MathCastle

Putnam

Session A and Session B

12 problems • 6 hours

Name _____

INSTRUCTIONS

1. Two sessions, A and B, of 6 problems each. Each session lasts 3 hours.
2. Each problem is worth 10 points. Full credit needs a complete proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Session A is handed in before Session B is opened.
5. Two sittings of 6 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

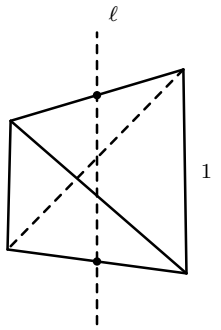
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

A1

Let M be the 2026×2026 matrix whose entry in row i and column j is $m_{ij} = \dfrac{1}{\max(i,j)}$ for $1 \le i, j \le 2026$. (So the first row and the first column are both $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2026}$, and the last row and last column are constant, equal to $\frac{1}{2026}$.) The matrix M is invertible. Find the trace of M^{-1}.

A2

A solid regular tetrahedron has edge length 1. Let ℓ be the line through the midpoints of two opposite edges (edges with no common vertex). The tetrahedron is rotated through a full turn about ℓ. Find the volume of the solid region swept out by the tetrahedron during the rotation, in simplest exact form.



A3

A sequence of rational numbers is defined by $a_0 = 1$, $a_1 = 2$ and $(n+1)\,a_{n+1} = (n+2)\,a_n + a_{n-1}$ for every $n \ge 1$. (So $a_2 = \frac{7}{2}$, $a_3 = \frac{16}{3}$, $a_4 = \frac{181}{24}$.) There are real constants α, β, γ such that $a_n - (\alpha n^2 + \beta n + \gamma) \to 0$ as $n \to \infty$. Find $\alpha + \beta + \gamma$ in exact form.

A4

Evaluate the improper integral $\displaystyle\int_0^\infty \left(\frac{e^{-x} - e^{-2x}}{x}\right)^3 dx$. Give your answer as an exact combination of natural logarithms of integers with rational coefficients.

A5

Let $n = 30$ and index the cells of an $n \times n$ array by pairs (i, j) with $i, j \in \{0, 1, \ldots, n-1\}$, read modulo n. For $s, c \in \{0, 1, \ldots, n-1\}$, the *line* $L_{s,c}$ is the set of the n cells (i, j) with $j \equiv c + si \pmod n$, and the *column* K_c is the set of the n cells (c, j), $j = 0, \ldots, n-1$. An assignment of a real number to each cell is called *balanced* if, for each fixed s, the n lines $L_{s,0}, L_{s,1}, \ldots, L_{s,n-1}$ all have the same sum, and the n columns $K_0, \ldots, K_{n-1}$ all have the same sum. The balanced arrays form a real vector space. Find its dimension.

A6

A population begins with a single founder. Every individual who ever lives, independently of all others, has a random number of children, equal to k with probability $\dfrac{e^{-1}}{k!}$ for $k = 0, 1, 2, \ldots$ (a

Poisson distribution with mean 1). Children then have children in the same way, and so on. Let N be the total number of individuals who ever live, counting the founder; with probability 1 the family dies out, so N is a finite positive integer. Find the exact value of $\mathbb{E}\left[\dfrac{1}{N^2}\right]$.

B1

A point P is chosen uniformly at random from the interior of a solid regular tetrahedron whose edges all have length 1. Let D be the distance from P to the nearest of the four faces (the planes of the faces). Find the expected value of D. Give your answer in simplest radical form.

B2

For each positive integer n, let σ be a permutation of $\{1, 2, \ldots, 2n\}$ chosen uniformly at random, and let C_n be the number of cycles of σ (fixed points count as cycles of length 1) all of whose elements are at most n. Find $\lim_{n\to\infty} E[C_n]$.

B3

Let $p = 2027$, which is a prime with $p \equiv 3 \pmod 4$. Count the 2×2 matrices with first row (a, b) and second row (c, d), with entries $a, b, c, d \in \{0, 1, \ldots, p-1\}$ such that $ad - bc \equiv 1 \pmod p$ and $a^2 + b^2 \equiv c^2 + d^2 \pmod p$.

B4

Let S be the set of 2×2 matrices A all four of whose entries lie in $\{-3, -2, -1, 0, 1, 2, 3\}$. A matrix A has *bounded powers* if there is a constant M such that every entry of A^n has absolute value at most M for every positive integer n. How many matrices in S have bounded powers?

B5

For each positive integer n, let $B_n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ be the closed unit ball in $\mathbb{R}^n$ (so B_1 is the segment $[-1, 1]$). Two points P and Q are chosen independently and uniformly at random from B_n, and let p_n be the probability that the closed ball having the segment PQ as a diameter is contained in B_n. Evaluate $\displaystyle\sum_{n=1}^{\infty} p_n$ in exact form.

B6

Let T be the infinite tree in which every vertex has exactly three neighbours, and fix a vertex r of T. Every vertex of T receives an independent label, uniformly distributed on $[0, 1]$. A set S of vertices is then defined by the greedy rule: a vertex v belongs to S if and only if no neighbour of v whose label is smaller than the label of v belongs to S. (With probability 1 all labels are distinct and every path of neighbouring vertices with decreasing labels is finite, so this rule determines S uniquely; it is the set obtained by scanning the vertices in increasing order of label and adding each one that has no neighbour already in S.) Find the probability that exactly two of the three neighbours of r belong to S.

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Answer Key — Putnam Practice Contest 3

A1 5544049726

A2 $\frac{\sqrt{2}\,\pi}{12}$

A3 $\frac{11}{2e}$

A4 $66\ln 2 + \frac{27}{2}\ln 3 - \frac{75}{2}\ln 5$

A5 431

A6 $\frac{5}{12}$

B1 $\frac{\sqrt{6}}{48}$

B2 $\ln 2$

B3 4108728

B4 181

B5 $1 + \frac{\pi}{2}$

B6 $\frac{3}{2}\ln 2 - \frac{3}{4}$

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Solutions — Putnam Practice Contest 3

A1 **Answer:** 5544049726. **The idea.** Inverting M directly is hopeless, but M factors as CC^T with a very simple lower-triangular C, and C^{-1} is bidiagonal. Then $\operatorname{tr} M^{-1} = \operatorname{tr}\left((C^{-1})^T C^{-1}\right)$ is just the sum of the squares of the entries of C^{-1}.

Work it out. Let $n = 2026$ and let C be the $n \times n$ lower-triangular matrix with $c_{ik} = \frac{1}{i}$ for $k \le i$ and $c_{ik} = 0$ for $k > i$. Then $(CC^T)_{ij} = \sum_{k \le \min(i,j)} \frac{1}{i}\cdot\frac{1}{j} = \dfrac{\min(i,j)}{ij} = \dfrac{1}{\max(i,j)}$, so $M = CC^T$.

Write $C = D^{-1}L$, where $D = \operatorname{diag}(1, 2, \ldots, n)$ and L is the lower-triangular all-ones matrix. The inverse of L is $I - S$, where S has ones just below the diagonal (differencing undoes partial sums). Hence $C^{-1} = L^{-1}D = (I - S)D$: its column j has the entry j on the diagonal and the entry $-j$ just below it (for $j < n$), and zeros elsewhere.

Now $M^{-1} = (C^{-1})^T C^{-1}$, and for any matrix B, $\operatorname{tr}(B^T B) = \sum_{i,j} b_{ij}^2$. Therefore $\operatorname{tr} M^{-1} = \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n-1} j^2 = 2\sum_{j=1}^{n-1} j^2 + n^2$.

With $n = 2026$: $\sum_{j=1}^{2025} j^2 = \dfrac{2025 \cdot 2026 \cdot 4051}{6} = 2769969225$, so the trace is $2 \cdot 2769969225 + 2026^2 = 5539938450 + 4104676 = 5544049726$.

Check. For $n = 2$, M has rows $(1, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$, determinant $\frac{1}{4}$, and M^{-1} has rows $(2, -2)$ and $(-2, 4)$, of trace $6 = 2 \cdot 1^2 + 2^2$, as the formula says. Exact inverses for all $n \le 12$ agree with $2\sum_{j<n} j^2 + n^2$.

A2 **Answer:** $\frac{\sqrt{2}\,\pi}{12}$. **The idea.** Slice perpendicular to the axis. Each slice of the tetrahedron is a rectangle centred on the axis, so rotating it sweeps out a full disc whose radius is the distance from the axis to a corner of the rectangle. Integrate the areas of those discs.

Work it out. Use the tetrahedron with vertices $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, of edge $2\sqrt{2}$; rescale at the end. The edges $(1,1,1)(-1,-1,1)$ and $(1,-1,-1)(-1,1,-1)$ are opposite, with midpoints $(0,0,1)$ and $(0,0,-1)$, so ℓ is the z-axis. The four faces are the planes through three vertices, and the solid is $x + y - z \le 1$, $-x - y - z \le 1$, $x - y + z \le 1$, $-x + y + z \le 1$ (each plane contains three vertices and the fourth vertex gives -3). At height $z \in [-1, 1]$ these read $|x + y| \le 1 + z$ and $|x - y| \le 1 - z$: a rectangle centred on the axis, with half-diagonals $\frac{1+z}{\sqrt{2}}$ and $\frac{1-z}{\sqrt{2}}$ along the perpendicular directions $x = y$ and $x = -y$.

Its farthest points from the axis are its corners, at squared distance $\frac{(1+z)^2+(1-z)^2}{2} = 1 + z^2$. The rectangle contains the axis point and is star-shaped about it, so turning it sweeps out the whole disc of radius $\sqrt{1+z^2}$, and no point of the rotated solid at height z is farther out. Hence the swept volume is $\int_{-1}^{1} \pi(1 + z^2)\,dz = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$. Scaling the edge from $2\sqrt{2}$ down to 1 multiplies volumes by $(2\sqrt{2})^{-3} = \frac{1}{16\sqrt{2}}$, giving $\dfrac{8\pi}{3 \cdot 16\sqrt{2}} = \dfrac{\pi}{6\sqrt{2}} = \dfrac{\sqrt{2}\,\pi}{12}$.

Check. The corners of the slice are where the four slanted edges cross height z; for instance the edge from $(1,1,1)$ to $(1,-1,-1)$ meets it at $(1, z, z)$, at squared distance $1 + z^2$, as claimed. The swept volume must exceed the tetrahedron's own volume $\frac{\sqrt{2}}{12} \approx 0.118$, and indeed it is exactly π times that, about 0.370.

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A3 **Answer:** $\frac{11}{2e}$. **The idea.** Put the terms into a generating function $A(x)=\sum_{n\ge 0} a_n x^n$. The recurrence has coefficients linear in n, so it becomes a first-order differential equation for A, which separates and gives $A(x)=\dfrac{e^{-x}}{(1-x)^3}$. Multiplying out the two series writes a_n as a finite sum whose three leading terms are exactly quadratic in n, with coefficients that are partial sums of series for e^{-1}.

Work it out. *The differential equation.* Work with formal power series and set $a_{-1}=0$; then the recurrence also holds for $n=0$, where it says $a_1=2a_0$. The coefficient of x^n in $(1-x)A'(x)$ is $(n+1)a_{n+1}-na_n$, and in $(2+x)A(x)$ it is $2a_n+a_{n-1}$. So the recurrence for all $n\ge 0$ says precisely $(1-x)A'=(2+x)A, \qquad A(0)=1$. Conversely this equation determines the series uniquely from $A(0)=1$, because comparing coefficients of x^n gives a_{n+1} in terms of a_n and a_{n-1}. Now let $B(x)=e^{-x}(1-x)^{-3}$ (the product of the two formal series). Then $B'=-B+\frac{3B}{1-x}$, so $(1-x)B'=(-(1-x)+3)B=(2+x)B$, and $B(0)=1$. Hence $A=B$.

A formula for a_n. Since $(1-x)^{-3}=\sum_{m\ge 0}\binom{m+2}{2}x^m$, $a_n=\sum_{j=0}^{n}\frac{(-1)^j}{j!}\binom{n-j+2}{2}$, $\binom{n-j+2}{2}=\frac{(n+2)(n+1)-(2n+3)j+j^2}{2}$. (For instance $a_2=6-3+\frac12=\frac72$.) Write $s_k(n)=\sum_{j=0}^{n}\frac{(-1)^j j^k}{j!}$ for $k=0,1,2$, so that $a_n=\frac12\left[(n+2)(n+1)\,s_0(n)-(2n+3)\,s_1(n)+s_2(n)\right]$.

The limits of the partial sums. The full series are $\sum_{j\ge 0}\frac{(-1)^j}{j!}=e^{-1}$, $\sum_{j\ge 0}\frac{(-1)^j j}{j!}=-\sum_{i\ge 0}\frac{(-1)^i}{i!}=-e^{-1}$, and $\sum_{j\ge 0}\frac{(-1)^j j^2}{j!}=\sum_{j\ge 0}\frac{(-1)^j j(j-1)}{j!}+\sum_{j\ge 0}\frac{(-1)^j j}{j!}=e^{-1}-e^{-1}=0$. The tails are tiny: for $k\le 2$ and $n\ge 2$, $\left|\sum_{j>n}\frac{(-1)^j j^k}{j!}\right|\le\sum_{j>n}\frac{j^2}{j!}\le\sum_{j>n}\frac{2}{(j-2)!}\le\frac{4}{(n-1)!}$. Multiplying by factors of size at most $(n+2)^2$ still gives a quantity tending to 0. Therefore $a_n=\frac12\left[(n+2)(n+1)e^{-1}+(2n+3)e^{-1}+0\right]+o(1)=\frac{n^2+5n+5}{2e}+o(1)$. So $\alpha=\frac{1}{2e}$, $\beta=\frac{5}{2e}$, $\gamma=\frac{5}{2e}$ work, and they are the only constants that work, since a nonzero quadratic polynomial in n cannot tend to 0. Hence $\alpha+\beta+\gamma=\dfrac{11}{2e}\approx 2.0233$.

Check. From the recurrence, $a_6=\frac{9403}{720}\approx 13.0597$, while $\frac{36+30+5}{2e}=\frac{71}{2e}\approx 13.0597$ already agrees to four decimal places, as the factorial-size error predicts. Running the recurrence to 80 digits and reading α,β,γ off exact second differences at $n=200$ gives $2e\alpha=1$, $2e\beta=5$, $2e\gamma=5$ to more than 50 digits.

A4 **Answer:** $66\ln 2+\frac{27}{2}\ln 3-\frac{75}{2}\ln 5$. **The idea.** The quotient $\dfrac{e^{-x}-e^{-2x}}{x}$ is itself an integral, $\int_1^2 e^{-tx}\,dt$. Cubing it turns the problem into a triple integral over the cube $[1,2]^3$; integrating in x first leaves $\iiint\frac{ds\,dt\,du}{s+t+u}$, which is a third finite difference of an antiderivative.

Work it out. For $x>0$, $\dfrac{e^{-x}-e^{-2x}}{x}=\int_1^2 e^{-tx}\,dt$, so $\left(\dfrac{e^{-x}-e^{-2x}}{x}\right)^3=\iiint_{[1,2]^3}e^{-(s+t+u)x}\,ds\,dt\,du$. Everything is positive, so by Tonelli's theorem we may integrate over $x\in(0,\infty)$ first, using $\int_0^\infty e^{-vx}dx=\frac{1}{v}$: $I=\iiint_{[1,2]^3}\dfrac{ds\,dt\,du}{s+t+u}$. Let G be any function on $(0,\infty)$ with $G^{(3)}(v)=\frac{1}{v}$. Integrating in u, then t, then s: $\int_1^2 G^{(3)}(s+t+u)\,du=G''(s+t+2)-G''(s+t+1)$, and repeating gives $I=G(6)-3G(5)+3G(4)-G(3)$.

Take $G(v)=\frac{v^2}{2}\ln v-\frac34 v^2$: then $G'(v)=v\ln v-v$, $G''(v)=\ln v$, $G^{(3)}(v)=\frac{1}{v}$. The third difference of the quadratic v^2 is $36-75+48-9=0$, so only the logarithmic part

survives: $I = \frac{1}{2}(36\ln 6 - 75\ln 5 + 48\ln 4 - 9\ln 3) = \frac{1}{2}(132\ln 2 + 27\ln 3 - 75\ln 5)$, that is $I = 66\ln 2 + \frac{27}{2}\ln 3 - \frac{75}{2}\ln 5 \approx 0.2250581$.

Check. Since $\frac{e^{-x}-e^{-2x}}{x} = \int_1^2 e^{-tx}dt$ lies between e^{-2x} and e^{-x}, the integral lies between $\int_0^\infty e^{-6x}dx = \frac{1}{6}$ and $\int_0^\infty e^{-3x}dx = \frac{1}{3}$, and 0.2251 does. Direct high-precision numerical integration of the original integral agrees with the closed form to 60 digits. The same method with the first power instead of the cube gives $\int_1^2 \frac{dt}{t} = \ln 2$, the classical Frullani value.

A5 Answer: 431**. The idea.** Take the two-dimensional discrete Fourier transform. Each family of parallel lines having equal sums says exactly that the transform vanishes on one explicit set of frequencies, so the dimension is n^2 minus the number of frequencies that some family kills. That count only depends on which multiples each residue has modulo n.

Work it out. *Fourier transform.* Let $\zeta = e^{2\pi i/n}$ and, for a complex array M, put $\widehat{M}(u,v) = \sum_{i,j} M(i,j)\,\zeta^{-(ui+vj)}$ for $u, v \in \mathbb{Z}/n$. The map $M \mapsto \widehat{M}$ is a linear bijection of $\mathbb{C}^{n\times n}$ (its inverse is $M(i,j) = \frac{1}{n^2}\sum_{u,v}\widehat{M}(u,v)\zeta^{ui+vj}$). A function f on $\mathbb{Z}/n$ is constant exactly when $\sum_c f(c)\zeta^{-wc} = 0$ for all $w \neq 0$.

Each family kills one set of frequencies. For fixed s let $f_s(c)$ be the sum of $L_{s,c}$. Every cell lies on exactly one line $L_{s,c}$, namely $c = j - si$, so $\sum_c f_s(c)\zeta^{-wc} = \sum_{i,j} M(i,j)\zeta^{-w(j-si)} = \widehat{M}(-sw, w)$. Likewise the column sums $g(c)$ give $\sum_c g(c)\zeta^{-wc} = \widehat{M}(w,0)$. Hence M is balanced if and only if $\widehat{M}$ vanishes on $Z = \{(-sw, w) : s \in \mathbb{Z}/n,\ w \neq 0\} \cup \{(w,0) : w \neq 0\}$. The complex solution space therefore has dimension $n^2 - |Z|$. The defining equations have real (integer) coefficients, so a real basis of the real solutions is also a complex basis of the complex solutions (row reduction gives the same rank over $\mathbb{R}$ and over $\mathbb{C}$), and the real dimension is also $n^2 - |Z|$.

Counting Z. For $v \neq 0$, the numbers $-sv$ ($s \in \mathbb{Z}/n$) run over the multiples of v modulo n, which are the multiples of $d = \gcd(v,n)$: there are n/d of them. The pairs $(w,0)$ are the $n-1$ nonzero points with second coordinate 0. So $|Z| = (n-1) + \sum_{v=1}^{n-1}\frac{n}{\gcd(v,n)}$ and the dimension is $n^2 - |Z| = 1 + \sum_{v=1}^{n-1}\left(n - \frac{n}{\gcd(v,n)}\right)$.

The case $n = 30$. Group $v \in \{1,\dots,29\}$ by $d = \gcd(v,30)$; exactly $\varphi(30/d)$ values of v have gcd equal to d, and each contributes $30 - 30/d$: $d = 1$: $8 \cdot 0$; $d = 2$: $8 \cdot 15 = 120$; $d = 3$: $4 \cdot 20 = 80$; $d = 5$: $2 \cdot 24 = 48$; $d = 6$: $4 \cdot 25 = 100$; $d = 10$: $2 \cdot 27 = 54$; $d = 15$: $1 \cdot 28 = 28$. These add to 430, so the dimension is $1 + 430 = 431$.

Check. The constant arrays are always balanced, which accounts for the 1. When n is prime every $v \neq 0$ has $\gcd(v,n) = 1$, so the formula gives dimension 1: only constant arrays are balanced, and nonconstant ones need a composite n. For $n = 4$ the formula gives $1 + (4-2) = 3$; a nonconstant example is $M(0,j) = (-1)^j$, $M(2,j) = -(-1)^j$ and $M(1,j) = M(3,j) = 0$: every column sums to 0, and every line meets rows 0 and 2 in cells $(0,c)$ and $(2,c+2s)$ whose entries $(-1)^c$ and $-(-1)^c$ cancel, so every line sums to 0 too. Elimination confirms the dimension 3. Gaussian elimination on the $31 \cdot 29 = 899$ equations in 900 unknowns, done modulo two large primes, gives dimension 431 for $n = 30$, and it also reproduces $1, 1, 3, 1, 11, 1$ for $n = 2, \dots, 7$.

A6 Answer: $\frac{5}{12}$**. The idea.** Use the generating function $G(s) = \mathbb{E}[s^N]$, which satisfies

a functional equation from the first generation. The quantity $\frac{1}{N^2}$ is an integral of s^{N-1} against $-\ln s$, so $\mathbb{E}[1/N^2]$ is an integral of G. We cannot solve for G in elementary terms, but we never need to: changing the variable of integration from s to $g = G(s)$ turns the integral into an elementary one.

Work it out. *The functional equation.* Condition on the founder's number of children K: the total progeny is $1 + N_1 + \cdots + N_K$ with N_i independent copies of N. Hence for $0 \le s \le 1$, $G(s) = s\sum_{k\ge 0} \frac{e^{-1}}{k!} G(s)^k = s\, e^{G(s)-1}$. The function $\varphi(g) = g\, e^{1-g}$ has $\varphi'(g) = (1-g)e^{1-g} > 0$ on $[0,1)$, so it is an increasing bijection of $[0,1]$ onto $[0,1]$. Since $0 \le G(s) \le 1$ and $\varphi(G(s)) = s$, we get $G = \varphi^{-1}$: G is continuous and increasing from $G(0) = 0$ to $G(1) = 1$ (the latter because $N < \infty$ almost surely).

An integral for $1/N^2$. For every positive integer n, $\int_0^1 (-\ln s)\, s^{n-1} ds = \frac{1}{n^2}$ (substitute $s = e^{-u}$ to get $\int_0^\infty u\, e^{-nu} du$). All terms are nonnegative, so by Tonelli, $\mathbb{E}\big[\frac{1}{N^2}\big] = \int_0^1 (-\ln s)\, \mathbb{E}[s^{N-1}]\, ds = \int_0^1 (-\ln s)\, \dfrac{G(s)}{s}\, ds$.

Change variables. Put $g = G(s)$, so $s = \varphi(g) = g\, e^{1-g}$, with g running from 0 to 1. Then $-\ln s = -\ln g + g - 1$ and $\frac{ds}{s} = \left(\frac{1}{g} - 1\right) dg$, so $\mathbb{E}\big[\frac{1}{N^2}\big] = \int_0^1 (-\ln g + g - 1)\, g \left(\frac{1}{g} - 1\right) dg = \int_0^1 (1-g)(-\ln g)\, dg - \int_0^1 (1-g)^2\, dg$. Now $\int_0^1 -\ln g\, dg = 1$ and $\int_0^1 g \ln g\, dg = -\frac{1}{4}$, so the first integral is $1 - \frac{1}{4} = \frac{3}{4}$; the second is $\frac{1}{3}$. Therefore $\mathbb{E}[1/N^2] = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$.

Check. The same substitution gives $\mathbb{E}[1/N] = \int_0^1 \frac{G(s)}{s} ds = \int_0^1 (1-g)\, dg = \frac{1}{2}$, a useful warm-up. Independently, the total progeny has the Borel law $P(N = n) = \frac{e^{-n} n^{n-1}}{n!}$, so $P(N = 1) = e^{-1} \approx 0.368$ and $P(N = 2) = e^{-2} \approx 0.135$; the series $\sum_n \frac{e^{-n} n^{n-3}}{n!}$ starts $0.368 + 0.034 + 0.008 + \cdots$ and, summed far out with its tail, approaches $0.41667 = \frac{5}{12}$. A 60-digit numerical integration of $\int_0^1 (-\ln s) G(s)\, \frac{ds}{s}$, with G computed from its equation, gives $\frac{5}{12}$ to every digit.

B1 Answer: $\frac{\sqrt{6}}{48}$. The idea. The four distances from P to the faces always add up to the height h of the tetrahedron, so they are $h\lambda_1, \ldots, h\lambda_4$ for barycentric coordinates $\lambda_i \ge 0$ with $\sum \lambda_i = 1$, and a uniform point of the tetrahedron has uniformly distributed barycentric coordinates. The nearest face is at distance $h \min \lambda_i$, and the tail of $\min \lambda_i$ is a volume ratio of two similar tetrahedra.

Work it out. Split the tetrahedron into four pyramids with apex P, one on each face. Each face has area A, so the volumes give $\frac{1}{3}A(d_1 + d_2 + d_3 + d_4) = \frac{1}{3}Ah$, i.e. $d_1 + d_2 + d_3 + d_4 = h$. Write $d_i = h\lambda_i$. The map $P \mapsto (\lambda_1, \lambda_2, \lambda_3)$ is affine and invertible, so it sends the uniform distribution on the tetrahedron to the uniform distribution on the simplex $\{\lambda_i \ge 0,\ \sum \lambda_i = 1\}$.

For $0 \le t \le \frac{1}{4}$, the event $\min \lambda_i > t$ is the set of points at distance more than th from every face. It is the tetrahedron cut out by the four planes parallel to the faces at distance th inside, which is a regular tetrahedron with barycentric coordinates $\lambda_i = t + (1-4t)\mu_i$, i.e. a copy scaled by the factor $1-4t$. Its volume ratio is $(1-4t)^3$, so $\Pr(\min \lambda_i > t) = (1-4t)^3$. Hence

$$E[\min \lambda_i] = \int_0^{1/4} \Pr(\min \lambda_i > t)\, dt = \int_0^{1/4} (1-4t)^3\, dt = \tfrac{1}{16}.$$

The height of a regular tetrahedron with edge 1: the centroid of a face is at distance $\frac{\sqrt{3}}{3}$ from each vertex of that face, so $h = \sqrt{1 - \frac{1}{3}} = \frac{\sqrt{6}}{3}$. Therefore $E[D] = \frac{h}{16} = \frac{\sqrt{6}}{48} \approx 0.0510$.

Check. The same argument in one dimension (a segment of length 1, nearest endpoint) gives $\Pr(\min > t) = 1 - 2t$ and $E = \int_0^{1/2}(1-2t)\,dt = \frac{1}{4}$, which is right. Here, a direct triple integral of λ_1 over the region where λ_1 is the smallest coordinate, times 4 and divided by the simplex volume $\frac{1}{6}$, also gives $\frac{1}{16}$. And $\frac{\sqrt{6}}{48}$ is a quarter of the inradius $\frac{\sqrt{6}}{12}$, a sensible size for the typical depth of a point.

B2 **Answer:** $\ln 2$. **The idea.** Count cycles by linearity of expectation: for every set $S \subseteq \{1,\dots,n\}$ compute the probability that S is exactly one cycle of σ. The sum over $|S| = k$ turns into a term that tends to $\frac{2^{-k}}{k}$, and the series $\sum \frac{2^{-k}}{k}$ is $\ln 2$.

Work it out. Fix $S \subseteq \{1,\dots,n\}$ with $|S| = k$. The permutations in which S forms one cycle are made by choosing a cyclic order on S, in $(k-1)!$ ways, and an arbitrary permutation of the other $2n-k$ elements, in $(2n-k)!$ ways. So the probability is $\frac{(k-1)!\,(2n-k)!}{(2n)!}$. Every cycle counted by C_n is exactly one such S, so

$$E[C_n] = \sum_{k=1}^{n} \tbinom{n}{k} \frac{(k-1)!\,(2n-k)!}{(2n)!} = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{n(n-1)\cdots(n-k+1)}{2n(2n-1)\cdots(2n-k+1)} = \sum_{k=1}^{n} \frac{1}{k} \prod_{i=0}^{k-1} \frac{n-i}{2n-i}.$$

Each factor satisfies $\frac{n-i}{2n-i} \le \frac{1}{2}$ (cross-multiply: $2n - 2i \le 2n - i$), so the k-th term is at most $\frac{2^{-k}}{k}$, and $\sum_k \frac{2^{-k}}{k}$ converges. For fixed k each factor tends to $\frac{1}{2}$, so the k-th term tends to $\frac{2^{-k}}{k}$. By dominated convergence for series (Tannery's theorem), $\lim E[C_n] = \sum_{k\ge1} \frac{(1/2)^k}{k} = -\ln\left(1-\frac{1}{2}\right) = \ln 2$.

Check. For $n = 1$: of the two permutations of $\{1,2\}$ only the identity has a cycle inside $\{1\}$, so $E[C_1] = \frac{1}{2}$, and the formula gives $\frac{1}{1}\cdot\frac{1}{2}$. Brute force over all permutations gives $E[C_2], E[C_3], E[C_4] = \frac{7}{12}, \frac{37}{60}, \frac{533}{840}$, matching the formula, and exact values at $n = 10, 100, 400$ increase to $0.6688, 0.6907, 0.6925$, approaching $\ln 2 \approx 0.6931$ from below, as the bound $\frac{n-i}{2n-i} \le \frac{1}{2}$ predicts.

B3 **Answer:** 4108728. **The idea.** Work in the field $\mathbb{F}_p$. Fix the first row $r = (a,b)$. The second rows with determinant 1 form a line $s_0 + t\,r$, and along this line the norm is a quadratic in t whose linear term vanishes. So each first row contributes $1+\chi(\cdot)$ solutions, where χ is the Legendre symbol, and the total is a character sum that can be evaluated exactly.

Work it out. Write $N(x,y) = x^2 + y^2$. Since $p \equiv 3 \pmod 4$, -1 is not a square, so $x^2 + y^2 = 0$ forces $x = y = 0$. The first row cannot be $(0,0)$ (the determinant would be 0), so $N = N(a,b) \neq 0$.

The second rows (c,d) with $ad - bc = 1$: the vector $s_0 = \frac{1}{N}(-b,a)$ works, since $a\cdot\frac{a}{N} - b\cdot\frac{-b}{N} = \frac{a^2+b^2}{N} = 1$. The condition $ad - bc = 1$ is one nontrivial linear equation, so the solutions are exactly $s_0 + t(a,b)$, $t \in \mathbb{F}_p$. Because $(-b,a)\cdot(a,b) = 0$, the norm is $N(s_0 + tr) = t^2 N + \frac{N}{N^2} = t^2 N + \frac{1}{N}$. It equals N exactly when $t^2 = \frac{N^2-1}{N^2}$, which has $1 + \chi(N^2-1)$ solutions t (with $\chi(0) = 0$).

Next, each $N \neq 0$ is $x^2 + y^2$ in exactly $p - \chi(-1) = p + 1$ ways: the number of solutions is $\sum_{u+v=N}(1+\chi(u))(1+\chi(v)) = p + \sum_u \chi(u)\chi(N-u)$, and for $u \neq 0$, $\chi(u)\chi(N-u) = \chi(Nu^{-1}-1)$, where $Nu^{-1}-1$ runs over all values except -1; so the sum is $0 - \chi(-1) = 1$. Therefore the count is

$$(p+1)\sum_{N\neq0}\big(1+\chi(N^2-1)\big) = (p+1)\Big[(p-1) + \sum_{N\in\mathbb{F}_p}\chi(N^2-1) - \chi(-1)\Big].$$

Finally, $\sum_N \chi(N^2-1) = \#\{(N,y) : y^2 = N^2-1\} - p$, and $N^2 - y^2 = 1$ means $(N-y)(N+y) = 1$: choose $N - y = u \neq 0$, then $N + y = u^{-1}$, and since p is odd this determines (N, y); so there are $p-1$ points and the sum is -1. With $\chi(-1) = -1$ the bracket is $(p-1) - 1 + 1 = p - 1$, and the count is $(p+1)(p-1) = p^2 - 1 = 2027^2 - 1 = 4108728$.

Check. An exhaustive count over all (a, b, c) with d solved from the determinant gives $p^2 - 1$ for $p = 3, 7, 11, 19$ $(8, 48, 120, 360)$. For $p = 2027$, a direct tally of how often each N occurs as $a^2 + b^2$ and how many t solve the quadratic gives 4108728. The hypothesis $p \equiv 3 \pmod 4$ matters: for $p = 5$ and $p = 13$ the counts are 16 and 144, not 24 and 168, because -1 is then a square, rows of norm 0 exist and $a^2 + b^2 = N$ has $p - 1$ solutions.

B4 Answer: 181. **The idea.** Only the characteristic polynomial $x^2 - tx + \delta$ (trace t, determinant δ, both integers) matters, plus one Jordan-block exception. Bounded powers force both eigenvalues into the closed unit disc, so $|\delta| \leq 1$, and that leaves a short list of pairs (δ, t). Then count matrices with given trace and determinant by counting factorisations $bc = m$.

Work it out. *Necessity.* If $Av = \lambda v$ with $v \neq 0$ and $|\lambda| > 1$, then $A^n v = \lambda^n v$ is unbounded, so A^n is too. Hence $|\lambda_1|, |\lambda_2| \leq 1$ and $|\delta| = |\lambda_1 \lambda_2| \leq 1$, so $\delta \in \{-1, 0, 1\}$.

$\delta = 0$: the eigenvalues are 0 and t, so $|t| \leq 1$. By Cayley–Hamilton $A^2 = tA$, so $A^n = t^{n-1}A$ is bounded for $t \in \{-1, 0, 1\}$. All three traces work.

$\delta = 1$: $\lambda_1 \lambda_2 = 1$ with both moduli at most 1 forces both moduli to be 1, and then $|t| = |\lambda_1 + \lambda_2| \leq 2$. If $|t| \leq 1$, $A^2 = tA - I$ gives $A^4 = I$ $(t = 0)$, $A^3 = -I$ $(t = 1)$ or $A^3 = I$ $(t = -1)$, so A has finite order and is bounded. If $t = \pm 2$, the only eigenvalue is ± 1 and $(A \mp I)^2 = 0$. Write $A = \pm(I + N)$ with $N^2 = 0$; then $A^n = (\pm 1)^n (I + nN)$, bounded only if $N = 0$. So only $A = I$ and $A = -I$.

$\delta = -1$: the roots of $x^2 - tx - 1$ are real (discriminant $t^2 + 4 > 0$) with product -1, so both have modulus 1 only if they are 1 and -1, i.e. $t = 0$; then $A^2 = I$ and A is bounded. For $t \neq 0$ one root has modulus greater than 1.

Counting. Let $f(m)$ be the number of pairs (b, c) with entries in $\{-3, \ldots, 3\}$ and $bc = m$: $f(0) = 13$, $f(\pm 1) = 2$, $f(\pm 2) = f(\pm 3) = 4$, $f(\pm 4) = 2$, $f(\pm 6) = 4$, $f(\pm 9) = 2$, and $f(\pm 5) = f(\pm 7) = f(\pm 8) = f(\pm 10) = 0$. With $d = t - a$ (and d in range), $\delta = a(t-a) - bc$, so $bc = a(t-a) - \delta$.

$(\delta, t) = (0, 0)$: $bc = -a^2$; $a = 0, \pm 1, \pm 2, \pm 3$ give $13 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 = 25$.

$(0, 1)$: $a \in \{-2, \ldots, 3\}$ and $bc = a(1-a)$, i.e. $-6, -2, 0, 0, -2, -6$: $4+4+13+13+4+4 = 42$. By $A \mapsto -A$, $(0, -1)$ also gives 42.

$(1, 0)$: $bc = -a^2 - 1$, i.e. -1 $(a = 0)$, -2 $(a = \pm 1)$, $-5, -10$: $2 + 4 + 4 = 10$.

$(1, 1)$: $bc = a(1-a) - 1$ for $a = -2, \ldots, 3$: $-7, -3, -1, -1, -3, -7$: $0+4+2+2+4+0 = 12$; likewise $(1, -1)$ gives 12.

$(1, \pm 2)$: $\pm I$, 2 matrices. $(-1, 0)$: $bc = 1 - a^2$, i.e. 1 $(a = 0)$, 0 $(a = \pm 1)$, -3 $(a = \pm 2)$, -8 $(a = \pm 3)$: $2 + 26 + 8 + 0 = 36$.

Total: $25 + 42 + 42 + 10 + 12 + 12 + 2 + 36 = 181$.

Check. An integer matrix has bounded powers exactly when it has only finitely many distinct powers, i.e. when some power repeats. Computing powers of all $7^4 = 2401$

matrices until a repeat appears (within 40 steps) finds exactly 181. The largest possible period is 6 (for $t = \pm 1, \delta = 1$ up to sign), consistent with the orders found above.

B5 **Answer:** $1 + \frac{\pi}{2}$. **The idea.** Change variables to $u = P + Q$ and $v = P - Q$. The small ball has centre $\frac{u}{2}$ and radius $\frac{\|v\|}{2}$, so it lies in B_n exactly when $\|u\| + \|v\| \le 2$, and by the triangle inequality this condition already forces $P, Q \in B_n$. So p_n is the volume of the region $\|u\| + \|v\| \le 2$ in $\mathbb{R}^{2n}$, up to known constants, and a Beta integral gives $p_n = 2^n / \binom{2n}{n}$. The series $\sum 2^n / \binom{2n}{n}$ is then summed with the Beta integral again.

Work it out. *When is the small ball inside?* A closed ball with centre m and radius $\rho \ge 0$ lies in B_n if and only if $\|m\| + \rho \le 1$: if so, any x with $\|x - m\| \le \rho$ has $\|x\| \le \|m\| + \rho \le 1$; conversely the point $m + \rho e$, where e is $m/\|m\|$ (or any unit vector if $m = 0$), lies in the ball and has norm $\|m\| + \rho$. The ball on the diameter PQ has $m = \frac{P+Q}{2}$ and $\rho = \frac{\|P-Q\|}{2}$, so the event is $\|u\| + \|v\| \le 2$ with $u = P + Q$, $v = P - Q$.

Change of variables. The linear map $(P, Q) \mapsto (u, v)$ of $\mathbb{R}^{2n}$ has determinant of absolute value 2^n (it is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ acting in each of the n coordinates). If $\|u\| + \|v\| \le 2$ then $\|P\| = \frac{\|u+v\|}{2} \le 1$ and $\|Q\| = \frac{\|u-v\|}{2} \le 1$, so the whole region $R = \{\|u\| + \|v\| \le 2\}$ corresponds to pairs inside $B_n \times B_n$. Writing ω_n for the volume of B_n, $p_n = \dfrac{\mathrm{vol}(R)/2^n}{\omega_n^2}$.

The volume of R. For fixed u with $\|u\| = r < 2$, the admissible v fill a ball of radius $2 - r$, of volume $\omega_n (2 - r)^n$. The sphere of radius r has area $n\omega_n r^{n-1}$, so $\mathrm{vol}(R) = \int_0^2 n\omega_n r^{n-1} \cdot \omega_n (2 - r)^n \, dr = n\omega_n^2 \, 2^{2n} \int_0^1 t^{n-1}(1 - t)^n \, dt = n\omega_n^2 \, 4^n \, \frac{(n-1)!\, n!}{(2n)!}$, using $r = 2t$ and the Beta integral $\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{(a-1)!(b-1)!}{(a+b-1)!}$. Hence $p_n = \dfrac{4^n (n!)^2}{2^n (2n)!} = \dfrac{2^n}{\binom{2n}{n}}$.

The sum. By the Beta integral again, $\frac{1}{\binom{2n}{n}} = \frac{(n!)^2}{(2n)!} = (2n + 1) \int_0^1 t^n (1 - t)^n \, dt$. Put $y = 2t(1 - t) \in [0, \frac{1}{2}]$. All terms are nonnegative, so we may swap sum and integral (Tonelli), and $\sum_{n \ge 0} (2n + 1) y^n = \frac{1+y}{(1-y)^2}$ gives $\sum_{n=1}^{\infty} p_n = \int_0^1 \left(\frac{1+y}{(1-y)^2} - 1 \right) dt$. Substitute $t = \frac{1+s}{2}$: then $1 - y = t^2 + (1 - t)^2 = \frac{1+s^2}{2}$ and $1 + y = \frac{3-s^2}{2}$, so $\frac{1+y}{(1-y)^2} = \frac{2(3-s^2)}{(1+s^2)^2}$ and $dt = \frac{ds}{2}$: $\int_0^1 \frac{1+y}{(1-y)^2} \, dt = \int_{-1}^1 \frac{3-s^2}{(1+s^2)^2} \, ds = 2 \int_0^1 \left(\frac{4}{(1+s^2)^2} - \frac{1}{1+s^2} \right) ds$. With $\int_0^1 \frac{ds}{1+s^2} = \frac{\pi}{4}$ and $\int_0^1 \frac{ds}{(1+s^2)^2} = \left[\frac{s}{2(1+s^2)} + \frac{1}{2} \arctan s \right]_0^1 = \frac{1}{4} + \frac{\pi}{8}$, this is $2 \left(1 + \frac{\pi}{2} - \frac{\pi}{4} \right) = 2 + \frac{\pi}{2}$. Subtracting $\int_0^1 1 \, dt = 1$, $\sum_{n=1}^{\infty} p_n = 1 + \dfrac{\pi}{2} \approx 2.5708$.

Check. For $n = 1$ the "ball" on PQ is the segment PQ itself, which always lies in $[-1, 1]$, and indeed $p_1 = \frac{2}{1} = 1$. Next $p_2 = \frac{4}{6} = \frac{2}{3}$, $p_3 = \frac{8}{20} = \frac{2}{5}$, $p_4 = \frac{16}{70} = \frac{8}{35}$; simulations with 200,000 random pairs in the disc and in the ball give 0.666 and 0.399. The partial sums $1,\ 1.667,\ 2.067,\ 2.295,\ 2.422, \ldots$ increase to 2.5708, and summing $2^n / \binom{2n}{n}$ directly to 50 digits (the terms shrink roughly like $2^{-n}\sqrt{\pi n}$) agrees with $1 + \frac{\pi}{2}$.

B6 **Answer:** $\frac{3}{2} \ln 2 - \frac{3}{4}$. **The idea.** Deleting an edge splits T into two identical halves, and a vertex only ever consults neighbours with *smaller* labels. So the chance that a vertex with label t gets into S "on its own side" satisfies a self-similar equation, which becomes the differential equation $G' = (1 - G)^2$ with solution $G(t) = \frac{t}{1+t}$. Then condition on the

label t of r and sort its neighbours into *early* ones (label below t) and *late* ones (label above t): the early ones decide whether r is in S, and when r is out, the late ones may join as well.

Work it out. *Branches.* For an edge uw, the branch at w (away from u) is the component of w after deleting the edge uw; in it w has two neighbours, and each of them heads its own branch, again a copy of the same infinite binary tree. Say w is *in on its branch* if w belongs to the set defined by the same greedy rule applied inside the branch alone. The rule, started at w, only looks along paths of decreasing labels. If the label of u is larger than that of w, such a path from w can never use u, so then $w \in S$ exactly when w is in on its branch.

The branch equation. Let $G(t)$ be the probability that a vertex w has label less than t and is in on its branch, and let $g(s)$ be the probability that w is in on its branch given that its label is s, so $G(t) = \int_0^t g(s)\,ds$. Given label s, w is in on its branch exactly when neither of its two children is (a child with label below s that is in on its own branch). The two child branches are independent copies of the whole situation, so $g(s) = (1 - G(s))^2$, i.e. $G' = (1-G)^2$ with $G(0) = 0$. The right-hand side is Lipschitz on $[0,1]$, so the solution is unique: $\frac{d}{dt}\frac{1}{1-G} = 1$ gives $\frac{1}{1-G} = 1 + t$, so $G(t) = \frac{t}{1+t}$ and $G(1) = \frac{1}{2}$.

Classifying the neighbours. Condition on the label t of r. For each neighbour n_j ($j = 1,2,3$) let E_j be the event that n_j has label less than t and is in on its branch away from r (probability $G(t)$), and L_j the event that it has label greater than t and is in on its branch (probability $G(1) - G(t) = \frac{1}{2} - G(t)$); with probability $\frac{1}{2}$ neither happens. The three branches at r are disjoint, so these are independent across j. By the branch fact, an early neighbour n_j is in S exactly on E_j, so $r \in S$ exactly when no E_j occurs; and then no neighbour of r is in S, since a neighbour with a larger label is blocked by r and one with a smaller label is not in S. If some E_j occurs, $r \notin S$. A late neighbour n_j then sees the smaller-labelled neighbours r (not in S) and those inside its own branch, so $n_j \in S$ exactly on L_j. Hence, whenever $r \notin S$, the number K of neighbours of r in S is the number of j for which E_j or L_j occurs.

Exactly two. Write $a = G(t)$ and $b = \frac{1}{2} - G(t)$, so $a + b = \frac{1}{2}$. $K = 2$ means that two neighbours have E or L, at least one of them E (otherwise $r \in S$ and $K = 0$), and the third has neither:

$$\Pr(K = 2 \mid t) = 3 \cdot \tfrac{1}{2}\left[(a+b)^2 - b^2\right] = \tfrac{3}{2}\,a(a+2b) = \tfrac{3}{2}\,G(t)\big(1 - G(t)\big) = \tfrac{3}{2} \cdot \tfrac{t}{(1+t)^2}.$$

Averaging over the uniform label t: $\int_0^1 \frac{t\,dt}{(1+t)^2} = \int_0^1 \left(\frac{1}{1+t} - \frac{1}{(1+t)^2}\right) dt = \ln 2 - \frac{1}{2}$, so the probability is $\frac{3}{2}\ln 2 - \frac{3}{4} \approx 0.2897$.

Check. The same bookkeeping gives the whole distribution of K: $\Pr(K=0) = \int_0^1 (1-G)^3\,dt = \frac{3}{8}$, $\Pr(K=1) = \frac{3}{4}(1 - \ln 2)$, $\Pr(K=2) = \frac{3}{2}\ln 2 - \frac{3}{4}$, $\Pr(K=3) = \frac{5}{8} - \frac{3}{4}\ln 2$. These add up to 1, and $E[K] = \frac{9}{8} = 3 \cdot \frac{3}{8}$, as symmetry demands (each neighbour of r is in S with the same probability $\frac{3}{8}$ as r). On the finite tree where each neighbour of r has a single further leaf, the method gives $G = t - \frac{t^2}{2}$ and $\Pr(K=2) = \frac{3}{10}$, which matches a direct count over all 7! label orders. Solving $G' = (1-G)^2$ numerically to 50 digits reproduces $\frac{3}{2}\ln 2 - \frac{3}{4}$, and a simulation on the infinite tree gives about 0.29.