MathCastle

Putnam

Session A and Session B

12 problems • 6 hours

Name _____

INSTRUCTIONS

1. Two sessions, A and B, of 6 problems each. Each session lasts 3 hours.
2. Each problem is worth 10 points. Full credit needs a complete proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Session A is handed in before Session B is opened.
5. Two sittings of 6 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

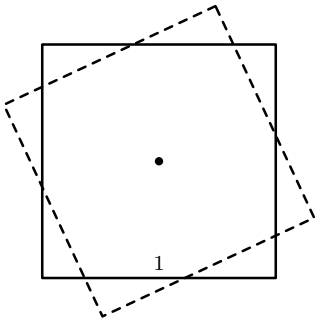
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

A1

How many ordered pairs (P, Q) of polynomials with integer coefficients satisfy $P(x)^2 + Q(x)^2 = (x^2+1)^2(x^2+4)$ identically in x? (Pairs are ordered: two pairs count as different whenever they differ at all, for example by swapping P and Q or by changing the sign of one of them.)

A2

A unit square is rotated about its centre through an angle chosen uniformly at random from $[0, 2\pi)$. Find the expected area of the region common to the rotated square and the original square. Give your answer in exact form.



A3

Let φ denote Euler's totient function. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n + 1}$. Give your answer as a fraction in lowest terms.

A4

Let A be a 5×5 matrix with real entries such that $A^2 = A^T$, where A^T is the transpose of A. As A ranges over all such matrices, the trace of A takes only finitely many distinct values. Find the sum of all of these values.

A5

For $n \geq 2$, let σ be a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random. Let L be the length of the cycle of σ that contains 1, and let M be the length of the cycle of σ that contains 2 (so $L = M$ when 1 and 2 lie in the same cycle). Find $\displaystyle\lim_{n\to\infty} \frac{\mathbb{E}[LM]}{n^2}$.

A6

For each positive integer n, let D_n be the determinant of the $n \times n$ matrix whose entry in row i and column j is $\dfrac{1}{i^2 + j^2}$ (for $1 \leq i, j \leq n$). It can be shown that $D_n > 0$ for all n. Find $\displaystyle\lim_{n\to\infty} \frac{\ln D_n}{n^2}$. Give your answer in exact form.

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B1

Evaluate the double series $S = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}$.
Give S in closed form, in terms of $\ln 2$.

B2

Let $\mathbb{F}_5$ be the field with 5 elements. A matrix A is chosen uniformly at random from $GL_4(\mathbb{F}_5)$, the set of all invertible 4×4 matrices with entries in $\mathbb{F}_5$. Let $N(A)$ be the number of vectors $v \in \mathbb{F}_5^4$ (the zero vector included) with $Av = v$.
Find the expected value $\mathbb{E}\big[N(A)^3\big]$.

B3

Consider the 5×5 matrices A every one of whose 25 entries is 0 or 1. How many of these 2^{25} matrices satisfy $A^3 = 0$, where the product is computed with ordinary integer arithmetic (not modulo 2)?

B4

The cube $[-1,1]^3$ is inscribed in the ball $B = \{(x,y,z) : x^2+y^2+z^2 \le 3\}$, so all eight vertices of the cube lie on the boundary sphere. The planes $x = \pm 1$ and $y = \pm 1$ of four of the cube's faces cut B into nine pieces. Find the volume of the piece that contains the cube, namely $\{(x,y,z) \in B : |x| \le 1,\ |y| \le 1\}$.
Give the exact value in terms of π and radicals.

B5

A rotation $A \in SO(3)$ (a real 3×3 orthogonal matrix of determinant 1) is chosen at random according to the Haar measure, the unique probability measure on $SO(3)$ invariant under multiplication on the left and on the right by rotations. Equivalently, A is the rotation $x \mapsto qxq^{-1}$ of the pure quaternions, where q is a uniformly random unit quaternion.
Find the expected value $\mathbb{E}\big[(\operatorname{tr} A)^6\big]$.

B6

Let R be the set of points $(a,b,c,d) \in \mathbb{R}^4$ such that all four roots of $x^4 + ax^3 + bx^2 + cx + d$ are real and lie in the interval $[-1,1]$ (roots counted with multiplicity). Find the four-dimensional volume of R, as a fraction in lowest terms.

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Answer Key — Putnam Practice Contest 2

A1 24

A2 $\frac{4\ln 2}{\pi}$

A3 $\frac{10}{9}$

A4 12

A5 $\frac{7}{24}$

A6 $\ln 2 - \frac{\pi}{2}$

B1 $2(1-\ln 2)^2$

B2 64

B3 8161

B4 $\frac{16\pi - 4\sqrt{3}\,\pi + 8}{3}$

B5 15

B6 $\frac{64}{1575}$

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Solutions — Putnam Practice Contest 2

A1 **Answer:** 24. **The idea.** Over the Gaussian integers a sum of two squares factors: $P^2+Q^2=(P+iQ)(P-iQ)$. The ring $\mathbb{Z}[i][x]$ has unique factorisation, so $P+iQ$ must be built from the Gaussian linear factors of the right-hand side, and conjugation pins down how.

Work it out. Over $\mathbb{Z}[i]$, $f(x)=(x^2+1)^2(x^2+4)=(x+i)^2(x-i)^2(x+2i)(x-2i)$, a product of monic (hence primitive and irreducible) linear factors, and $\mathbb{Z}[i]$ is a UFD, so by Gauss's lemma $\mathbb{Z}[i][x]$ is a UFD. Put $G=P+iQ\in\mathbb{Z}[i][x]$. Its complex conjugate (conjugating coefficients) is $\overline{G}=P-iQ$, and $G\overline{G}=f$. Since f is monic, the content of G is a unit, so $G=u\,(x+i)^a(x-i)^b(x+2i)^c(x-2i)^d$ with $u\in\{\pm1,\pm i\}$ and $a+b\le 2$, $c+d\le 1$ matching factors of f. Then $\overline{G}=\bar{u}\,(x-i)^a(x+i)^b(x-2i)^c(x+2i)^d$ and $G\overline{G}=(x+i)^{a+b}(x-i)^{a+b}(x+2i)^{c+d}(x-2i)^{c+d}$. Comparing with f forces $a+b=2$ and $c+d=1$. So $a\in\{0,1,2\}$, $c\in\{0,1\}$ and u has 4 choices.

Conversely each such G has Gaussian-integer coefficients, so $P=\operatorname{Re}G$ and $Q=\operatorname{Im}G$ (taken coefficientwise) have integer coefficients, and $P^2+Q^2=G\overline{G}=f$. Different triples (u,a,c) give different G by unique factorisation, and G determines (P,Q). The count is $4\cdot 3\cdot 2=24$.

Check. Two sample pairs: $a=1,c=1,u=1$ gives $G=(x^2+1)(x+2i)$, i.e. $(P,Q)=(x^3+x,\,2x^2+2)$, and clearly $P^2+Q^2=(x^2+1)^2(x^2+4)$. With $a=2,c=1,u=1$, $G=(x+i)^2(x+2i)=x^3-5x+i(4x^2-2)$, and $(x^3-5x)^2+(4x^2-2)^2=x^6+6x^4+9x^2+4=(x^2+1)^2(x^2+4)$. Independently, a computer search confirms 24: $\deg P,\deg Q\le 3$ (leading terms cannot cancel), so P,Q are determined by their values at $0,1,2,3$, which must be lattice points on the circles of radii $\sqrt{4},\sqrt{20},\sqrt{200},\sqrt{1300}$; interpolating every combination and testing the identity exactly leaves exactly 24 pairs.

A2 **Answer:** $\frac{4\ln 2}{\pi}$. **The idea.** The overlap is the original square with four congruent right-triangular corners cut off. Along one side of the original square, the two legs and the hypotenuse of the cut-off triangles fill the whole side, which gives the hypotenuse, hence the area, in closed form; the area turns out to be the surprisingly simple $2/(1+\sin\theta+\cos\theta)$.

Work it out. Rotating by $\pi/2$ maps the square to itself, so the overlap area $A(\theta)$ has period $\pi/2$, and the expected value is $\frac{2}{\pi}\int_0^{\pi/2}A(\theta)\,d\theta$. Also $A(0)=A(\pi/2)=1$, so take $0<\theta<\pi/2$. Both squares have inradius $\frac{1}{2}$ about the common centre, and by symmetry (the configuration is invariant under the rotation by $\pi/2$) each side of the rotated square cuts off one corner of the original square, and vice versa, so the overlap is an octagon: the original square minus four congruent right triangles. In one such triangle, the hypotenuse c lies along a side of the rotated square and the legs lie along two sides of the original square; its acute angles are θ and $\frac{\pi}{2}-\theta$, so its legs are $c\cos\theta$ and $c\sin\theta$.

Now walk along one side of the original square. From corner to corner it meets: a leg of one cut triangle, then a side of the octagon, then a leg of the next cut triangle. The octagon side lying on the original square equals the hypotenuse c of a triangle cut from the *rotated* square, and by the symmetry swapping the two squares (reflection in a line through the centre at angle $\theta/2$ to the sides) that triangle is congruent to ours.

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Adjacent triangles meet this side with their two different legs, so $c\cos\theta + c + c\sin\theta = 1$, i.e. $c = \dfrac{1}{1+\sin\theta+\cos\theta}$. Therefore, writing $s = \sin\theta + \cos\theta$ and using $\sin 2\theta = s^2 - 1$, $A(\theta) = 1 - 4\cdot\frac{1}{2}c^2\sin\theta\cos\theta = 1 - \frac{s^2-1}{(1+s)^2} = 1 - \frac{s-1}{s+1} = \frac{2}{1+\sin\theta+\cos\theta}$.

With $t = \tan(\theta/2)$ we have $1+\sin\theta+\cos\theta = \frac{(1+t^2)+2t+(1-t^2)}{1+t^2} = \frac{2(1+t)}{1+t^2}$ and $d\theta = \frac{2\,dt}{1+t^2}$, so $\int_0^{\pi/2}\frac{d\theta}{1+\sin\theta+\cos\theta} = \int_0^1 \frac{dt}{1+t} = \ln 2$. The expected area is $\frac{2}{\pi}\cdot 2\ln 2 = \frac{4\ln 2}{\pi} \approx 0.8825$.

Check. At $\theta = 0$ the formula gives $2/2 = 1$, correct. At $\theta = \pi/4$ it gives $\frac{2}{1+\sqrt{2}} = 2(\sqrt{2}-1)$, the area of a regular octagon with inradius $\frac{1}{2}$, namely $8\cdot(\frac{1}{2})^2\tan 22.5^\circ = 2(\sqrt{2}-1)$. Averaging a quantity that lies between $2(\sqrt{2}-1) \approx 0.828$ and 1 should give something in between, and 0.8825 is. Numerically clipping the rotated square against the original and integrating gives the same value to 58 digits.

A3 **Answer:** $\frac{10}{9}$. **The idea.** Generating functions of the Lambert type $\sum a_n\frac{x^n}{1-x^n}$ turn divisor sums into ordinary power series. For φ, the divisor sum $\sum_{d\mid m}\varphi(d) = m$ makes the Lambert series collapse to $\frac{x}{(1-x)^2}$. Our summand is $\frac{x^n}{1+x^n}$ at $x = \frac{1}{2}$ (a plus sign, not a minus), and a one-line identity converts it into two Lambert series of the good kind.

Work it out. For $0 < x < 1$, expanding each geometric series and grouping by $m = nk$ (all terms are nonnegative, so rearranging is allowed): $L(x) = \sum_{n\ge1}\varphi(n)\frac{x^n}{1-x^n} = \sum_{n\ge1}\sum_{k\ge1}\varphi(n)x^{nk} = \sum_{m\ge1}x^m\sum_{d\mid m}\varphi(d) = \sum_{m\ge1}mx^m = \frac{x}{(1-x)^2}$. Here we used $\sum_{d\mid m}\varphi(d) = m$: classify the fractions $\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m}$ by their reduced denominator d; exactly $\varphi(d)$ of them have denominator d.

Next, $\frac{y}{1+y} = \frac{y}{1-y} - \frac{2y^2}{1-y^2}$, since the right side is $\frac{y(1+y)-2y^2}{1-y^2} = \frac{y(1-y)}{(1-y)(1+y)}$. With $y = x^n$ and summing (both series converge absolutely), $\sum_{n\ge1}\varphi(n)\frac{x^n}{1+x^n} = L(x) - 2L(x^2) = \frac{x}{(1-x)^2} - \frac{2x^2}{(1-x^2)^2}$. Since $\frac{1}{2^n+1} = \frac{x^n}{1+x^n}$ at $x = \frac{1}{2}$, the sum is $\frac{1/2}{1/4} - \frac{2\cdot\frac{1}{4}}{9/16} = 2 - \frac{8}{9} = \frac{10}{9}$.

Check. The first terms are $\frac{1}{3}+\frac{1}{5}+\frac{2}{9}+\frac{2}{17}+\frac{4}{33}+\frac{2}{65}+\frac{6}{129}+\cdots \approx 0.333+0.200+0.222+0.118+0.121+0.031+0.047 = 1.072$, and the tail is tiny (the terms decay like $n/2^n$), consistent with $10/9 \approx 1.111$; a 260-term partial sum agrees with 10/9 to 60 digits.

A4 **Answer:** 12. **The idea.** The condition says A^T is a polynomial in A, so A commutes with its transpose: A is normal. Normal real matrices are orthogonally block-diagonal with 1×1 and rotation-type 2×2 blocks, and the condition cuts the possible eigenvalues down to four.

Work it out. *A is normal:* $AA^T = A\cdot A^2 = A^3 = A^2\cdot A = A^TA$.

Eigenvalues. For a normal matrix, $Av = \lambda v$ implies $A^Tv = \bar\lambda v$ (because $\|(A^T-\bar\lambda I)v\| = \|(A-\lambda I)v\|$ when A is normal; here $A^T = A^*$ as A is real). But $A^Tv = A^2v = \lambda^2 v$, so $\bar\lambda = \lambda^2$. Taking absolute values, $|\lambda| = |\lambda|^2$, so $\lambda = 0$ or $|\lambda| = 1$; in the latter case $\lambda^2 = \bar\lambda = 1/\lambda$, so $\lambda^3 = 1$. Hence $\lambda \in \{0, 1, \omega, \bar\omega\}$ with $\omega = e^{2\pi i/3}$, and $\omega, \bar\omega$ occur with equal multiplicity since A is real.

Structure. A real normal matrix is orthogonally similar to a block-diagonal matrix with blocks $[\lambda]$ for real eigenvalues and, for each pair $p \pm qi$, a 2×2 block with rows $(p, -q)$ and (q, p). So A has a eigenvalues 1, b pairs $\omega, \bar\omega$, and $5-a-2b$ zeros, with $\operatorname{tr}A = a + b(\omega+\bar\omega) = a - b$ and $a + 2b \le 5$.

All of these occur. The rotation by 120°, R, satisfies $R^2 = R^{-1} = R^T$, and [1], [0] trivially satisfy the condition, so any block-diagonal matrix built from a blocks [1], b blocks R and zeros satisfies $A^2 = A^T$ (squaring and transposing both act blockwise).

Values. $b = 0$: $a - b \in \{0, \dots, 5\}$. $b = 1$ ($a \le 3$): $\{-1, 0, 1, 2\}$. $b = 2$ ($a \le 1$): $\{-2, -1\}$. The union is $\{-2, -1, 0, 1, 2, 3, 4, 5\}$, whose sum is 12.

Check. The extremes make sense: $\operatorname{tr} A = 5$ for $A = I$, and $\operatorname{tr} A = -2$ for $A = R \oplus R \oplus [0]$; a trace of -3 would need three rotation blocks, which do not fit in dimension 5. Solving $\bar{\lambda} = \lambda^2$ with $\lambda = u + vi$ directly ($u^2 - v^2 = u$, $2uv = -v$) gives exactly $0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

A5 Answer: $\frac{7}{24}$. The idea. The cycle through a fixed point of a random permutation has uniformly distributed length, and once that cycle is removed, what remains is again a uniformly random permutation of the leftover points. So first reveal the cycle of 1, then ask where 2 is.

Work it out. *L is uniform.* Trace the cycle of 1: $1 \to \sigma(1) \to \sigma^2(1) \to \cdots$. The number of permutations in which this cycle has length exactly k is $(n-1)(n-2)\cdots(n-k+1) \cdot (n-k)! = (n-1)!$ (choose the $k-1$ successive images, then permute the rest arbitrarily). So $P(L = k) = \frac{1}{n}$ for $1 \le k \le n$.

Given $L = k$. By the same count, conditioned on the cycle of 1 being a particular k-cycle, the rest of σ is a uniform permutation of the remaining $n-k$ points; and by symmetry the $k-1$ other points of the cycle of 1 form a uniformly random $(k-1)$-subset of $\{2, \dots, n\}$. Hence 2 lies in the cycle of 1 with probability $\frac{k-1}{n-1}$, in which case $M = k$. Otherwise 2 is one of the $n-k$ remaining points, and M is the cycle length of a fixed point in a uniform permutation of $n-k$ points: uniform on $\{1, \dots, n-k\}$, with mean $\frac{n-k+1}{2}$. Therefore $\mathbb{E}[LM] = \frac{1}{n} \sum_{k=1}^{n} \left[\frac{k-1}{n-1} k^2 + \frac{n-k}{n-1} \, k \, \frac{n-k+1}{2} \right]$.

The limit. Divide by n^2 and put $x = k/n$: the bracket is $n^2 \left[x^3 + \frac{1}{2} x (1-x)^2 \right] + O(n)$ uniformly in k, so $\frac{\mathbb{E}[LM]}{n^2}$ is a Riemann sum converging to $\int_0^1 \left(x^3 + \frac{1}{2} x (1-x)^2 \right) dx = \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{12} = \frac{7}{24}$.

Check. For $n = 5$ the bracketed terms for $k = 1, \dots, 5$ are $\frac{5}{2}$, 4, $\frac{27}{4}$, 13, 25, with sum $\frac{205}{4}$, so $\mathbb{E}[LM] = \frac{41}{4}$, which matches a brute-force average over all 120 permutations. Independently, writing $LM = \sum_{i,j} [i \sim 1][j \sim 2]$ and using the fact that the cycles of a random permutation split any k fixed points into blocks B with probability $\prod (|B|-1)!/k!$ gives $\mathbb{E}[LM]$ as a rational function of n whose limit over n^2 is again $\frac{7}{24}$. (Heuristically: with probability about $\frac{1}{2}$ in the limit the two points share a cycle, and L, M are of order n.)

A6 Answer: $\ln 2 - \frac{\pi}{2}$. The idea. The matrix is a Cauchy matrix $\left[\frac{1}{x_i + y_j} \right]$ with $x_i = y_i = i^2$, whose determinant is an explicit product. After taking logarithms, the $\ln n$ pieces cancel at order n^2, and what is left is a sum of Riemann sums for three double integrals of logarithms, each with an integrable singularity.

Work it out. *Cauchy's formula.* $\det \left[\frac{1}{x_i + y_j} \right] = \frac{\prod_{i<j} (x_j - x_i)(y_j - y_i)}{\prod_{i,j} (x_i + y_j)}$. (Proof sketch: multiplying by $\prod_{i,j} (x_i + y_j)$ gives a polynomial of degree $n^2 - n$ that vanishes when $x_i = x_j$ or $y_i = y_j$, so it is a constant multiple of the numerator, which also has degree $n^2 - n$; the constant is 1, by induction on n after multiplying by $x_n + y_n$ and

letting $x_n = -y_n$.) With $x_i = y_i = i^2$, $D_n = \frac{\prod_{1\le i<j\le n}(j^2-i^2)^2}{\prod_{i,j=1}^{n}(i^2+j^2)} > 0$, $\ln D_n = 2\sum_{i<j}[\ln(j-i)+\ln(j+i)] - \sum_{i,j}\ln(i^2+j^2)$.

Removing the scale. Write $\ln(j-i) = \ln n + \ln\frac{j-i}{n}$, $\ln(j+i) = \ln n + \ln\frac{j+i}{n}$ and $\ln(i^2+j^2) = 2\ln n + \ln\frac{i^2+j^2}{n^2}$. The $\ln n$ terms total $2\cdot\binom{n}{2}\cdot 2\ln n - n^2\cdot 2\ln n = -2n\ln n = o(n^2)$. So we need the limits of $S_1 = \frac{2}{n^2}\sum_{i<j}\ln\frac{j-i}{n}$, $S_2 = \frac{2}{n^2}\sum_{i<j}\ln\frac{i+j}{n}$, $S_3 = \frac{1}{n^2}\sum_{i,j}\ln\left(\left(\frac{i}{n}\right)^2+\left(\frac{j}{n}\right)^2\right)$.

The Riemann sums converge. (1) Grouping by $d = j-i$: $S_1 = \frac{2}{n}\sum_{d=1}^{n-1}h(\frac{d}{n})$ with $h(t) = (1-t)\ln t$. Since $h'(t) = \frac{1}{t} - 1 - \ln t > 0$ on $(0,1)$, h is increasing, so $\int_0^{1-1/n} h \le \frac{1}{n}\sum_{d=1}^{n-1}h(\frac{d}{n}) \le \int_{1/n}^1 h$, and both bounds tend to $\int_0^1(1-t)\ln t\,dt = -\frac{3}{4}$ because h is integrable. So $S_1 \to -\frac{3}{2}$. (2) and (3) Let $g(x,y)$ be $\ln(x+y)$ or $\ln(x^2+y^2)$; it is increasing in each variable. Comparing $g(\frac{i}{n},\frac{j}{n})$ with its average over the cells $[\frac{i-1}{n},\frac{i}{n}]\times[\frac{j-1}{n},\frac{j}{n}]$ (below it) and $[\frac{i}{n},\frac{i+1}{n}]\times[\frac{j}{n},\frac{j+1}{n}]$ (above it) gives $\iint_{[0,1]^2} g \le \frac{1}{n^2}\sum_{i,j=1}^{n} g(\frac{i}{n},\frac{j}{n}) \le \iint_{[1/n,1+1/n]^2} g$, and both bounds tend to $\iint_{[0,1]^2} g$ since g is integrable near the origin (the only singularity). For S_2, $\sum_{i<j} = \frac{1}{2}\big(\sum_{i,j} - \sum_{i=j}\big)$ and the diagonal contributes $\frac{1}{n^2}\sum_i \ln\frac{2i}{n} = O(\frac{1}{n})$. Hence $\lim\frac{\ln D_n}{n^2} = \iint_{[0,1]^2}\big[\ln|x-y| + \ln(x+y) - \ln(x^2+y^2)\big]\,dx\,dy$.

The integrals. $\iint\ln|x-y| = \int_{-1}^1(1-|t|)\ln|t|\,dt = -\frac{3}{2}$. The density of $x+y$ is s on $[0,1]$ and $2-s$ on $[1,2]$, so $\iint\ln(x+y) = \int_0^1 s\ln s\,ds + \int_1^2(2-s)\ln s\,ds = -\frac{1}{4} + (4\ln 2 - 2) - (2\ln 2 - \frac{3}{4}) = 2\ln 2 - \frac{3}{2}$. Finally $\int_0^1\ln(x^2+y^2)\,dy = \ln(1+x^2) - 2 + 2x\arctan\frac{1}{x}$ (integrate by parts), and $\int_0^1\ln(1+x^2)dx = \ln 2 - 2 + \frac{\pi}{2}$, $\int_0^1 2x\arctan\frac{1}{x}\,dx = \big[x^2\arctan\frac{1}{x}\big]_0^1 + \int_0^1\frac{x^2}{1+x^2}dx = \frac{\pi}{4} + 1 - \frac{\pi}{4} = 1$. So $\iint\ln(x^2+y^2) = \ln 2 - 3 + \frac{\pi}{2}$. The limit is $-\frac{3}{2} + \big(2\ln 2 - \frac{3}{2}\big) - \big(\ln 2 - 3 + \frac{\pi}{2}\big) = \ln 2 - \frac{\pi}{2} \approx -0.8776$.

Check. For $n = 6$ the determinant computed directly equals the Cauchy product exactly. The three double integrals, computed as genuine two-dimensional quadratures (the last in polar coordinates), reproduce $\ln 2 - \frac{\pi}{2}$ to 58 digits. At $n = 400$ the exact log-sum gives $\frac{\ln D_n}{n^2} \approx -0.889$, already close to -0.8776; the error is $O(\frac{\ln n}{n})$, and fitting $\ln D_n \approx an^2 + bn\ln n + cn + d\ln n + e$ to the exact values at $n = 250, 500, \dots, 4000$ gives $a \approx -0.8776491$, matching $\ln 2 - \frac{\pi}{2} = -0.87764914\ldots$ to seven digits.

B1 **Answer:** $2(1-\ln 2)^2$. **The idea.** The awkward factor $\frac{1}{m+n+1}$ is an integral, $\int_0^1 x^{m+n}\,dx$. Once it is written that way the double sum splits into a product of two identical one-variable sums, each of which is a logarithm.

Work it out. First, the series converges absolutely: $mn(m+n+1) \ge mn\cdot 2\sqrt{mn}$, and $\sum_{m,n}\frac{1}{2(mn)^{3/2}} = \frac{1}{2}\zeta(3/2)^2 < \infty$. So we may rearrange freely, and by Tonelli (the terms $\frac{x^{m+n}}{mn}$ are nonnegative and their integrals sum to a finite value) we may swap sum and integral: $S = \int_0^1\sum_{m,n\ge1}\frac{(-x)^m(-x)^n}{mn}\,dx = \int_0^1\Big(\sum_{m\ge1}\frac{(-x)^m}{m}\Big)^2 dx$. For $0 \le x < 1$, $\sum_{m\ge1}\frac{(-x)^m}{m} = -\ln(1+x)$, so $S = \int_0^1\ln^2(1+x)\,dx = \int_1^2\ln^2 u\,du$. An antiderivative of $\ln^2 u$ is $u\ln^2 u - 2u\ln u + 2u$ (differentiate to check: $\ln^2 u + 2\ln u - 2\ln u - 2 + 2$). Hence $S = (2\ln^2 2 - 4\ln 2 + 4) - (0 - 0 + 2) = 2\ln^2 2 - 4\ln 2 + 2 = 2(1-\ln 2)^2 \approx 0.18832$.

Check. Group the terms by $s = m+n$. Since $\frac{1}{mn} = \frac{1}{s}\big(\frac{1}{m} + \frac{1}{n}\big)$, $\sum_{m+n=s}\frac{1}{mn} = \frac{2H_{s-1}}{s}$, so

$S = \sum_{s\ge 2} \frac{(-1)^s\, 2H_{s-1}}{s(s+1)} = \frac{2}{6} - \frac{2\cdot\frac{3}{2}}{12} + \frac{2\cdot\frac{11}{6}}{20} - \cdots$. Its partial sums 0.333, 0.083, 0.267, 0.128, 0.237, ... oscillate and close in on 0.18832; summed to 60 digits it agrees with $2(1-\ln 2)^2$.

B2 **Answer:** 64. **The idea.** $N(A)^3$ is the number of ordered triples (v_1, v_2, v_3) of vectors that A fixes, i.e. the number of fixed points of A acting on $X = (\mathbb{F}_5^4)^3$. By Burnside's lemma the average number of fixed points of a group element equals the number of orbits. So we must count the orbits of $G = GL_4(\mathbb{F}_5)$ on triples of vectors, and a triple is determined up to G by its linear relations.

Work it out. Burnside: $\frac{1}{|G|}\sum_{A\in G} |\mathrm{Fix}(A)| = \#\{\text{orbits of } G \text{ on } X\}$, and $|\mathrm{Fix}(A)| = N(A)^3$, so $\mathbb{E}[N(A)^3]$ is the number of orbits.

For a triple $\mathbf{v} = (v_1, v_2, v_3)$ let $R(\mathbf{v}) = \{c \in \mathbb{F}_5^3 : c_1v_1 + c_2v_2 + c_3v_3 = 0\}$, a subspace of $\mathbb{F}_5^3$ (the kernel of $c \mapsto \sum c_iv_i$). Invertible A preserves it: $\sum c_iAv_i = A\sum c_iv_i$ is zero exactly when $\sum c_iv_i$ is. Conversely, suppose $R(\mathbf{v}) = R(\mathbf{w})$. Define $\phi : \mathrm{span}(\mathbf{v}) \to \mathrm{span}(\mathbf{w})$ by $\phi(\sum c_iv_i) = \sum c_iw_i$. It is well defined and injective because $\sum c_iv_i = \sum c_i'v_i$ iff $c - c' \in R(\mathbf{v}) = R(\mathbf{w})$ iff $\sum c_iw_i = \sum c_i'w_i$; it is onto, so it is a linear isomorphism between spaces of the same dimension d. Extend a basis of each span to a basis of $\mathbb{F}_5^4$ and extend ϕ to an invertible A; then $Av_i = w_i$. So orbits correspond injectively to relation spaces.

Every subspace $R \subseteq \mathbb{F}_5^3$ occurs: $\mathbb{F}_5^3/R$ has dimension at most $3 \le 4$, so there is a linear map $T : \mathbb{F}_5^3 \to \mathbb{F}_5^4$ with kernel exactly R, and $v_i = T(e_i)$ has $R(\mathbf{v}) = R$. Hence the number of orbits equals the number of subspaces of $\mathbb{F}_5^3$: one of dimension 0, $\frac{5^3-1}{5-1} = 31$ lines, 31 planes (dual to lines), and one of dimension 3. The expectation is $1 + 31 + 31 + 1 = 64$.

Check. Count directly instead. A fixes a triple iff it is the identity on $U = \mathrm{span}(\mathbf{v})$; if $\dim U = d$, writing A in a basis extending one of U shows there are $5^{d(4-d)}\,|GL_{4-d}(\mathbb{F}_5)|$ such A. Summing $\Pr(A \text{ fixes } \mathbf{v})$ over all 5^{12} triples, sorted by d, gives exactly 64. The same reasoning with $GL_3(\mathbb{F}_3)$ predicts the number of subspaces of $\mathbb{F}_3^3$, namely $1 + 13 + 13 + 1 = 28$, and a brute-force average over all 11232 matrices of $GL_3(\mathbb{F}_3)$ gives 28.

B3 **Answer:** 8161. **The idea.** Read A as a directed graph on vertices $1, \ldots, 5$ with an edge $i \to j$ when $a_{ij} = 1$ (loops allowed). Because all entries are nonnegative, $(A^3)_{ij}$ is exactly the number of directed walks of length 3 from i to j, so $A^3 = 0$ says: there is no walk with 3 edges. Such a graph is acyclic, and labelling each vertex by the length of the longest path leaving it cuts the graph into three levels with a rigid structure that can be counted directly.

Work it out. Suppose $A^3 = 0$. A cycle (a loop included) could be walked around for 3 steps, so the graph has no cycles, and every path has at most 2 edges. Let $h(v) \in \{0, 1, 2\}$ be the number of edges in a longest path starting at v, and let $H_k = \{v : h(v) = k\}$, of sizes a, b, c with $a + b + c = 5$. Then:

(i) $v \in H_0$ has no out-edges.

(ii) $v \in H_1$ has at least one out-edge, and every out-neighbour w has $h(w) = 0$ (if $h(w) \ge 1$ we would get a path of length 2 from v).

(iii) $v \in H_2$: every out-neighbour w has $h(w) \le 1$ (otherwise a path of length 3), and at least one has $h(w) = 1$ (the second vertex of a longest path).

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Conversely, take any split of the vertices into labelled sets H_0, H_1, H_2 and any edges obeying (i)–(iii). Every edge goes to a strictly smaller label, so every walk has at most 2 edges and $A^3 = 0$; and the labels are the longest-path lengths (a vertex of H_1 has an edge into H_0 and nothing further; a vertex of H_2 has an edge to some vertex of H_1, which has an edge onward, while no walk exceeds 2 edges), so the graph determines the split. Thus the matrices correspond bijectively to (split, edges obeying (i)–(iii)). For a given split each vertex of H_1 picks a nonempty subset of H_0: 2^a-1 ways; each vertex of H_2 picks a subset of $H_0 \cup H_1$ meeting H_1: $2^a(2^b-1)$ ways. Hence $\# = \sum_{a+b+c=5} \binom{5}{a,\,b,\,c} (2^a-1)^b \big(2^a(2^b-1)\big)^c$.

The terms with $c = 0$ (these are the matrices with $A^2 = 0$) are $1+75+490+270+5 = 841$ for $(a,b) = (5,0), (4,1), (3,2), (2,3), (1,4)$. With $c = 1$: $(3,1)$: $20 \cdot 7 \cdot 8 = 1120$; $(2,2)$: $30 \cdot 9 \cdot 12 = 3240$; $(1,3)$: $20 \cdot 1 \cdot 14 = 280$; total 4640. With $c = 2$: $(2,1)$: $30 \cdot 3 \cdot 16 = 1440$; $(1,2)$: $30 \cdot 1 \cdot 36 = 1080$; total 2520. With $c = 3$: $(1,1)$: $20 \cdot 1 \cdot 2^3 = 160$. (Terms with $a = 0$ vanish, and $b = 0 < c$ vanishes.) Altogether $841 + 4640 + 2520 + 160 = 8161$.

Check. Small sizes: for 2×2 the formula gives 3 (the zero matrix and the two off-diagonal units), and for 3×3 it gives $25 = 13 + 12$: the 13 matrices with $A^2 = 0$ plus, for each of the 6 orderings $x \to y \to z$, the path with or without the shortcut edge $x \to z$. An exhaustive computer check of all 2^{25} matrices, multiplying the 0/1 matrices directly, also finds 8161.

B4 Answer: $\frac{16\pi - 4\sqrt{3}\,\pi + 8}{3}$. **The idea.** Integrate over the square $Q = [-1,1]^2$ in the xy-plane: above and below every point of Q the piece is a full vertical chord of the ball. Polar coordinates on one eighth of the square reduce everything to a single integral $\int_0^1 \frac{(2-t^2)^{3/2}}{1+t^2}\,dt$, and the identity $2 - t^2 = 3 - (1 + t^2)$ splits it into a circular area and an arctangent that happens to be $\arctan\sqrt{3} = \frac{\pi}{3}$.

Work it out. Every $(x,y) \in Q$ has $x^2 + y^2 \le 2 < 3$, so the vertical line through it meets B in the segment $|z| \le \sqrt{3 - x^2 - y^2}$, and a point with $|x| > 1$ or $|y| > 1$ is not in the piece. Hence $V = \iint_Q 2\sqrt{3 - x^2 - y^2}\,dA$. The integrand and Q are invariant under the 8 symmetries of the square, and the triangle $0 \le y \le x \le 1$ is, in polar coordinates, $0 \le \theta \le \frac{\pi}{4}$, $0 \le \rho \le \sec\theta$. So $V = 16\int_0^{\pi/4}\!\int_0^{\sec\theta} \sqrt{3 - \rho^2}\,\rho\,d\rho\,d\theta = \frac{16}{3}\int_0^{\pi/4} \Big(3\sqrt{3} - (3 - \sec^2\theta)^{3/2}\Big)d\theta = 4\sqrt{3}\,\pi - \frac{16}{3}J$, where $J = \int_0^{\pi/4}(2 - \tan^2\theta)^{3/2}\,d\theta$ (using $3 - \sec^2\theta = 2 - \tan^2\theta \ge 1$ on this range). Put $t = \tan\theta$, $d\theta = \frac{dt}{1+t^2}$, and write $2 - t^2 = 3 - (1 + t^2)$: $J = \int_0^1 \frac{(2-t^2)^{3/2}}{1+t^2}\,dt = 3K - L, \qquad K = \int_0^1 \frac{\sqrt{2-t^2}}{1+t^2}\,dt, \quad L = \int_0^1 \sqrt{2 - t^2}\,dt.$

L is the area under the circle of radius $\sqrt{2}$ over $[0,1]$: a right triangle with legs $1, 1$ (area $\frac{1}{2}$) plus a sector of angle $\frac{\pi}{4}$ (area $\frac{1}{2} \cdot 2 \cdot \frac{\pi}{4}$), so $L = \frac{1}{2} + \frac{\pi}{4}$.

For K put $t = \sqrt{2}\sin\varphi$, $0 \le \varphi \le \frac{\pi}{4}$: then $\sqrt{2 - t^2}\,dt = 2\cos^2\varphi\,d\varphi$ and $2\cos^2\varphi = 3 - (1 + 2\sin^2\varphi)$, so $K = \int_0^{\pi/4} \frac{3\,d\varphi}{1 + 2\sin^2\varphi} - \frac{\pi}{4}$. Dividing top and bottom by $\cos^2\varphi$ and setting $u = \tan\varphi$ gives $\int_0^{\pi/4} \frac{d\varphi}{1 + 2\sin^2\varphi} = \int_0^1 \frac{du}{1 + 3u^2} = \frac{\arctan\sqrt{3}}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$. Hence $K = \frac{\pi}{\sqrt{3}} - \frac{\pi}{4}$ and $J = \sqrt{3}\,\pi - \frac{3\pi}{4} - \frac{1}{2} - \frac{\pi}{4} = \sqrt{3}\,\pi - \pi - \frac{1}{2}$. Therefore $V = 4\sqrt{3}\,\pi - \frac{16}{3}\Big(\sqrt{3}\,\pi - \pi - \frac{1}{2}\Big) = \frac{16\pi - 4\sqrt{3}\,\pi + 8}{3} \approx 12.16663$.

Check. Sanity bounds: the piece contains the cube (volume 8) and lies in B (volume $4\sqrt{3}\,\pi \approx 21.77$). Inclusion–exclusion gives an independent route: the piece is B minus the four caps $x > 1$, $x < -1$, $y > 1$, $y < -1$, plus the four places where two caps overlap (such

as $x, y > 1$); no point lies in three caps, because any three of them include an opposite pair such as $x > 1$ and $x < -1$. Each cap has height $h = \sqrt{3}-1$ and volume $\frac{\pi h^2(3\sqrt{3}-h)}{3} = \frac{\pi(6\sqrt{3}-8)}{3}$, so the formula predicts that each overlap has volume $\frac{2\sqrt{3}\,\pi-4\pi+2}{3} \approx 0.10548$; a direct double integral of $2\sqrt{3-x^2-y^2}$ over $\{x, y \geq 1,\ x^2+y^2 \leq 3\}$ agrees to 40 digits. Slicing horizontally instead (at height z the section is Q cut by a disk of radius $\sqrt{3-z^2}$) and integrating numerically also gives $12.1666300288\ldots$.

B5 Answer: 15. **The idea.** Write $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,u$ with u a unit pure quaternion; then $x \mapsto qxq^{-1}$ is the rotation by θ about u, whose trace is $1 + 2\cos\theta = 4\cos^2\frac{\theta}{2} - 1$. So $\operatorname{tr} A = 4t^2 - 1$ where t is the real coordinate of a uniform point on the sphere S^3. That coordinate has the semicircle density, and the moment becomes the constant term of a Laurent polynomial.

Work it out. *Distribution of t.* For a uniform point of $S^3 \subset \mathbb{R}^4$, the slice at first coordinate t is a 2-sphere of radius $\sqrt{1-t^2}$, area $4\pi(1-t^2)$, met at slant factor $1/\sqrt{1-t^2}$; so t has density proportional to $\sqrt{1-t^2}$ on $[-1, 1]$, which normalises to $\frac{2}{\pi}\sqrt{1-t^2}$.

Put $t = \cos\varphi$, $\varphi \in [0, \pi]$: the density becomes $\frac{2}{\pi}\sin^2\varphi\,d\varphi$, and $4t^2-1 = 1+2\cos 2\varphi$. With $\psi = 2\varphi \in [0, 2\pi]$ and $\sin^2\varphi = \frac{1-\cos\psi}{2}$: $\mathbb{E}[(\operatorname{tr} A)^6] = \frac{1}{2\pi}\int_0^{2\pi}(1+2\cos\psi)^6(1-\cos\psi)\,d\psi$. (Equivalently the rotation angle has density $\frac{1-\cos\psi}{\pi}$ on $[0, \pi]$.) With $z = e^{i\psi}$, $1+2\cos\psi = 1+z+z^{-1}$ and $1-\cos\psi = 1-\frac{1}{2}(z+z^{-1})$, and $\frac{1}{2\pi}\int_0^{2\pi}$ extracts the constant term. If T_k is the coefficient of z^k in $(1+z+z^{-1})^6$, the answer is $T_0 - \frac{1}{2}(T_1+T_{-1}) = T_0 - T_1$.

Counting words of six letters from $\{+1, 0, -1\}$: $T_0 = \sum_j \frac{6!}{j!\,j!\,(6-2j)!} = 1+30+90+20 = 141$ (j = number of $+1$s), and $T_1 = \sum_j \frac{6!}{(j+1)!\,j!\,(5-2j)!} = 6+60+60 = 126$. So the expectation is $141 - 126 = 15$.

Check. $\operatorname{tr} A$ is the character of the 3-dimensional (spin-1) representation V_1, so $\mathbb{E}[\chi^6]$ is the number of trivial summands in $V_1^{\otimes 6}$. Using $V_l \otimes V_1 = V_{l-1} \oplus V_l \oplus V_{l+1}$ ($l \geq 1$) and $V_0 \otimes V_1 = V_1$, the trivial multiplicities of $V_1^{\otimes k}$ for $k = 0, \ldots, 6$ are $1, 0, 1, 1, 3, 6, 15$ (the Riordan numbers); for instance $k = 2$ gives 1, matching $\mathbb{E}[(\operatorname{tr} A)^2] = 1$. A direct symbolic integration of $\frac{1}{\pi}\int_0^{\pi}(1+2\cos\psi)^6(1-\cos\psi)\,d\psi$ also gives 15.

B6 Answer: $\frac{64}{1575}$. **The idea.** Parametrise R by the roots. On the region $-1 \leq r_1 < r_2 < r_3 < r_4 \leq 1$ the map from roots to coefficients is one-to-one, and its Jacobian determinant is, up to sign, the Vandermonde product $\Delta = \prod_{i<j}(r_j - r_i)$. So the volume is $\int \Delta$ over the ordered region, and de Bruijn's identity turns that fourfold integral of a determinant into a 4×4 Pfaffian of double integrals.

Work it out. *Change of variables.* The coefficient map is $\Phi(r) = (-e_1, e_2, -e_3, e_4)$ in the elementary symmetric functions. Its Jacobian determinant J is a polynomial in the r_i of degree $0+1+2+3 = 6$. If $r_i = r_j$ the columns $\partial\Phi/\partial r_i$ and $\partial\Phi/\partial r_j$ coincide (swap symmetry), so J vanishes; hence $(r_i - r_j) \mid J$ for every pair and $\Delta \mid J$. Both have degree 6, so $J = \pm\Delta\cdot\text{const}$, and comparing the coefficient of $r_2 r_3^2 r_4^3$ gives $|J| = |\Delta|$. Every point of R whose roots are distinct has exactly one preimage in the open ordered region $O = \{-1 < r_1 < \cdots < r_4 < 1\}$, and Φ is a diffeomorphism there ($J \neq 0$). Points of R with a repeated root or a root at ± 1 form the image of a set of lower dimension under a polynomial map, hence have volume 0. So $\operatorname{vol}(R) = \int_O \Delta\,dr$, $\quad \Delta = \det\left[r_i^j\right]_{1\leq i\leq 4,\ 0\leq j\leq 3} > 0$ on O.

De Bruijn. For an even number of functions, $\int_{x_1<\cdots<x_4} \det[\phi_j(x_i)]\,dx = \operatorname{Pf}[a_{jk}]$ with $a_{jk} = \iint \operatorname{sgn}(y-x)\phi_j(x)\phi_k(y)\,dx\,dy$. Here $\phi_j(x) = x^j$ on $[-1,1]$. The inner integral is $\int_{-1}^{1} \operatorname{sgn}(y-x)x^j\,dx = \frac{2y^{j+1}-1-(-1)^{j+1}}{j+1}$, and integrating against y^k gives $a_{01} = \frac{4}{3}$, $a_{03} = \frac{4}{5}$, $a_{12} = -\frac{4}{15}$, $a_{23} = \frac{4}{21}$, $a_{02} = a_{13} = 0$ (the zero entries have $j+k$ even, where the integrand is odd under $(x,y) \mapsto (-y,-x)$). The Pfaffian of a 4×4 antisymmetric matrix is $a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}$, so $\operatorname{vol}(R) = \frac{4}{3}\cdot\frac{4}{21} - 0 + \frac{4}{5}\cdot\left(-\frac{4}{15}\right) = \frac{16}{63} - \frac{16}{75} = \frac{16\cdot 12}{4725} = \frac{64}{1575}$.

(Alternatively, Selberg's integral with $\alpha = \beta = 1$, $\gamma = \frac{1}{2}$ gives $\int_{[0,1]^4} |\Delta| = \frac{1}{1050}$; rescaling to $[-1,1]$ multiplies by 2^{4+6}, and dividing by 4! for the ordering gives $\frac{1024}{25200} = \frac{64}{1575}$ again.)

Check. The same method in lower degree: for x^2+ax+b it gives $\int_{-1<r_1<r_2<1}(r_2-r_1) = \frac{4}{3}$, and indeed the region $\{|a| \le 1+b,\ b \le a^2/4,\ |a| \le 2\}$ between the parabola and its two tangent lines $b = \pm a - 1$ has area $\frac{4}{3}$; for cubics it gives $\frac{16}{45}$. Expanding Δ and integrating iteratively in r_1, then r_2, r_3, r_4 with exact arithmetic gives $\frac{64}{1575} \approx 0.0406$ directly.