two endpoints Q is at least as large; there one more step equals ± 1. Repeating, we reach a maximiser with at most one step outside $\{-1, 1\}$. That step equals minus the sum of the other 2025 steps, an odd integer, and it lies in $[-1, 1]$, so it is ± 1 too. Hence the maximum is attained with every step equal to ± 1 (and then 1013 of each).

Step 2: levels are repeated. With all steps ± 1, shift the numbers so that the smallest is 0; they are then integers from 0 to some r, and let c_h be the number of indices with value h. Going around the circle from a position P of value 0 to a position R of value r one way, and from R back to P the other way, each of the two arcs passes through every level strictly between 0 and r (steps are ± 1), and the two arcs share only P and R. So $c_h \geq 2$ for $0 < h < r$, while $c_0, c_r \geq 1$. Adding, $n \geq 2r$, so $r \leq m$.

Step 3: the bound. Let $\mu = \frac{1}{n} \sum_h c_h h$ be the mean of the shifted values. Expanding both sides gives the identity

$$Q = \sum_h c_h (h - \mu)^2 = n\mu(r - \mu) - \sum_h c_h \, h(r - h).$$

Here $n\mu(r-\mu) \leq \frac{nr^2}{4} = \frac{mr^2}{2}$, and since $h(r-h) \geq 0$ for $0 \leq h \leq r$, $\sum_h c_h h(r-h) \geq 2\sum_{h=1}^{r-1} h(r-h) = \frac{r^3 - r}{3}$. So $Q \leq G(r) := \frac{mr^2}{2} - \frac{r^3 - r}{3}$. For $0 \leq r \leq m - 1$,

$$G(r+1)-G(r)=\frac{m(2r+1)}{2}-r(r+1)\ge\frac{(r+1)(2r+1)}{2}-r(r+1)=\frac{r+1}{2}>0,$$

so G increases on $0,1,\dots,m$ and $Q\le G(m)=\dfrac{m^3+2m}{6}$.

Step 4: equality. Take the tent $x_i=\min(i-1,\,n+1-i)-\frac{m}{2}$ for $i=1,\dots,n$: the shifted values run $0,1,\dots,m,\dots,1$, all steps are ±1, the multiplicities are $1,2,2,\dots,2,1$, and the mean is $\frac{m}{2}$ (so $\sum x_i=0$). Both estimates in Step 3 are equalities, so $Q=\frac{m^3+2m}{6}$.

With $m=1013$: $\dfrac{1013\,(1013^2+2)}{6}=\dfrac{1013\cdot1026171}{6}=\dfrac{1013\cdot342057}{2}=\dfrac{346503741}{2}$.

Check. For $n=4$ the formula gives $\frac{8+4}{6}=2$, attained by $-1,0,1,0$; the other balanced pattern $-\frac12,-\frac12,\frac12,\frac12$ gives only 1. For $n=6$ it gives $\frac{11}{2}$, attained by $-\frac32,-\frac12,\frac12,\frac32,\frac12,-\frac12$. A computer search over all balanced ±1 step patterns for every even $n\le16$ reproduces $\frac{m^3+2m}{6}$, and random real sequences obeying the constraints never exceed it.

B3 **Answer:** $\frac43-\frac{2\sqrt3\,\pi}{9}$. **The idea.** The lcm of four consecutive integers is their product divided by a small factor, 2 or 6, and which one depends only on k mod 3. So the sum is a combination of $\sum\frac{1}{k(k+1)(k+2)(k+3)}$ over all k and over multiples of 3. The Beta integral $\frac{6}{k(k+1)(k+2)(k+3)}=\int_0^1x^{k-1}(1-x)^3\,dx$ turns the sum over multiples of 3 into a geometric series under an integral, and that integral is elementary.

Work it out. Let $P_k=k(k+1)(k+2)(k+3)$ and $L_k=\operatorname{lcm}(k,\dots,k+3)$. For each prime p, $v_p(P_k)-v_p(L_k)$ is the sum of the p-adic valuations of the four numbers minus the largest one. A prime $p\ge5$ divides at most one of four consecutive integers, so it contributes nothing. Exactly two of the four are even, and they differ by 2, so one of them is 2 mod 4: the prime 2 contributes exactly 1. The prime 3 divides two of them exactly when $3\mid k$ (namely k and $k+3$), and then one of those two is not divisible by 9 because they differ by 3; so 3 contributes 1 if $3\mid k$ and 0 otherwise. Hence $\frac{1}{L_k}=\frac{2}{P_k}+[3\mid k]\,\frac{4}{P_k}$, $\qquad S=2\sum_{k\ge1}\frac{1}{P_k}+4\sum_{m\ge1}\frac{1}{P_{3m}}$. Integrals. By the Beta integral, $\int_0^1x^{k-1}(1-x)^3\,dx=\frac{(k-1)!\,3!}{(k+3)!}=\frac{6}{P_k}$. Summing over all $k\ge1$ (nonnegative terms, so the swap is fine) gives $\sum_k\frac{1}{P_k}=\frac16\int_0^1\frac{(1-x)^3}{1-x}\,dx=\frac16\int_0^1(1-x)^2\,dx=\frac{1}{18}$. Summing over $k=3m$ uses $\sum_{m\ge1}x^{3m-1}=\frac{x^2}{1-x^3}$: $\sum_{m\ge1}\frac{1}{P_{3m}}=\frac16\int_0^1\frac{x^2(1-x)^3}{1-x^3}\,dx=\frac16\int_0^1\frac{x^2(1-x)^2}{x^2+x+1}\,dx$. Divide: $x^4-2x^3+x^2=(x^2+x+1)(x^2-3x+3)-3$. So the integral is $\int_0^1(x^2-3x+3)\,dx-3\int_0^1\frac{dx}{x^2+x+1}=\frac{11}{6}-3\cdot\frac{2}{\sqrt3}\Big[\arctan\frac{2x+1}{\sqrt3}\Big]_0^1=\frac{11}{6}-\frac{6}{\sqrt3}\Big(\frac{\pi}{3}-\frac{\pi}{6}\Big)=\frac{11}{6}-\frac{\pi}{\sqrt3}$. Therefore $S=\frac{2}{18}+\frac46\Big(\frac{11}{6}-\frac{\pi}{\sqrt3}\Big)=\frac19+\frac{11}{9}-\frac{2\pi}{3\sqrt3}=\frac43-\frac{2\sqrt3\,\pi}{9}\approx0.1241338$.

Check. The first six terms listed in the statement add to about 0.12222, and the remaining terms are all below $\frac{6}{P_k}$, whose sum from $k=7$ on is $6\big(\frac{1}{3\cdot7\cdot8\cdot9}\big)=\frac{1}{252}\approx0.004$; so S lies in $[0.1222,0.1262]$, consistent with 0.12413. As a sanity check on the integral, $\sum_m\frac{1}{P_{3m}}=\frac16\big(\frac{11}{6}-\frac{\pi}{\sqrt3}\big)\approx0.003270$, and its first two terms are $\frac{1}{360}+\frac{1}{3024}\approx0.003108$. A computer check with math.lcm for $k\le3000$ exactly, plus each residue class mod 3 summed to 55 digits, agrees.

B4 **Answer:** 9874. **The idea.** Over $\mathbb{F}_2$, $x^4-x=x(x+1)(x^2+x+1)$, and these three factors are distinct and irreducible. So $A^4=A$ says exactly that A splits $\mathbb{F}_2^4$ into three pieces: one where $A=0$, one where $A=I$, and one where $A^2+A+I=0$. The third

piece is a vector space over $\mathbb{F}_4$. Counting the matrices is then counting a conjugacy class for each dimension pattern, which is $|GL_4(\mathbb{F}_2)|$ divided by the size of a centraliser.

Work it out. Structure. If $A^4 = A$, the minimal polynomial of A divides $x(x+1)(x^2+x+1)$. These factors are pairwise coprime (x^2+x+1 has no root in $\mathbb{F}_2$), so by the primary decomposition $V = \mathbb{F}_2^4 = V_0 \oplus V_1 \oplus W$, with $V_0 = \ker A$, $V_1 = \ker(A+I)$ and $W = \ker(A^2+A+I)$, all three A-invariant. On W, $T = A|_W$ satisfies $T^2+T+1 = 0$, so W is a module over $\mathbb{F}_2[T]/(T^2+T+1) \cong \mathbb{F}_4$. So W is an $\mathbb{F}_4$-vector space, and $\dim_{\mathbb{F}_2} W = 2m$ is even. Conversely, any splitting $V = V_0 \oplus V_1 \oplus W$ together with any T on W satisfying $T^2+T+1 = 0$ defines an A (equal to $0, I, T$ on the three pieces) with $A^4 = A$. That is because $x^4 - x$ is divisible by x, by $x+1$ and by x^2+x+1.

Classes. Let $a = \dim V_0$ and $b = \dim V_1$, so $a + b + 2m = 4$. Two such matrices with the same (a, b, m) are conjugate in $GL_4(\mathbb{F}_2)$: map V_0 to V_0' and V_1 to V_1' by any isomorphisms, and W to W' by an $\mathbb{F}_4$-linear isomorphism (two $\mathbb{F}_4$-spaces of equal dimension are isomorphic). So each triple (a, b, m) gives one conjugacy class. The centraliser of A is the set of invertible maps that preserve each piece and commute with T on W, namely $GL_a(\mathbb{F}_2) \times GL_b(\mathbb{F}_2) \times GL_m(\mathbb{F}_4)$. The class therefore has $\frac{|GL_4(\mathbb{F}_2)|}{|GL_a(\mathbb{F}_2)|\,|GL_b(\mathbb{F}_2)|\,|GL_m(\mathbb{F}_4)|}$ elements.

Orders: $|GL_1(\mathbb{F}_2)| = 1$, $|GL_2(\mathbb{F}_2)| = 6$, $|GL_3(\mathbb{F}_2)| = 168$, $|GL_4(\mathbb{F}_2)| = 15 \cdot 14 \cdot 12 \cdot 8 = 20160$, $|GL_1(\mathbb{F}_4)| = 3$ and $|GL_2(\mathbb{F}_4)| = 15 \cdot 12 = 180$.

$m = 0$: $(a, b) = (4, 0), (3, 1), (2, 2), (1, 3), (0, 4)$ give $1 + 120 + 560 + 120 + 1 = 802$.

$m = 1$: $(a, b) = (2, 0), (1, 1), (0, 2)$ give $\frac{20160}{3}\left(\frac{1}{6} + 1 + \frac{1}{6}\right) = 1120 + 6720 + 1120 = 8960$.

$m = 2$: $\frac{20160}{180} = 112$.

Total: $802 + 8960 + 112 = 9874$.

Check. The same count for 2×2 matrices gives $(1 + 6 + 1) + \frac{6}{3} = 10$. Listing all 16 matrices confirms it: the 8 that are 0, I or have $A^2 = A$ with rank 1, plus the two of order 3, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. For the 4×4 case, checking all $2^{16} = 65536$ matrices by computer finds exactly 9874.

B5 **Answer:** $\frac{6211}{14580}$**. The idea.** Give each cycle of length divisible by 3 a weight y. The exponential formula turns the weighted count into exp of a sum of logarithms, $\frac{(1-x^3)^{(1-y)/3}}{1-x}$. A roots-of-unity filter over $y \in \{1, \omega, \omega^2\}$ (with $\omega = e^{2\pi i/3}$) picks out $C \equiv 0 \pmod 3$. The coefficient we need is a partial sum of a binomial series with a complex exponent, and that partial sum telescopes into a finite product in $\mathbb{Z}[\omega]$.

Work it out. Generating function. Let $c(\sigma)$ be the number of cycles of σ with length divisible by 3. There are $(k-1)!$ cycles on a given k-set, so by the exponential formula (a permutation is a set of cycles), as formal power series in x, $G(x, y) = \sum_{n \geq 0} \frac{x^n}{n!} \sum_{\sigma \in S_n} y^{c(\sigma)} = \exp\left(\sum_{3 \nmid k} \frac{x^k}{k} + y \sum_{3 \mid k} \frac{x^k}{k} \right)$. Since $\sum_{k \geq 1} \frac{x^k}{k} = -\ln(1-x)$ and $\sum_{3 \mid k} \frac{x^k}{k} = -\frac{1}{3}\ln(1-x^3)$, the exponent is $-\ln(1-x) - \frac{y-1}{3}\ln(1-x^3)$, so $G(x, y) = \frac{(1-x^3)^\beta}{1-x}$ with $\beta = \frac{1-y}{3}$, where $(1-x^3)^\beta$ means $\exp(\beta \ln(1-x^3)) = \sum_j \binom{\beta}{j}(-x^3)^j$ for any complex β.

Filter. For an integer c, $\frac{1}{3}(1+\omega^c+\omega^{2c})$ is 1 if $3 \mid c$ and 0 otherwise. Let $e_n(y) = [x^n]G(x, y)$, which is the expected value of y^C for a random $\sigma \in S_n$. Then $p_n = P(3 \mid C) =$

$\frac{1}{3}\big(e_n(1)+e_n(\omega)+e_n(\omega^2)\big)=\frac{1}{3}+\frac{2}{3}\operatorname{Re}e_n(\omega)$, because $e_n(1)=1$ and $e_n(\omega^2)=\overline{e_n(\omega)}$ (all coefficients of G in y are real).

Coefficient. Dividing by $1-x$ takes partial sums, so with $m=\lfloor n/3\rfloor$, $e_n(y)=\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}$. For any complex β, $\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}=(-1)^m\binom{\beta-1}{m}=\prod_{j=1}^{m}\left(1-\frac{\beta}{j}\right)$: this holds for $m=0$, and the step $m\to m+1$ is Pascal's rule, $(-1)^m\binom{\beta-1}{m}+(-1)^{m+1}\binom{\beta}{m+1}=(-1)^{m+1}\binom{\beta-1}{m+1}$. For $y=\omega$, $\beta=\frac{1-\omega}{3}$ and $1-\frac{\beta}{j}=\frac{3j-1+\omega}{3j}$. So for $n=18$, $m=6$: $e_{18}(\omega)=\frac{(2+\omega)(5+\omega)(8+\omega)(11+\omega)(14+\omega)(17+\omega)}{3^6\cdot 6!}$. Multiply out in $\mathbb{Z}[\omega]$ using $\omega^2=-1-\omega$, so $(a+b\omega)(c+\omega)=(ac-b)+(a+bc-b)\omega$: the partial products are $2+\omega$, $9+6\omega$, $66+51\omega$, $675+576\omega$, $8874+8163\omega$, $142695+139482\omega$. Since $\operatorname{Re}\omega=-\frac{1}{2}$, $\operatorname{Re}(a+b\omega)=a-\frac{b}{2}$, and $142695-69741=72954$. With $3^6\cdot 6!=524880$, $p_{18}=\frac{1}{3}+\frac{2}{3}\cdot\frac{72954}{524880}=\frac{524880+145908}{1574640}=\frac{670788}{1574640}=\frac{6211}{14580}$. Here $670788=108\cdot 6211$ and $1574640=108\cdot 14580$; $14580=2^2\cdot 3^6\cdot 5$ and 6211 is odd, is not divisible by 3 (digit sum 10) and is not divisible by 5, so the fraction is in lowest terms (≈ 0.42599).

Check. Small case $n=3$ ($m=1$): $e_3(\omega)=\frac{2+\omega}{3}$, with real part $\frac{1}{2}$, so $p_3=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$. Directly, the two 3-cycles in S_3 have $C=1$ and the other four permutations have $C=0$, giving $\frac{4}{6}$. Small case $n=6$ ($m=2$): $e_6(\omega)=\frac{9+6\omega}{18}$ has real part $\frac{1}{3}$, so $p_6=\frac{1}{3}+\frac{2}{9}=\frac{5}{9}$, which matches a brute-force count of 400 of the 720 permutations. For $n=18$, an independent exact recursion (remove the cycle through 1: a k-cycle through 1 can be chosen in $\frac{(n-1)!}{(n-k)!}$ ways, and it shifts C by one when $3\mid k$) gives the same fraction, and that recursion matches a brute-force count over S_3, S_6, S_7 and S_8.

B6 Answer: $\frac{2-2\ln 2-\ln^2 2}{2\,(1-\ln 2)^2}$. **The idea.** Pass to logarithms: each break multiplies the length by an independent factor M, uniform on $[\frac{1}{2},1]$, so $-\ln(\text{length})$ is a sum of independent steps $Y=-\ln M$ with density $2e^{-y}$ on $[0,\ln 2]$. Conditioning on the first break gives an integral equation for $U(a)=E[N(e^{-a})]$; differentiating turns it into the delay equation $U'(a)=1+U(a)-U(a-\ln 2)$. After removing the linear growth, a quantity is conserved, which pins down the constant, and a contraction shows the limit exists.

Work it out. *Set-up.* Write $h=\ln 2$ and $\mu=1-h$. If the break point of a piece is at a uniform fraction V of it, the kept fraction is $M=\max(V,1-V)$, uniform on $[\frac{1}{2},1]$ (density 2). So $Y=-\ln M$ lies in $[0,h]$ with density $2e^{-y}$, and $E[Y]=\int_0^h 2ye^{-y}dy=2-2e^{-h}(1+h)=1-h=\mu$. With $S_n=Y_1+\cdots+Y_n$ and $a=\ln(1/t)$, the kept piece after n breaks is shorter than t exactly when $S_n>a$. Let $U(a)=E[N(e^{-a})]=\sum_{n\ge 0}P(S_n\le a)$; this is finite, because $P(S_n\le a)\le e^aE[e^{-S_n}]=e^a\left(\frac{3}{4}\right)^n$, as $E[e^{-Y}]=E[M]=\frac{3}{4}$.

The integral equation. Condition on the first break: if $Y_1>a$ we stop after one break, and otherwise the remaining process is a fresh copy started from level $a-Y_1$. Hence

$$U(a)=1+\int_0^{\min(a,h)}2e^{-y}\,U(a-y)\,dy=1+2\int_{\max(0,a-h)}^{a}e^{s-a}\,U(s)\,ds.$$

U is nondecreasing and locally bounded, so the right side is continuous in a; therefore U is continuous, and then the right side is continuously differentiable on $(0,h)$ and on (h,∞). Multiplying by e^a and differentiating: for $0<a<h$, $U'=U+1$, so with $U(0)=1$ we get $U(a)=2e^a-1$ on $[0,h]$ (in particular $U(h)=3$); for $a>h$, since $2e^{-h}=1$,

$$U'(a)=1+U(a)-U(a-h).$$

A conserved quantity. Let $W(a)=U(a)-\frac{a}{\mu}$. Since $U(a)-U(a-h)=W(a)-W(a-h)+\frac{h}{\mu}$

and $1+\frac{h}{\mu}-\frac{1}{\mu}=\frac{\mu+h-1}{\mu}=0$, the delay equation becomes $W'(a)=W(a)-W(a-h)$ for $a>h$. Then $\Phi(a)=W(a)-\int_{a-h}^{a}W(s)\,ds$ has $\Phi'(a)=W'(a)-W(a)+W(a-h)=0$, so Φ is constant on $[h,\infty)$. Its value at $a=h$, using $U=2e^{s}-1$ on $[0,h]$ and $e^{h}=2$:

$$\Phi=\Big(3-\frac{h}{\mu}\Big)-\Big(2-h-\frac{h^{2}}{2\mu}\Big)=\frac{\mu(1+h)-h+\frac{h^{2}}{2}}{\mu}=\frac{1-h-\frac{h^{2}}{2}}{\mu}.$$

The limit exists. Put $L=\dfrac{\Phi}{1-h}=\dfrac{1-h-\frac{h^{2}}{2}}{\mu^{2}}$ and $D(a)=W(a)-L$. Since $L-hL=\Phi$, for $a\ge h$ we get $D(a)=\int_{a-h}^{a}D(s)\,ds$. Let $m_k=\max_{[kh,(k+1)h]}|D|$ (finite, D is continuous). For $a\in[(k+1)h,(k+2)h]$ the window $[a-h,a]$ lies in $[kh,(k+2)h]$, so $|D(a)|\le h\max(m_k,m_{k+1})$, i.e. $m_{k+1}\le h\max(m_k,m_{k+1})$. As $h<1$, this forces $m_{k+1}\le h\,m_k$, so $m_k\le h^{k}m_0\to 0$ and $W(a)\to L$. Therefore

$$\lim_{t\to 0^{+}}\left(E[N(t)]-\frac{\ln(1/t)}{1-\ln 2}\right)=L=\frac{2-2\ln 2-\ln^{2}2}{2(1-\ln 2)^{2}}\approx 0.70760.$$

Check. Renewal theory predicts the constant $\frac{E[Y^{2}]}{2E[Y]^{2}}$, and $E[Y^{2}]=\int_{0}^{h}2y^{2}e^{-y}dy=2\big[2-e^{-h}(h^{2}+2h+2)\big]=2-2h-h^{2}$, so it agrees with L. The formula is already accurate for moderate t: solving the delay equation on $[h,2h]$ gives $E[N(\frac{1}{4})]=U(2h)=8-4\ln 2\approx 5.2274$, while $\frac{2\ln 2}{1-\ln 2}+L\approx 5.2254$; and a numerical solution of the integral equation at $t=\frac{1}{64}$ differs from $\frac{\ln 64}{1-\ln 2}+L$ by less than 10^{-7}.