MathCastle

Putnam

Session A and Session B

12 problems • 6 hours

Name _____

INSTRUCTIONS

1. Two sessions, A and B, of 6 problems each. Each session lasts 3 hours.
2. Each problem is worth 10 points. Full credit needs a complete proof.
3. Here every problem also has a numeric target, so your answer can be checked.
4. Session A is handed in before Session B is opened.
5. Two sittings of 6 problems; the first is handed in before the second is opened. One generating-function problem appears on every set.

DO NOT TURN THE PAGE UNTIL YOU ARE TOLD TO BEGIN.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

A1

Define a sequence by $x_1 = 1$ and $x_{n+1} = x_n + \dfrac{1}{x_1 + x_2 + \cdots + x_n}$ for $n \geq 1$ (so $x_2 = 2$, $x_3 = \frac{7}{3}$). Find $\displaystyle\lim_{n\to\infty} \frac{x_n}{\sqrt{\ln n}}$.

A2

Let M be the 1024×1024 matrix whose rows and columns are both indexed by the 1024 subsets of $\{1, 2, \ldots, 10\}$ (in the same order), with $M_{S,T} = 1$ if $S \cap T = \varnothing$ and $M_{S,T} = 0$ otherwise. M is real symmetric, so it has 1024 real eigenvalues counted with multiplicity. Find the sum of the absolute values of these eigenvalues.

A3

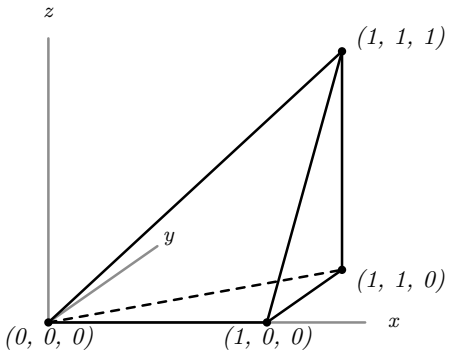
A $2 \times n$ rectangular board (2 rows, n columns) is tiled by 1×2 dominoes. A domino is *vertical* if it covers both cells of one column. One tiling is chosen uniformly at random from all tilings of the $2 \times n$ board, and V_n denotes the number of vertical dominoes in it. Find $\displaystyle\lim_{n\to\infty} \frac{\mathbb{E}[V_n]}{n}$. Give the exact value.

A4

A function $f : \mathbb{Z}/100\mathbb{Z} \to \mathbb{Z}/100\mathbb{Z}$ is called *polynomial* if there is a polynomial P with integer coefficients such that $f(x) \equiv P(x) \pmod{100}$ for every x. (Different polynomials may give the same function; count functions, not polynomials.) How many polynomial functions $\mathbb{Z}/100\mathbb{Z} \to \mathbb{Z}/100\mathbb{Z}$ are there?

A5

Let T be the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(1,1,1)$. A unit vector u is chosen uniformly at random on the unit sphere, and T is projected orthogonally onto a plane perpendicular to u. The shadow is a convex polygon, which (with probability 1) is either a triangle or a quadrilateral. Find the probability that the shadow is a quadrilateral.



A6

Let O be the solid octahedron $\{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \leq 1\}$. Evaluate $\displaystyle\iiint_O \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$.

B1

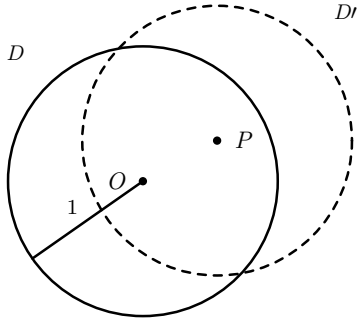
Three numbers X, Y, Z are chosen independently and uniformly at random from the interval $[0,1]$. Find the probability that $X^Y > Z$. (Give the exact value.)

B2

Evaluate $\displaystyle\int_0^1\int_0^1\int_0^1 \left|xz - y^2\right|\,dx\,dy\,dz$, the average absolute determinant of the symmetric matrix $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ with entries in $[0,1]$. Give the answer as a fraction in lowest terms.

B3

Let D be the closed disk of radius 1 centred at the origin O. A point P is chosen uniformly at random from D, and D' is the closed disk of radius 1 centred at P (the translate of D by the vector $\overrightarrow{OP}$). Find the expected value of the area of $D \cap D'$. (Give the exact value.)



B4

The order of a root of unity ω is the least positive integer k with $\omega^k = 1$. Let P be the product of $\zeta - \eta$ taken over all ordered pairs (ζ, η) of complex numbers with $\zeta^{12} = \eta^{12} = 1$ such that the order of ζ is strictly less than the order of η. (There are 57 factors.) Find P.

B5

For $n \geq 1$ let T_n be the $n \times n$ matrix whose (i,j) entry is 4 if $i = j$, is -1 if $|i-j| \in \{1,2\}$, and is 0 if $|i-j| \geq 3$. (So $\det T_1 = 4$, $\det T_2 = 15$, $\det T_3 = 50$.) Find $\displaystyle\lim_{n\to\infty} \frac{\ln \det T_n}{n}$. (Give the exact value.)

B6

Let $V_1, V_2, V_3, \ldots$ be independent random variables, where each V_i is the larger of two independent numbers chosen uniformly from $[0,1]$. Define $S = V_1 + V_1V_2 + V_1V_2V_3 + \cdots$ (the series converges with probability 1). Find the exact value of $\dfrac{P(S \leq 2)}{P(S \leq 1)}$.

Answer Key — Putnam Practice Contest 1

A1 $\sqrt{2}$

A2 3125

A3 $\frac{\sqrt{5}}{5}$

A4 1953125000000

A5 $\frac{2}{3}$

A6 $\frac{\pi^2}{12}$

B1 $\ln 2$

B2 $\frac{89}{300}$

B3 $\pi - \frac{3\sqrt{3}}{4}$

B4 41472

B5 $\ln \frac{3+\sqrt{5}}{2}$

B6 $9 - 8\ln 2$

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Solutions — Putnam Practice Contest 1

A1 **Answer:** $\sqrt{2}$. **The idea.** Write $S_n = x_1 + \cdots + x_n$, so $x_{n+1} - x_n = 1/S_n$ and $S_{n+1} - S_n = x_{n+1}$. In continuous form this is $x' = 1/S$, $S' = x$, and then $(x^2)' = 2x/S = 2(\ln S)'$: the quantity $x^2 - 2\ln S$ is conserved. We show the discrete version is conserved up to a summable error, and then crude bounds pin $\ln S_n$ to $\ln n$.

Work it out. *Crude bounds.* Every increment $1/S_n$ is positive, so $x_n \geq x_1 = 1$ and $S_n \geq n$. Hence $x_{k+1} - x_k = 1/S_k \leq 1/k$, so $x_n \leq 1 + \sum_{k=1}^{n-1} \frac{1}{k} \leq 2 + \ln n$, and since x_n is increasing, $n \leq S_n \leq n\, x_n \leq n(2 + \ln n)$.

Near-conservation. First, $x_{n+1}^2 - x_n^2 = (x_{n+1} - x_n)(x_{n+1} + x_n) = \frac{1}{S_n}\big(2x_n + \frac{1}{S_n}\big) = \frac{2x_n}{S_n} + \frac{1}{S_n^2}$. Second, with $t_n = x_{n+1}/S_n > 0$ we have $2\ln\frac{S_{n+1}}{S_n} = 2\ln(1 + t_n)$, and $t - \frac{t^2}{2} \leq \ln(1+t) \leq t$ for $t \geq 0$ gives $2t_n - t_n^2 \leq 2\ln\frac{S_{n+1}}{S_n} \leq 2t_n$, where $2t_n = \frac{2x_n}{S_n} + \frac{2}{S_n^2}$. Subtracting, the defect $d_n = (x_{n+1}^2 - x_n^2) - 2\ln\frac{S_{n+1}}{S_n}$ satisfies $-\frac{1}{S_n^2} \leq d_n \leq t_n^2 - \frac{1}{S_n^2}$, so $|d_n| \leq \frac{1}{S_n^2} + t_n^2 \leq \frac{1 + (3 + \ln n)^2}{n^2}$, using $S_n \geq n$ and $x_{n+1} \leq 2 + \ln(n+1) \leq 3 + \ln n$. This bound is summable, so $x_n^2 - 2\ln S_n = (x_1^2 - 2\ln S_1) + \sum_{k<n} d_k = 1 + \sum_{k<n} d_k$ converges to a finite constant C; in particular $|x_n^2 - 2\ln S_n| \leq M$ for some M and all n.

The limit. From the crude bounds, $\ln n \leq \ln S_n \leq \ln n + \ln(2 + \ln n)$, so $\frac{\ln S_n}{\ln n} \to 1$. Therefore $\frac{x_n^2}{\ln n} = \frac{2\ln S_n}{\ln n} + \frac{x_n^2 - 2\ln S_n}{\ln n} \to 2 + 0 = 2$, and since $x_n > 0$, $\lim_{n\to\infty} \dfrac{x_n}{\sqrt{\ln n}} = \sqrt{2}$.

Check. Running the recurrence, $x_n^2 - 2\ln S_n$ is 2.5741 to four places already at $n = 10^5$ and does not move by $n = 2 \cdot 10^6$, as the summable defect predicts. The ratio itself approaches $\sqrt{2} \approx 1.414$ only slowly (it is about 1.561 at $n = 10^6$), because $\ln S_n$ exceeds $\ln n$ by roughly $\ln x_n \approx \frac{1}{2}\ln(2\ln n)$; indeed $\sqrt{(2\ln n + 2\ln x_n + C)/\ln n}$ with $n = 10^6$, $x_n \approx 5.80$, $C \approx 2.574$ gives 1.562, matching.

A2 **Answer:** 3125. **The idea.** Disjointness is decided one element at a time: $S \cap T = \varnothing$ exactly when, for every i, the element i is not in both. So M is a Kronecker (tensor) power of the 2×2 matrix that records "not both", and the eigenvalues of a Kronecker power are products of the small eigenvalues.

Work it out. Identify a subset S with its indicator vector $(s_1, \ldots, s_{10}) \in \{0,1\}^{10}$. Then $M_{S,T} = \prod_{i=1}^{10} A_{s_i, t_i}$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (rows and columns indexed by $0, 1$): the factor is 0 exactly when $s_i = t_i = 1$. Hence $M = A^{\otimes 10}$. The eigenvalues of A are the roots of $\lambda^2 - \lambda - 1 = 0$, namely $\varphi = \frac{1+\sqrt{5}}{2}$ and $-\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}$. If $Au = \alpha u$ and $Av = \beta v$ then $(A \otimes A)(u \otimes v) = \alpha\beta\,(u \otimes v)$, and tensor products of eigenbases form an eigenbasis, so the eigenvalues of M are $\varphi^k(-1/\varphi)^{10-k}$ with multiplicity $\binom{10}{k}$, $k = 0, \ldots, 10$.

Their absolute values are $\varphi^k\varphi^{-(10-k)}$, and by the binomial theorem $\sum_k \binom{10}{k}\varphi^k(1/\varphi)^{10-k} = (\varphi + 1/\varphi)^{10}$. Now $\varphi + \frac{1}{\varphi} = \frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5}$, so the sum is $(\sqrt{5})^{10} = 5^5 = 3125$.

Check. For the ground set $\{1\}$ the matrix is A itself, and $|\varphi| + |-1/\varphi| = \sqrt{5}$, matching $(\sqrt{5})^1$. For $\{1,2\}$ the eigenvalues are $\varphi^2, -1, -1, \varphi^{-2}$, with absolute sum $\varphi^2 + \varphi^{-2} + 2 = 3 + 2 = 5 = (\sqrt{5})^2$. A numerical diagonalisation of the full 1024×1024 matrix also gives 3125.

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A3 Answer: $\frac{\sqrt{5}}{5}$**. The idea.** Read a tiling left to right as a word in two letters: a vertical domino v (one column) and a stacked pair of horizontal dominoes h (two columns). Mark each v with a variable y; the bivariate generating function is rational, and the average number of v's grows like n times the logarithmic derivative of its dominant singularity.

Work it out. Horizontal dominoes must come in stacked pairs: the top-left cell of the first column not covered by a vertical domino must be covered by a horizontal domino, and then so must the cell below it. So tilings are words in v (weight xy) and h (weight x^2), and $\sum_{n,k} t(n,k)x^n y^k = \frac{1}{1-yx-x^2}$, where $t(n,k)$ counts tilings of length n with k verticals.

Let $T_n(y) = \sum_k t(n,k)y^k$. For y near 1 the denominator $1 - yx - x^2$ has two roots $\rho(y)$ and $\sigma(y)$ with $|\rho| < |\sigma|$; with $\rho(1) = \frac{\sqrt{5}-1}{2} = 1/\varphi$ and $\sigma(1) = -\varphi$. Partial fractions give $T_n(y) = A(y)\rho(y)^{-n}\big(1 + O(\theta^n)\big)$ for some $\theta < 1$, uniformly near $y = 1$, with A analytic and $A(1) \neq 0$. Then $\mathbb{E}[V_n] = T_n'(1)/T_n(1) = \frac{A'(1)}{A(1)} - n\frac{\rho'(1)}{\rho(1)} + o(1)$, so the limit is $-\rho'(1)/\rho(1)$.

Differentiate $y\rho + \rho^2 = 1$: $\rho + y\rho' + 2\rho\rho' = 0$, so $\rho' = -\rho/(y + 2\rho)$ and $-\rho'(1)/\rho(1) = \frac{1}{1+2\rho(1)} = \frac{1}{1+(\sqrt{5}-1)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

Check. Exactly: with T_n tilings and W_n the total count of vertical dominoes over all tilings, $T_n = T_{n-1} + T_{n-2}$ and $W_n = W_{n-1} + T_{n-1} + W_{n-2}$ (condition on the first piece). Then $\mathbb{E}[V_n] = W_n/T_n$ and $\mathbb{E}[V_n] - \mathbb{E}[V_{n-1}]$ converges geometrically; at $n = 400$ it equals $0.4472135954\ldots = 1/\sqrt{5}$ to far more than 60 digits. Intuitively, a random tiling uses verticals at density $1/\sqrt{5} \approx 0.447$ per column and pairs at density $(1 - 1/\sqrt{5})/2 \approx 0.276$.

A4 Answer: 1953125000000**. The idea.** By the Chinese Remainder Theorem a polynomial function mod 100 is the same thing as a pair (polynomial function mod 4, polynomial function mod 25): a single integer polynomial reduces to both, and conversely two polynomials P_4, P_{25} can be glued coefficientwise by CRT into one P with $P \equiv P_4 \pmod 4$ and $P \equiv P_{25} \pmod{25}$. So we count mod p^e for $p^e = 4$ and 25 and multiply. Mod p^e, the right basis is the binomial polynomials $\binom{x}{k}$, not the monomials.

Work it out. Every integer polynomial is an integer combination of $k!\binom{x}{k}$ (since x^n is an integer combination of the falling factorials $x(x-1)\cdots(x-k+1) = k!\binom{x}{k}$), and conversely each $k!\binom{x}{k}$ is an integer polynomial. So the polynomial functions mod $m = p^e$ are exactly the functions $x \mapsto \sum_k c_k\, k!\binom{x}{k} \bmod m$, $c_k \in \mathbb{Z}$.

Claim: the function $\sum_k b_k\binom{x}{k} \bmod m$ (with integers b_k, $0 \le k \le m-1$) determines each $b_k \bmod m$. Indeed, evaluating at $x = 0, 1, 2, \ldots$ gives $f(0) = b_0$, $f(1) = b_0 + b_1$, and in general $f(n) = \sum_{k \le n}\binom{n}{k}b_k$, a unitriangular system, so $b_0, \ldots, b_{m-1}$ are recovered mod m from $f(0), \ldots, f(m-1)$. Terms with $k \ge m$ never matter: $m \mid m! \mid k!$, so $c_k\, k!\binom{x}{k} \equiv 0 \pmod m$ for every integer x.

So the polynomial functions correspond bijectively to tuples $(b_k \bmod m)$ with b_k ranging over the multiples of $k!$ mod m. The multiples of $k!$ mod m form a subgroup of size $m/\gcd(m, k!)$. Hence the count is $\prod_{k \ge 0}\frac{m}{\gcd(m,k!)}$ (the factors become 1 once $m \mid k!$).

Mod 4: $k = 0, 1, 2, 3$ give $\gcd(4, k!) = 1, 1, 2, 2$ and $4 \mid k!$ for $k \ge 4$, so $4 \cdot 4 \cdot 2 \cdot 2 = 64 = 2^6$.

Mod 25: $\gcd(25, k!) = 1$ for $k \le 4$, $= 5$ for $5 \le k \le 9$, and $25 \mid k!$ for $k \ge 10$. So $25^5 \cdot 5^5 = 5^{15}$.

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Total: $2^6 \cdot 5^{15} = 64 \cdot 30517578125 = 1953125000000$.

Check. The formula for mod 4 can be confirmed by brute force: listing all 4^8 polynomials of degree at most 7 mod 4 yields exactly 64 distinct functions (out of $4^4 = 256$). For $m = p$ prime the formula gives p^p, i.e. every function is polynomial, which is Lagrange interpolation.

A5 Answer: $\frac{2}{3}$. The idea. The shadow is a triangle exactly when one vertex projects inside the triangle formed by the other three, i.e. when the line through that vertex in direction u passes through the opposite face. That happens exactly when u or $-u$ lies in the solid-angle cone of T at that vertex. So the probability is a sum of solid angles, and the solid angles of a tetrahedron are controlled by its dihedral angles.

Work it out. The four projected points are in general position with probability 1. Their convex hull is a triangle iff one of them lies inside the triangle of the other three. The projection of vertex A lies inside the projected opposite triangle iff the line $A + tu$ meets the opposite face, iff u or $-u$ lies in the cone K_A spanned by the three edges at A. The eight cones $\pm K_A$ have disjoint interiors: if $u \in K_A$ and $u \in K_B$ ($A \neq B$), then the shadow would have two interior points, impossible for four points; and $u \in K_A$, $-u \in K_A$ is impossible for a pointed cone; a line in direction u with $u \in K_A$ and $-u \in K_B$ would put both A and B inside the hull, again impossible. So $P(\text{triangle}) = \frac{2\sum_A \Omega_A}{4\pi}$, where Ω_A is the solid angle at A.

For a trihedral corner, the spherical triangle cut out on the unit sphere has angles equal to the three dihedral angles at the edges through A, so by Girard's theorem $\Omega_A =$ (sum of those three dihedral angles) $- \pi$. Summing over the four vertices counts each of the six edges twice: $\sum_A \Omega_A = 2\sum_{\text{edges}} \theta_e - 4\pi$.

Dihedral angles of T: label $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,1,0)$, $C = (1,1,1)$. The faces are OAB (plane $z = 0$), ABC (plane $x = 1$), OAC (plane $y = z$) and OBC (plane $x = y$). Using normals: at edge AB (faces $z = 0$, $x = 1$): $\pi/2$; at edge BC (faces $x = 1$, $x = y$): $\pi/4$; at edge OA (faces $z = 0$, $y = z$): $\pi/4$; at edge AC (faces $x = 1$, $y = z$): $\pi/2$; at edge OB (faces $z = 0$, $x = y$): $\pi/2$; at edge OC (faces $y = z$, $x = y$): the normals $(0,1,-1)$ and $(1,-1,0)$ make an angle whose cosine is $\pm\frac{1}{2}$, so the dihedral angle is $\pi/3$ or $2\pi/3$; it is $\pi/3$ because the six tetrahedra $\{1 \geq x_{\sigma(1)} \geq x_{\sigma(2)} \geq x_{\sigma(3)} \geq 0\}$, one per permutation σ, are congruent copies of T (reflected in coordinates) that tile the unit cube and all share the edge OC, so six equal angles fill 2π around it. The sum is $\frac{3\pi}{2} + \frac{\pi}{2} + \frac{\pi}{3} = \frac{7\pi}{3}$, so $\sum \Omega_A = \frac{14\pi}{3} - 4\pi = \frac{2\pi}{3}$.

Therefore $P(\text{triangle}) = \frac{2 \cdot 2\pi/3}{4\pi} = \frac{1}{3}$ and $P(\text{quadrilateral}) = \frac{2}{3}$.

Check. Consistency: six copies of T (one for each ordering $x \geq y \geq z$ of coordinates, reflected) tile the unit cube around the diagonal OC, and indeed six dihedral angles of $\pi/3$ fill 2π around OC, while four copies of $\pi/2$ at AB and eight of $\pi/4$ at OA fit around the cube edges. Independently, the Van Oosterom–Strackee formula $\tan\frac{\Omega}{2} = \frac{|a\cdot(b\times c)|}{abc+(a\cdot b)c+(a\cdot c)b+(b\cdot c)a}$ for each vertex gives solid angles summing to $2\pi/3$ numerically.

A6 Answer: $\frac{\pi^2}{12}$. The idea. Direct integration is hopeless: after the z-integral one is left with arctangents of square roots over a triangle. Instead, read the integrand as a solid-angle density one dimension up. Place O in the hyperplane $x_4 = 1$ of $\mathbb{R}^4$; then the

integral is exactly the 3-dimensional measure of the set of directions (points of the unit sphere S^3) seen through O from the origin. The octahedron at height 1 turns out to be one facet of the regular 24-cell, whose 24 facets are all alike, so it subtends $\frac{1}{24}$ of the whole sphere.

Work it out. *Step 1: solid angle.* Let F be a region of the hyperplane $x_4 = 1$ and $C_F = \{t(p,1) : p \in F,\ t > 0\}$ the cone over it. We claim $\sigma(C_F \cap S^3) = \int_F \frac{dV(p)}{(1+|p|^2)^2}$, where σ is surface measure on S^3. Consider a small piece dV of F around p and the thin cone over it. The part of that cone between the origin and the hyperplane is a pyramid of height 1, with 4-volume $\frac{1}{4} \cdot 1 \cdot dV$ (the apex-to-base formula $\frac{1}{d}$ height $\cdot$ base in dimension $d = 4$). The part inside the unit ball is a pyramid over the spherical patch $d\sigma$ with 4-volume $\frac{1}{4}d\sigma$, and it is the first pyramid scaled by the factor $1/r$ with $r = |(p,1)| = \sqrt{1+|p|^2}$, so its volume is r^{-4} times as large. Hence $d\sigma = dV/(1+|p|^2)^2$, and integrating proves the claim. In particular the requested integral I equals the measure of the directions seen through $O \times \{1\}$.

Step 2: the 24*-cell.* Let W be the 24 unit vectors $\pm e_i$ $(i = 1, \dots, 4)$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$, and let $P = \{x \in \mathbb{R}^4 : \langle x, w\rangle \le 1 \text{ for all } w \in W\}$, i.e. $|x_i| \le 1$ for all i and $|x_1| + |x_2| + |x_3| + |x_4| \le 2$. P is a bounded convex body containing a ball around 0. For $w \in W$ let $F_w = \{x \in P : \langle x, w\rangle = 1\}$. On F_{e_4} we have $x_4 = 1$, and then the constraints reduce to $|x_1| + |x_2| + |x_3| \le 1$ (the others follow). So $F_{e_4} = O \times \{1\}$.

Every ray from 0 leaves P at exactly one boundary point, and that point lies in some F_w. Two different F_w meet only in a set of lower dimension, which projects to a σ-null set. So the radial projections of the 24 sets F_w tile S^3 up to measure zero, and $\sum_w \sigma(\text{projection of } F_w) = \sigma(S^3) = 2\pi^2$.

Now the symmetry. Signed permutations of coordinates are orthogonal maps that permute W. They map e_4 to every $\pm e_i$, and $\frac{1}{2}(1,1,1,1)$ to every $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. Also let $H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, which is orthogonal. It sends $\pm e_i$ to $\pm$ its i-th column, which lies in W. For a sign vector ε, $2H\varepsilon$ is an integer vector with entries in $\{0, \pm 2, \pm 4\}$ and squared length 16 (since $2H$ is orthogonal up to the factor 2 and $|\varepsilon|^2 = 4$). So it is either $(\pm 2, \pm 2, \pm 2, \pm 2)$ or $\pm 4e_k$, and in both cases $H(\frac{1}{2}\varepsilon) = \frac{1}{4}(2H\varepsilon) \in W$. Hence H permutes W, and $He_1 = \frac{1}{2}(1,1,1,1)$. An orthogonal map Q that permutes W satisfies $Q(P) = P$, since $\langle Qx, Qw\rangle = \langle x, w\rangle$, and $Q(F_w) = F_{Qw}$. Together these maps carry F_{e_4} onto every F_w, and they preserve σ, so all 24 projections have equal measure $\frac{2\pi^2}{24}$.

Therefore $I = \sigma(\text{projection of } F_{e_4}) = \frac{2\pi^2}{24} = \frac{\pi^2}{12}$.

Check. The same argument one dimension down gives $\iint_{[-1,1]^2} \frac{dx\,dy}{(1+x^2+y^2)^{3/2}} = \frac{4\pi}{6}$ (a face of the cube $[-1,1]^3$ seen from its centre), and in $\mathbb{R}^4$ with the hypercube it gives $\iiint_{[-1,1]^3} \frac{dV}{(1+|p|^2)^2} = \frac{2\pi^2}{8}$. Numerically, integrating z in closed form over the simplex $x, y, z \ge 0$, $x + y + z \le 1$ and the rest by quadrature, then multiplying by 8, gives $0.8224670334\ldots$, matching $\pi^2/12$ to 45 digits. A sanity bound: the integrand lies between $\frac{1}{4}$ and 1 on O, whose volume is $\frac{4}{3}$, and $\frac{1}{3} < 0.822 < \frac{4}{3}$.

B1 **Answer:** $\ln 2$**. The idea.** Freeze X and Y. Then Z is still uniform on $[0,1]$, and the chance that it lands below the fixed number $X^Y \in [0,1]$ is exactly X^Y. So the probability

is just the average value of x^y over the unit square, and that double integral is easy in the right order.

Work it out. By independence, $P(X^Y > Z) = E\big[P(Z < X^Y \mid X, Y)\big] = E[X^Y] = \int_0^1\int_0^1 x^y\,dx\,dy$. For fixed $y \geq 0$ the inner integral is $\int_0^1 x^y\,dx = \frac{1}{1+y}$. Hence the probability is $\int_0^1 \frac{dy}{1+y} = \ln 2 \approx 0.693$. (The integrand is bounded and nonnegative, so the order of integration may be chosen freely.)

Check. Integrating in the other order gives $\int_0^1 x^y\,dy = \frac{x-1}{\ln x}$, and $\int_0^1 \frac{x-1}{\ln x}\,dx$ is a Frullani-type integral whose value is again $\ln 2$ (numerically $0.6931\ldots$). Sanity: $X^Y \geq X$ on $[0,1]$, so the answer must exceed $P(X > Z) = \frac{1}{2}$, and it does.

B2 Answer: $\frac{89}{300}$. The idea. Put $D = xz - y^2$ and use $|D| = 2D^+ - D$, where $D^+ = \max(D, 0)$. The signed integral of D is immediate. For the positive part, integrate over y first: for fixed x, z the set where $D > 0$ is the interval $0 \leq y < \sqrt{xz}$, which never leaves $[0,1]$ because $xz \leq 1$, so no case split is needed.

Work it out. Signed part: $\iiint D = \int_0^1 x\,dx \int_0^1 z\,dz - \int_0^1 y^2\,dy = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$. Positive part: for fixed $x, z \in [0,1]$, $\int_0^1 D^+\,dy = \int_0^{\sqrt{xz}} (xz - y^2)\,dy = (xz)^{3/2} - \frac{1}{3}(xz)^{3/2} = \frac{2}{3}(xz)^{3/2}$. Then $\iiint D^+ = \frac{2}{3}\int_0^1 x^{3/2}\,dx \int_0^1 z^{3/2}\,dz = \frac{2}{3}\cdot\frac{2}{5}\cdot\frac{2}{5} = \frac{8}{75}$. Hence $\iiint |D| = 2\cdot\frac{8}{75} + \frac{1}{12} = \frac{64}{300} + \frac{25}{300} = \frac{89}{300}$.

Check. Integrate over x first instead. For fixed y, z the zero of $xz - y^2$ is $x_0 = y^2/z$. If $z \leq y^2$ then $x_0 \geq 1$ and $\int_0^1 |D|\,dx = y^2 - \frac{z}{2}$; if $z > y^2$ then $\int_0^1 |D|\,dx = \int_0^{x_0}(y^2 - xz)\,dx + \int_{x_0}^1 (xz - y^2)\,dx = \frac{y^4}{z} + \frac{z}{2} - y^2$. The first region contributes $\int_0^1 \frac{3}{4}y^4\,dy = \frac{3}{20}$. The second contributes $\int_0^1 \big(-2y^4\ln y + \frac{1-y^4}{4} - y^2 + y^4\big)\,dy = \frac{2}{25} + \frac{1}{5} - \frac{2}{15} = \frac{11}{75}$. The total is $\frac{3}{20} + \frac{11}{75} = \frac{45+44}{300} = \frac{89}{300} \approx 0.2967$, and a midpoint-rule grid agrees.

B3 Answer: $\pi - \frac{3\sqrt{3}}{4}$. The idea. The overlap depends only on $d = |OP|$. Two unit disks whose centres are d apart overlap in a lens of area $L(d) = 2\arccos\frac{d}{2} - \frac{d}{2}\sqrt{4-d^2}$, and d has density $2d$ on $[0,1]$ (the chance that $|OP| \leq d$ is d^2). So we need $\int_0^1 2d\,L(d)\,dd$. The substitution $d = 2\cos\theta$ turns the arccos into θ.

Work it out. Lens: the common chord cuts each disk at distance $d/2$ from its centre. Each half of the lens is a circular segment with half-angle $\alpha = \arccos\frac{d}{2}$, of area $\alpha - \sin\alpha\cos\alpha = \arccos\frac{d}{2} - \frac{d}{2}\cdot\frac{\sqrt{4-d^2}}{2}$. Doubling gives $L(d)$. Now put $d = 2\cos\theta$, θ from $\frac{\pi}{2}$ down to $\frac{\pi}{3}$, so $dd = -2\sin\theta\,d\theta$, $\arccos\frac{d}{2} = \theta$ and $\frac{d}{2}\sqrt{4-d^2} = 2\sin\theta\cos\theta$. Then $E = \int_{\pi/3}^{\pi/2} 2(2\cos\theta)\big(2\theta - 2\sin\theta\cos\theta\big)\,2\sin\theta\,d\theta = \int_{\pi/3}^{\pi/2}\big(8\theta\sin 2\theta - 4\sin^2 2\theta\big)\,d\theta$. By parts, $\int 8\theta\sin 2\theta\,d\theta = -4\theta\cos 2\theta + 2\sin 2\theta$, which from $\frac{\pi}{3}$ to $\frac{\pi}{2}$ gives $(2\pi + 0) - \big(\frac{2\pi}{3} + \sqrt{3}\big) = \frac{4\pi}{3} - \sqrt{3}$. And $\int 4\sin^2 2\theta\,d\theta = 2\theta - \frac{\sin 4\theta}{2}$, which over the same range gives $\big(\pi - 0\big) - \big(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\big) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$. So $E = \frac{4\pi}{3} - \sqrt{3} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} = \pi - \frac{3\sqrt{3}}{4} \approx 1.8426$.

Check. Compute the lens by vertical strips instead: $L(d) = 4\int_{d/2}^1 \sqrt{1-x^2}\,dx$. Swapping the order of integration, $E = \int_0^{1/2} 16x^2\sqrt{1-x^2}\,dx + \int_{1/2}^1 4\sqrt{1-x^2}\,dx$, which evaluates to the same $\pi - \frac{3\sqrt{3}}{4}$. Sanity: the answer lies between $L(1) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.23$ and $L(0) = \pi$.

B4 Answer: 41472. The idea. Sort the twelve roots by order $d \in \{1, 2, 3, 4, 6, 12\}$; the roots of order d are exactly the roots of the cyclotomic polynomial Φ_d. For fixed ζ of

order d, the product of $\zeta-\eta$ over all η of order e is $\Phi_e(\zeta)$, because Φ_e is monic with those roots. So $P=\prod_{d<e} R(d,e)$ with $R(d,e)=\prod_{\zeta \text{ of order } d}\Phi_e(\zeta)$, fifteen small evaluations, each made easy by the relation $\Phi_d(\zeta)=0$.

Work it out. The polynomials are $\Phi_1=x-1$, $\Phi_2=x+1$, $\Phi_3=x^2+x+1$, $\Phi_4=x^2+1$, $\Phi_6=x^2-x+1$, $\Phi_{12}=x^4-x^2+1$. Order 1 ($\zeta=1$): $\Phi_2(1)\Phi_3(1)\Phi_4(1)\Phi_6(1)\Phi_{12}(1)=2\cdot3\cdot2\cdot1\cdot1=12$. Order 2 ($\zeta=-1$): $\Phi_3(-1)\Phi_4(-1)\Phi_6(-1)\Phi_{12}(-1)=1\cdot2\cdot3\cdot1=6$. Order 3 ($\zeta=\omega,\bar\omega$ with $\omega^2=-\omega-1$): $\Phi_4(\omega)=\omega^2+1=-\omega$, $\Phi_6(\omega)=\omega^2-\omega+1=-2\omega$, $\Phi_{12}(\omega)=\omega-\omega^2+1=-2\omega^2$. Multiplying each value by its conjugate (use $\omega\bar\omega=1$) gives 1, 4, 4, so order 3 contributes 16. Order 4 ($\zeta=\pm i$): $\Phi_6(i)=-i$ and $\Phi_6(-i)=i$ multiply to 1, and $\Phi_{12}(\pm i)=1+1+1=3$, so order 4 contributes 9. Order 6 ($\zeta^2=\zeta-1$, hence $\zeta^3=-1$ and $\zeta^4=-\zeta$): $\Phi_{12}(\zeta)=-\zeta-(\zeta-1)+1=2(1-\zeta)$, and over the two conjugate roots this gives $4(1-\zeta)(1-\bar\zeta)=4\,\Phi_6(1)=4$. Therefore $P=12\cdot6\cdot16\cdot9\cdot4=41472=2^9\cdot3^4$.

Check. Size, by a different route. Over all 132 ordered pairs of distinct 12th roots, $\prod_{\zeta\neq\eta}(\zeta-\eta)=\prod_\zeta f'(\zeta)$ for $f=x^{12}-1$, which has absolute value 12^{12}. The pairs with order of η less than order of ζ give $\pm P$ again, and the pairs of equal order give, for each d, $\pm\operatorname{disc}\Phi_d$, where $|\operatorname{disc}\Phi_d|=1,1,3,4,3,144$ for $d=1,2,3,4,6,12$. So $|P|^2=12^{12}/5184=2^{18}3^8$ and $|P|=2^9\cdot3^4=41472$, matching. The sign is + because every one of the fifteen partial products above came out positive. (Each $R(d,e)$ is either 1 or a power of the prime p with $e/d=p^k$, e.g. $R(4,12)=3^{\varphi(4)}=9$; a numerical multiplication of all 57 factors also returns 41472.)

B5 Answer: $\ln\frac{3+\sqrt5}{2}$. **The idea.** T_n is a Gram matrix. With $\varphi=\frac{1+\sqrt5}{2}$ and $q(z)=\varphi-z-\varphi^{-1}z^2=(1-z)(\varphi+\varphi^{-1}z)$, the autocorrelations of the coefficient list $(c_0,c_1,c_2)=(\varphi,-1,-\varphi^{-1})$ are $c_0^2+c_1^2+c_2^2=\varphi^2+1+\varphi^{-2}=4$, then $c_0c_1+c_1c_2=-\varphi+\varphi^{-1}=-1$, then $c_0c_2=-1$. So $T_n=C^{\mathsf T}C$ for an $(n+2)\times n$ banded matrix C. Most of C is a triangular matrix with diagonal φ, which gives $\det T_n=\varphi^{2n}$ times a factor that grows only polynomially. (The Szegő average $\frac{1}{2\pi}\int\ln(4-2\cos\theta-2\cos2\theta)\,d\theta$ predicts the same thing. The symbol vanishes at $\theta=0$, which is why a careful argument is needed.)

Work it out. Let C be the $(n+2)\times n$ matrix with $C_{kj}=c_{k-j}$ (rows $k=0,\dots,n+1$, columns $j=0,\dots,n-1$, and $c_m=0$ unless $m\in\{0,1,2\}$). Every product $c_{k-i}c_{k-j}$ with $k-i,k-j\in\{0,1,2\}$ occurs for some row $k\le n+1$, so $(C^{\mathsf T}C)_{ij}=\sum_m c_m c_{m+|i-j|}$, which is $4,-1,-1,0$ for $|i-j|=0,1,2,\ge3$. Hence $T_n=C^{\mathsf T}C$. Split C into its top n rows A (lower triangular Toeplitz, diagonal φ, so $\det A=\varphi^n$) and its last two rows R. Then $T_n=A^{\mathsf T}A+R^{\mathsf T}R$, and by the matrix determinant lemma $\det T_n=\varphi^{2n}\det(I_2+GG^{\mathsf T})$ with $G=RA^{-1}$, a $2\times n$ matrix. Lower bound: $GG^{\mathsf T}$ is positive semidefinite, so $\det(I_2+GG^{\mathsf T})\ge1$. Upper bound: A^{-1} is the lower triangular Toeplitz matrix of the power series $\frac{1}{q(z)}=\frac{1}{1-z}\cdot\frac{\varphi^{-1}}{1+\varphi^{-2}z}=\sum_k h_kz^k$, where $|h_k|\le\sum_{j\ge0}\varphi^{-1}\varphi^{-2j}=\frac{\varphi^{-1}}{1-\varphi^{-2}}=1$ (using $\varphi^2-1=\varphi$). The rows of R are $(0,\dots,0,-\varphi^{-1},-1)$ and $(0,\dots,0,-\varphi^{-1})$, so each row of G combines at most two rows of A^{-1} with coefficients of size at most 1. Every entry of G is therefore at most 2 in absolute value, and each row g has $\|g\|^2\le4n$. By Hadamard's inequality for positive definite matrices, $\det(I_2+GG^{\mathsf T})\le(1+4n)^2$. So $\varphi^{2n}\le\det T_n\le(1+4n)^2\varphi^{2n}$. Taking logarithms and dividing by n, the limit is $2\ln\varphi=\ln\varphi^2=\ln\frac{3+\sqrt5}{2}\approx0.9624$.

Check. The determinants run $4,15,50,160,494,1491,\dots$. They satisfy a linear recurrence with characteristic polynomial $(x+1)(x^2-3x+1)^2$, so $\det T_n\approx(an+b)\varphi^{2n}$ with a double root at $\varphi^2\approx2.618$. For example $\det T_6/\det T_5=1491/494\approx3.02$, and the ratio

drifts down toward 2.618 with an error of order $\frac{1}{n}$. The trap to avoid is the one-root heuristic "$4-1-1-1-1=0$": the symbol vanishes at $\theta=0$, yet the growth rate is φ^2, not 1.

B6 **Answer:** $9-8\ln 2$. **The idea.** Self-similarity: $S=V_1(1+S')$ where $S'=V_2+V_2V_3+\cdots$ has the same law as S and is independent of V_1. Conditioning on S' gives an explicit density for S in terms of the law of S itself. That density equation is a delay equation. On $[0,1]$ it forces the density to be a multiple of x, and one more step determines it on $[1,2]$. The unknown normalising constant cancels in the ratio.

Work it out. V has density $2v$ on $[0,1]$, since $P(V\le v)=v^2$. Also $E[S]=\sum_{k\ge1}(2/3)^k=2$, so $S<\infty$ almost surely. For any Borel set $B\subseteq(0,\infty)$, conditioning on S' and substituting $x=v(1+S')$ gives $P(S\in B)=E\Big[\int_0^1 1_B\big(v(1+S')\big)\,2v\,dv\Big]=\int_B E\Big[\frac{2x}{(1+S')^2}\,1\{S'>x-1\}\Big]dx$. So S has the density $f(x)=2x\,E\big[(1+S)^{-2};\,S>x-1\big]$. For $0<x\le1$ the event $S>x-1$ is certain, so $f(x)=2cx$ with $c=E[(1+S)^{-2}]>0$, and $P(S\le1)=c$. For $1\le x\le2$ we have $0\le x-1\le1$, where f is already known: $f(x)=2x\Big(c-\int_0^{x-1}\frac{2cy}{(1+y)^2}\,dy\Big)$. Since $\int_0^t\frac{y\,dy}{(1+y)^2}=\ln(1+t)+\frac{1}{1+t}-1$, putting $t=x-1$ gives $f(x)=2cx\big(3-2\ln x-\frac{2}{x}\big)=2c\,(3x-2x\ln x-2)$. This is positive on $[1,2]$ and continuous at $x=1$, where it equals $2c$. Then $\int_1^2(3x-2x\ln x-2)\,dx=\frac{9}{2}-2\big(2\ln2-\frac{3}{4}\big)-2=4-4\ln2$, so $P(1<S\le2)=c(8-8\ln2)$. Therefore $\frac{P(S\le2)}{P(S\le1)}=\frac{c+c(8-8\ln2)}{c}=9-8\ln2\approx3.4548$.

Check. Redo it with the distribution function and no densities: $F(x)=P(S\le x)=E\big[\min\big(1,\frac{x}{1+S}\big)^2\big]$. For $x\le1$ this is cx^2. For $1\le x\le2$ it is $F(x-1)+x^2\Big(c-\int_0^{x-1}\frac{dF(y)}{(1+y)^2}\Big)$, and at $x=2$ this gives $F(2)=c(9-8\ln2)$ again. Numerically, solving the fixed-point equation $F(s)=E\big[F\big(\frac{s}{V}-1\big)\big]$ on a fine grid by plain iteration gives $P(S\le1)\approx0.15762$, $P(S\le2)\approx0.54454$, and a ratio $3.4548225555\ldots$, agreeing with $9-8\ln2$ to ten digits; a simulation gives the same to three digits. (With plain uniforms instead of maxima, the same method gives the Dickman-function ratio $3-2\ln2$.)