MathCastle

STATE COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

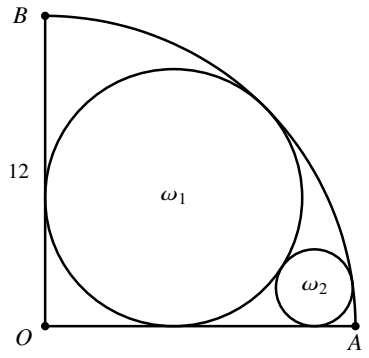
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. At 8:00 a.m. Priya's phone battery reads 100%. While she uses the phone, the charge falls at a constant rate, and at 11:30 a.m. it reads 58%. She then plugs the phone in: the charger adds 1 percentage point every 54 seconds until the battery reaches 100%, and after that the battery simply stays at 100% while plugged in. She unplugs it at 12:15 p.m. and starts using it again. For the first hour after unplugging she streams video, which drains the battery 1.5 times as fast as her morning use; after that hour her use drains it at exactly the morning rate. How many minutes after 12:00 noon does the battery first read 20%?

1. _____

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2. A quarter circle has center O, radius 12, and bounding radii $\overline{OA}$ and $\overline{OB}$. Circle ω_1 lies inside the quarter circle and is tangent to $\overline{OA}$, to $\overline{OB}$ and to arc AB. Circle ω_2 lies in the small region near A that is bounded by ω_1, radius $\overline{OA}$ and arc AB, and it is tangent to all three. What is the radius of ω_2? Express your answer in simplest radical form.



2. _____

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3. The number 99,999,999 (eight nines) has exactly five distinct prime factors. What is the sum of these five primes? (A calculator may be used, but it has no factoring key.)

3. _____

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4. Consider the thirty terms $\dfrac{k \cdot 2^k}{(k+1)(k+2)}$ for $k = 1, 2, 3, \ldots, 30$. The first few are $\frac{2}{6}$, $\frac{8}{12}$, $\frac{24}{20}$. What is the exact value of the sum of all thirty terms?

4. _____

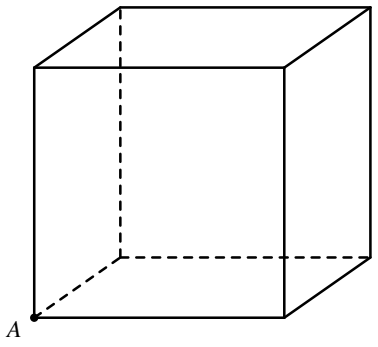
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5. A bike lock has three dials, and each dial shows one of the digits 0 through 9 arranged in a circle, so that 9 is next to 0. One click turns one dial by one digit, in either direction (for example, one click moves a dial from 9 to 0 or from 9 to 8). The lock opens at the combination 3-8-1. Of the 1000 possible settings of the lock, how many can be turned to 3-8-1 using at most 5 clicks in total? (The setting 3-8-1 itself counts; it needs 0 clicks.)

5. _____

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6. A bug starts at vertex A of a cube. Every minute it walks along one edge to an adjacent vertex. On its first move it chooses one of the three edges at A uniformly at random. After that, it never walks straight back along the edge it has just used: at each vertex it chooses uniformly at random between the two other edges there. What is the expected number of minutes until the bug first reaches the vertex of the cube opposite A?



6. _____

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7. A clock has an hour hand, a minute hand and a second hand, and all three move continuously at constant speeds. At exactly 3:00:00 the minute and second hands both point at 12. How many seconds after 3:00:00 does the second hand first point exactly along the bisector of the smaller angle formed by the hour hand and the minute hand? Express your answer as a common fraction.

7. _____

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8. A regular tetrahedron has edge length 12. Four congruent spheres are placed inside it, one at each vertex: each is tangent to the three faces that meet at its vertex, and each is tangent to the other three spheres. Then on each face a small sphere is placed inside the tetrahedron, tangent to that face and tangent to the three vertex spheres that touch that face. What is the radius of one of these small spheres? Express your answer in simplest radical form.

8. _____

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Answer Key — MathCounts State Target Practice Contest 3

1. 385

2. $\frac{60\sqrt{2}-12}{49}$

3. 325

4. 67108863

5. 228

6. 6

7. $\frac{10800}{1427}$

8. $\frac{2\sqrt{6}-2}{5}$

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Solutions — MathCounts State Target Practice Contest 3

1. Answer: 385**. The idea.** Everything is linear, piece by piece, so find the morning drain rate, then follow the battery through each stage. The one trap is the charger: 45 minutes of charging would add more than the 42 points the battery is missing, so the battery hits the 100% cap and waits there.

Work it out. From 8:00 to 11:30 is 210 minutes and the battery lost 42 points, so the morning rate is $\frac{42}{210} = 0.2$ points per minute. Charging: 42 points take $42 \times 54 = 2268$ seconds = 37.8 minutes, less than the 45 minutes from 11:30 to 12:15, so at 12:15 the battery reads exactly 100% (not $100 + \frac{2700}{54} - 42 = 108$). Streaming for 60 minutes at $1.5 \times 0.2 = 0.3$ points per minute uses 18 points, leaving 82% at 1:15 p.m. Falling from 82% to 20% at 0.2 points per minute takes $\frac{62}{0.2} = 310$ minutes. So the battery first reads 20% at $75 + 310 = 385$ minutes after noon (6:25 p.m.).

Check. Count the charge used after unplugging: 60 minutes of streaming is 18 points and 310 minutes of normal use is 62 points, a total of 80 points, exactly the drop from 100% to 20%. And $15 + 60 + 310 = 385$ minutes after noon. Ignoring the cap would start the afternoon at 108% and give 425 minutes, which is why the cap matters.

2. Answer: $\frac{60\sqrt{2}-12}{49}$**. The idea.** Two circles tangent to the same line and to each other have their touch points on the line $2\sqrt{Rr}$ apart (the centers differ by $R - r$ vertically and by $R + r$ in total, so the horizontal gap is $\sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$). That turns the tangency with ω_1 into a clean equation; tangency with the arc gives a second one. Writing $r = c\,t^2$, where c is the radius of ω_1, makes everything rational in t.

Work it out. Put O at the origin with A on the positive x-axis. ω_1 has center (c, c) and $c\sqrt{2} + c = 12$, so $c = 12(\sqrt{2} - 1)$. Let ω_2 have radius r and center (x, r) with $x > c$. Tangent to ω_1 (both touch the x-axis): $x = c + 2\sqrt{cr}$. Internally tangent to the arc: $x^2 + r^2 = (12 - r)^2$, so $x^2 = 144 - 24r$. Put $r = ct^2$, so $x = c(1 + 2t)$, and divide by c^2: $(1 + 2t)^2 = \frac{144}{c^2} - \frac{24}{c}t^2$. Here $\frac{12}{c} = \sqrt{2} + 1$, so $\frac{144}{c^2} = 3 + 2\sqrt{2}$ and $\frac{24}{c} = 2\sqrt{2} + 2$. Expanding: $(3 + \sqrt{2})t^2 + 2t - (1 + \sqrt{2}) = 0$. The discriminant is $4 + 4(3 + \sqrt{2})(1 + \sqrt{2}) = 24 + 16\sqrt{2} = 4(2 + \sqrt{2})^2$, so $t = \frac{-2+2(2+\sqrt{2})}{2(3+\sqrt{2})} = \frac{1+\sqrt{2}}{3+\sqrt{2}} = \frac{1+2\sqrt{2}}{7}$. Then $t^2 = \frac{9+4\sqrt{2}}{49}$ and $r = 12(\sqrt{2} - 1) \cdot \frac{9+4\sqrt{2}}{49} = \frac{12(5\sqrt{2}-1)}{49} = \frac{60\sqrt{2}-12}{49}$.

Check. Numerically $c \approx 4.971$, $r \approx 1.487$, $x = c + 2\sqrt{cr} \approx 10.408$. Then $\sqrt{x^2 + r^2} \approx 10.513 = 12 - r$ (on the arc) and the distance between centers is $\sqrt{(10.408 - 4.971)^2 + (1.487 - 4.971)^2} \approx 6.457 = c + r$. Also $x + r < 12$ and $x > c$, so ω_2 really sits in the corner near A.

3. Answer: 325**. The idea.** $99{,}999{,}999 = 10^8 - 1$, and a difference of squares splits it twice: $10^8 - 1 = (10^4 - 1)(10^4 + 1) = (10^2 - 1)(10^2 + 1)(10^4 + 1)$. The small pieces factor at sight; only 10001 needs a short trial-division search.

Work it out. $10^2 - 1 = 99 = 3^2 \cdot 11$ and $10^2 + 1 = 101$, which is prime (not divisible by 2, 3, 5, 7, and $11^2 > 101$). For 10001: it is odd, its digit sum is 2, it does not end in 0 or 5, and dividing by $7, 11, 13, \ldots$ with the calculator gives the first hit at $10001 = 73 \times 137$. Both 73 and 137 are prime (137 has no prime factor up to 11, and $13^2 > 137$). So $99{,}999{,}999 = 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137$, and the sum of the distinct primes is $3 + 11 + 73 + 101 + 137 = 325$.

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Check. Multiply back: $9 \cdot 11 = 99$, $99 \cdot 101 = 9999$, $73 \cdot 137 = 10001$, and $9999 \cdot 10001 = 99{,}999{,}999$. Exactly five distinct primes appear, as the statement says.

4. Answer: 67108863**. The idea.** Adding thirty fractions by calculator is slow and loses precision. Instead look for a telescoping form: the terms involve 2^k with denominators $k+1$ and $k+2$, which suggests comparing $\frac{2^{k+1}}{k+2}$ with $\frac{2^k}{k+1}$.

Work it out. Let $f(k) = \frac{2^k}{k+1}$. Then $f(k+1) - f(k) = \frac{2^{k+1}}{k+2} - \frac{2^k}{k+1} = \frac{2^k(2(k+1)-(k+2))}{(k+1)(k+2)} = \frac{k \cdot 2^k}{(k+1)(k+2)}$, which is exactly the k-th term. So the sum telescopes: $\sum_{k=1}^{30} (f(k+1) - f(k)) = f(31) - f(1) = \frac{2^{31}}{32} - \frac{2}{2} = 2^{26} - 1 = 67{,}108{,}863$.

Check. The same identity for the first three terms gives $f(4) - f(1) = \frac{16}{5} - 1 = \frac{11}{5}$, and directly $\frac{1}{3} + \frac{2}{3} + \frac{6}{5} = 1 + \frac{6}{5} = \frac{11}{5}$. The formula also explains why the answer is an integer: $32 = 2^5$ divides 2^{31}.

5. Answer: 228**. The idea.** The dials are independent. A dial whose digit is d steps around the circle from its target (with $0 \le d \le 5$) needs exactly d clicks, and the fewest clicks for a setting is the sum of the three distances. For each dial, distance 0 occurs for 1 digit, distances $1, 2, 3, 4$ each occur for 2 digits (one on each side), and distance 5 occurs for just 1 digit (the one directly opposite). So count the ordered triples of distances with sum at most 5, weighting each distance by its number of digits.

Work it out. Let N_s be the number of settings needing exactly s clicks. $N_0 = 1$. N_1: one dial at distance 1: $3 \cdot 2 = 6$. N_2: one dial at 2 ($3 \cdot 2 = 6$) or two dials at 1 ($3 \cdot 4 = 12$): 18. N_3: (3): 6; $(2,1)$: $6 \cdot 4 = 24$; $(1,1,1)$: 8; total 38. N_4: (4): 6; $(3,1)$: $6 \cdot 4 = 24$; $(2,2)$: $3 \cdot 4 = 12$; $(2,1,1)$: $3 \cdot 8 = 24$; total 66. N_5: (5): $3 \cdot 1 = 3$; $(4,1)$: $6 \cdot 4 = 24$; $(3,2)$: $6 \cdot 4 = 24$; $(3,1,1)$: $3 \cdot 8 = 24$; $(2,2,1)$: $3 \cdot 8 = 24$; total 99. (Here 6 counts the ordered choices of two different dials and 3 counts the ways to pick the odd dial out.) The answer is $1 + 6 + 18 + 38 + 66 + 99 = 228$.

Check. Replacing each dial's distance d by $5 - d$ keeps the weights $1, 2, 2, 2, 2, 1$ in the same order and turns a click total s into $15 - s$, so $N_s = N_{15-s}$. Continuing the count gives $N_6 = 128$ and $N_7 = 144$, and then $1 + 6 + 18 + 38 + 66 + 99 + 128 + 144 = 500$, exactly half of the 1000 settings, as this symmetry requires.

6. Answer: 6**. The idea.** Group vertices by their distance from A (0, 1, 2 or 3 edges). Because the bug remembers where it came from, a state must record both its distance and the distance of the vertex it just left. By symmetry only a few such states occur, and first-step analysis gives a small linear system.

Work it out. A vertex at distance 1 has one neighbour at distance 0 and two at distance 2; a vertex at distance 2 has two neighbours at distance 1 and one at distance 3 (the target). Let a = expected remaining time at distance 1 having come from distance 0; b = at distance 2 having come from distance 1; d = at distance 1 having come from distance 2; e = at A having come from distance 1. From a the bug cannot go back, so it moves to distance 2: $a = 1 + b$. From b it cannot return along its edge, so it chooses between the other distance-1 vertex and the target: $b = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} d$. From d its two choices are A and the other distance-2 vertex: $d = 1 + \frac{1}{2} e + \frac{1}{2} b$. From e it must go to a new distance-1 vertex: $e = 1 + a$. Substituting $e = 1 + a$ and $b = a - 1$ into the third equation gives $d = 1 + \frac{1+a}{2} + \frac{a-1}{2} = 1 + a$; the second equation gives $d = 2(b-1) = 2a - 4$. So $2a - 4 = 1 + a$, $a = 5$, and the answer is $1 + a = 6$.

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Check. From state b the walk finishes with probability $\frac{1}{2}$ each time, and a failed attempt brings it back to state b after 4 minutes if it goes through A ($b \to d \to e \to a \to b$, probability $\frac{1}{2}$) or after 2 minutes if it goes across ($b \to d \to b$, probability $\frac{1}{2}$), so a failure costs 3 minutes on average. The expected number of failures is 1, so the expected time is 2 (to reach b) +3 (failures) +1 (the winning step) = 6. For comparison, the ordinary random walk that may turn back takes 10 minutes on average.

7. Answer: $\frac{10800}{1427}$. **The idea.** Measure every hand's position in degrees clockwise from 12 as a linear function of the time t in seconds. While the minute hand is still behind the hour hand, the bisector of the smaller angle is simply the average of the two positions, so the question becomes one linear equation.

Work it out. The hour hand turns 30° per hour, which is $\frac{1}{120}$ degree per second; the minute hand turns 360° per hour, $\frac{1}{10}$ degree per second; the second hand turns 6° per second. At time t: hour hand at $90 + \frac{t}{120}$, minute hand at $\frac{t}{10}$, second hand at $6t$. The angle between the first two is less than 180°, so its bisector is at $\frac{1}{2}\left(90 + \frac{t}{120} + \frac{t}{10}\right) = 45 + \frac{13t}{240}$. Setting $6t = 45 + \frac{13t}{240}$ gives $\frac{1427t}{240} = 45$, so $t = \frac{10800}{1427}$ seconds.

Check. $t \approx 7.568$ s: the hour hand is at about 90.063°, the minute hand at about 0.757°, their smaller angle is about 89.31° with bisector at about 45.41°, and the second hand is at $6t \approx 45.41°$. Before this moment the second hand is behind the bisector (at $t = 0$ it is at 0° while the bisector is at 45°), so this is the first time.

8. Answer: $\frac{2\sqrt{6}-2}{5}$. **The idea.** A sphere tangent to the three faces at a vertex V has its center on the segment from V to the center G of the tetrahedron, and its distance from those faces grows in proportion to its distance from V. At G the distance to the faces is the inradius while VG is the circumradius, three times as long, so a vertex sphere of radius r has its center $3r$ from V. The four centers then form a smaller regular tetrahedron, which fixes r. Finally, the three vertex spheres on one face all rest on that face, so their centers lie at height r above it, forming an equilateral triangle of side $2r$; a sphere resting on the face in the middle is found by a 2D Pythagoras.

Work it out. For edge 12 the height is $12\sqrt{2/3} = 4\sqrt{6}$, the inradius is $\frac{1}{4}$ of that, $\sqrt{6}$, and the circumradius is $GV = 3\sqrt{6}$. The vertex-sphere centers are the images of the vertices under the scaling about G with ratio $\frac{3\sqrt{6}-3r}{3\sqrt{6}} = 1 - \frac{r}{\sqrt{6}}$, so they form a regular tetrahedron of edge $12 - \frac{12r}{\sqrt{6}} = 12 - 2\sqrt{6}\,r$. Adjacent spheres touch, so this edge equals $2r$: $12 = r(2 + 2\sqrt{6})$ and $r = \frac{6}{1+\sqrt{6}} = \frac{6(\sqrt{6}-1)}{5}$. Now fix one face. The three vertex spheres at its corners are tangent to it, so their centers are at height r above it; they are $2r$ apart, and by symmetry they sit above the face's center, at horizontal distance $\frac{2r}{\sqrt{3}}$ from it. The small sphere has radius ρ and center at height ρ directly above the face's center. Tangency with a vertex sphere: $\left(\frac{2r}{\sqrt{3}}\right)^2 + (r-\rho)^2 = (r+\rho)^2$, so $\frac{4r^2}{3} = 4r\rho$ and $\rho = \frac{r}{3} = \frac{2(\sqrt{6}-1)}{5} = \frac{2\sqrt{6}-2}{5}$.

Check. Numerically $r \approx 1.739$ and $\rho \approx 0.580$. Check the vertex spheres against the face directly: the corner centers lie $\frac{2r}{\sqrt{3}} \approx 2.008$ from the face center horizontally, while the face's vertices are $4\sqrt{3} \approx 6.928$ away; the ratio $\frac{2.008}{6.928} \approx 0.290 = 1 - \frac{r}{\sqrt{6}}$, consistent with the scaling. The small sphere reaches only to height $2\rho \approx 1.16$, far below the center of the tetrahedron at height $\sqrt{6} \approx 2.45$, and it clears the other three faces, so it really fits in the pocket.

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