MathCastle

STATE COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. On a test, the boys in a class averaged 72 points and the girls averaged 84 points, and the whole class averaged 79.2 points. Later, 10 new boys joined the class and took the same test; each of them scored 90. With the new boys included, the class average became 80.4 points. How many girls are in the class?

1. _____

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2. Each of 20 students joins exactly one of 5 clubs. Each student chooses independently, and each club is equally likely to be chosen. What is the expected number of clubs that end up with exactly 4 members? Express your answer as a decimal to the nearest hundredth.

2. _____

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3. A cube is inscribed in a sphere of radius 5, so that all 8 of its vertices lie on the sphere. A smaller cube rests on top of it: the bottom face of the small cube lies on the top face of the large cube, the two cubes have parallel edges, the centre of the small cube's bottom face is the centre of the large cube's top face, and the 4 top vertices of the small cube lie on the sphere. What is the edge length of the small cube? Express your answer in simplest radical form.

3. _____

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4. A positive integer n is called *lucky* if $n^2 + 1$ is a multiple of 1105. For example, $n = 47$ is lucky, because $47^2 + 1 = 2210 = 2 \cdot 1105$, but its neighbour $n = 48$ is not, because $48^2 + 1 = 2305$ leaves a remainder of 95 when divided by 1105. How many lucky integers n satisfy $1 \le n \le 10000$?

4. _____

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5. A standard 52-card deck (13 ranks in each of 4 suits, with one ace per suit) is shuffled so that all orders are equally likely, and the cards are turned over one at a time. What is the probability that the first ace to appear comes strictly before the first spade to appear? (The ace of spades is both an ace and a spade; if it is the first of these cards to appear, the first ace and the first spade appear together, which does not count.) Express your answer as a common fraction.

5. _____

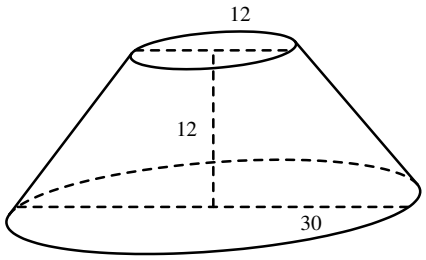
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6. A wire 60 cm long is cut into three pieces of positive length (the lengths may be chosen freely). One piece is bent into an equilateral triangle, one into a square and one into a circle. What is the least possible total area, in square centimetres, enclosed by the three shapes? Express your answer as a decimal to the nearest hundredth.

6. _____

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7. A lampshade is the curved side of a frustum (a cone with its tip cut off by a plane parallel to the base), with no top or bottom. Its top rim is a circle of diameter 12 cm, its bottom rim is a circle of diameter 30 cm, and the vertical distance between the two rims is 12 cm. The shade is cut along one straight seam running from the top rim to the bottom rim and laid flat without stretching. The flat piece is part of a ring between two concentric circles, cut off by two radii of those circles. How many degrees are in the angle between those two radii?



7. _____

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8. A positive integer is called *climbing* if its digits never decrease from left to right; for example 1137, 55 and 8 are climbing, but 1021 is not. How many climbing positive integers less than 100,000,000 are divisible by 99?

8. _____

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Answer Key — MathCounts State Target Practice Contest 2

1. 48

2. 1.09

3. $\frac{10\sqrt{3}}{9}$

4. 73

5. $\frac{3}{16}$

6. 72.95

7. 216

8. 80

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Solutions — MathCounts State Target Practice Contest 2

1. **Answer:** 48. **The idea.** A class average is a weighted average, so it tells us the *ratio* of girls to boys, not the counts. The second average then pins down the actual size.

   Work it out. Let there be b boys and g girls. From $72b + 84g = 79.2(b + g)$ we get $4.8g = 7.2b$, so $g = 1.5b$. (The class average 79.2 sits 7.2 above the boys and 4.8 below the girls, and the heavier group is the one closer to the average.) The total score is then $72b + 84(1.5b) = 198b$ and the class has $2.5b$ students. After the new boys join, $\frac{198b+900}{2.5b+10} = 80.4$, so $198b + 900 = 201b + 804$, giving $3b = 96$, $b = 32$ and $g = 48$.

   Check. The original class: $32 \cdot 72 + 48 \cdot 84 = 2304 + 4032 = 6336 = 79.2 \cdot 80$. The new class: $6336 + 900 = 7236 = 80.4 \cdot 90$. The number of girls is 48.

2. **Answer:** 1.09. **The idea.** The club sizes depend on each other (they add to 20), which makes the exact distribution of "how many clubs have 4 members" messy. But the expected value does not need that distribution: by linearity of expectation it is the sum, over the 5 clubs, of the probability that a given club has exactly 4 members, and that holds even though the events depend on each other.

   Work it out. Fix one club. Each student joins it with probability $\frac{1}{5}$, independently, so its size is binomial: the chance of exactly 4 members is $\binom{20}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^{16}$. Now $\binom{20}{4} = 4845$, $\left(\frac{1}{5}\right)^4 = 0.0016$ and $0.8^{16} \approx 0.0281475$, so the probability is about $4845 \cdot 0.0016 \cdot 0.0281475 \approx 0.21820$. Multiply by the 5 clubs: the expected number is about 1.0910, which is 1.09 to the nearest hundredth.

   Check. The expected size of a club is 4, and a binomial with mean 4 is most likely to be exactly 4 (about 22% of the time), so about $5 \cdot 0.22 \approx 1.1$ clubs of size exactly 4 is sensible. A direct summation over all $\binom{24}{4} = 10626$ size vectors, weighted by multinomial probabilities, gives the same 1.0910.

3. **Answer:** $\frac{10\sqrt{3}}{9}$. **The idea.** Put the centre of the sphere at the origin. Every vertex on the sphere is at distance 5 from it, and for a vertex we know two things: how far it is from the vertical axis and how high it is. One Pythagorean equation in the small cube's edge follows.

   Work it out. The large cube's space diagonal is a diameter: if its edge is a then $a\sqrt{3} = 10$, so $a = \frac{10}{\sqrt{3}}$ and its top face is at height $\frac{a}{2} = \frac{5}{\sqrt{3}}$. Let the small cube have edge s. A top vertex of it is at height $\frac{5}{\sqrt{3}} + s$, and at horizontal distance $\frac{s}{\sqrt{2}}$ from the vertical axis (half the diagonal of a face). So $\frac{s^2}{2} + \left(\frac{5}{\sqrt{3}} + s\right)^2 = 25$, that is $\frac{3}{2}s^2 + \frac{10}{\sqrt{3}}s - \frac{50}{3} = 0$. Multiply by 6: $9s^2 + 20\sqrt{3}\,s - 100 = 0$. The quadratic formula gives $s = \frac{-20\sqrt{3}+\sqrt{1200+3600}}{18} = \frac{-20\sqrt{3}+40\sqrt{3}}{18} = \frac{10\sqrt{3}}{9}$.

   Check. The large edge is $\frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$, so the small cube's edge is exactly one third of it. Directly: with $s = \frac{a}{3}$, a top vertex is at horizontal distance $\frac{a}{3\sqrt{2}}$ and height $\frac{a}{2} + \frac{a}{3} = \frac{5a}{6}$, and $\frac{a^2}{18} + \frac{25a^2}{36} = \frac{27a^2}{36} = \frac{3a^2}{4} = 25$, as it should be.

4. **Answer:** 73. **The idea.** Whether 1105 divides n^2+1 depends only on n modulo 1105. Factoring, $1105 = 5 \cdot 221 = 5 \cdot 13 \cdot 17$, a product of three different primes, so the Chinese Remainder Theorem splits the condition into three small ones, and each prime contributes two square roots of -1.

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Work it out. Modulo 5: $n^2 \equiv -1$ has solutions $n \equiv 2, 3$. Modulo 13: $n \equiv 5, 8$ ($25 = 26 - 1$). Modulo 17: $n \equiv 4, 13$ ($16 = 17 - 1$). Each of the $2 \cdot 2 \cdot 2 = 8$ choices glues, by CRT, into exactly one residue modulo 1105. Working them out (a calculator helps) gives the residues $47, 242, 268, 463, 642, 837, 863, 1058$; they come in pairs adding to 1105 (n and $-n$). Now $10000 = 9 \cdot 1105 + 55$. The numbers 1 to 9945 are 9 complete blocks of 1105 consecutive integers, each containing all 8 residues once: 72 values. The leftover 9946 to 10000 have residues 1 to 55, and only 47 is on the list (that is $n = 9945 + 47 = 9992$). Total: $72 + 1 = 73$.

Check. The example $n = 47$ is on the list and 48 is not, matching the statement. For $n = 9992$: $9992 \equiv 47 \pmod{1105}$, so it works too. And $242^2 + 1 = 58565 = 53 \cdot 1105$.

5. **Answer:** $\frac{3}{16}$. **The idea.** Only the cards that are aces or spades matter; the other 36 cards can be ignored entirely, because removing them from the shuffled order does not change which ace or spade comes first. Among the relevant cards, each is equally likely to be the first one.

   Work it out. The relevant cards are the 13 spades and the 3 aces that are not spades: 16 cards. In a random shuffle, the order of these 16 cards among themselves is uniformly random, so each is equally likely to come first. The first ace strictly beats the first spade exactly when the first of these 16 cards is one of the 3 non-spade aces. (If it is the ace of spades, the two "firsts" tie; if it is any other spade, the spade wins.) The probability is $\frac{3}{16}$.

   Check. A common slip is to count the ace of spades as a win, giving $\frac{4}{16}$; the tie rule excludes it. As a sanity check, compare the chance that the first spade strictly beats the first ace: $\frac{12}{16}$. The three outcomes, ace first, tie, spade first, have probabilities $\frac{3}{16} + \frac{1}{16} + \frac{12}{16} = 1$. A card-by-card recursion over the whole deck gives $\frac{3}{16}$ as well.

6. **Answer:** 72.95. **The idea.** Each shape's area is a constant times the square of its piece's length. Minimising $k_1x^2 + k_2y^2 + k_3z^2$ with $x + y + z$ fixed is a weighted "spread it out" problem: the Cauchy–Schwarz inequality, or simply the balance condition that moving a little wire from one piece to another cannot help, gives the answer $\frac{(x+y+z)^2}{1/k_1+1/k_2+1/k_3}$.

   Work it out. A triangle of perimeter x has side $\frac{x}{3}$ and area $\frac{\sqrt{3}}{4} \cdot \frac{x^2}{9} = \frac{\sqrt{3}}{36}x^2$. A square of perimeter y has area $\frac{y^2}{16}$. A circle of circumference z has radius $\frac{z}{2\pi}$ and area $\frac{z^2}{4\pi}$. So $k_1 = \frac{\sqrt{3}}{36}$, $k_2 = \frac{1}{16}$, $k_3 = \frac{1}{4\pi}$. By Cauchy–Schwarz, $(x+y+z)^2 = \left(\sqrt{k_1}x \cdot \frac{1}{\sqrt{k_1}} + \sqrt{k_2}y \cdot \frac{1}{\sqrt{k_2}} + \sqrt{k_3}z \cdot \frac{1}{\sqrt{k_3}}\right)^2 \le (k_1x^2 + k_2y^2 + k_3z^2)\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$, with equality when each length is proportional to $\frac{1}{k_i}$ (all three pieces positive, as required). Here $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3} = \frac{36}{\sqrt{3}}+16+4\pi = 12\sqrt{3}+16+4\pi \approx 20.7846+16+12.5664 = 49.3510$. The least area is $\frac{3600}{49.3510} \approx 72.947$, or 72.95 to the nearest hundredth.

   Check. At the optimum the pieces are $60 \cdot \frac{20.7846}{49.3510} \approx 25.27$ cm (triangle), ≈ 19.45 cm (square) and ≈ 15.28 cm (circle). Their areas are about 30.72, 23.65 and 18.57, adding to 72.947. Every area has the form $k_i\ell_i^2$ with $2k_i\ell_i$ the same for all three (≈ 2.432), which is exactly the balance condition. Using the whole wire for the circle alone would give $\frac{3600}{4\pi} \approx 286.5$, far more, as expected: splitting helps.

7. **Answer:** 216. **The idea.** Complete the frustum to the full cone it was cut from. Laid flat, a cone's side becomes a sector whose radius is the cone's slant height and whose arc is the base circumference; the flat lampshade is that sector with the flattened small top cone removed, so both have the same angle.

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Work it out. The radii are 6 and 15, which differ by 9, and the rims are 12 apart vertically, so the slant length of the shade is $\sqrt{9^2 + 12^2} = 15$ cm. The top rim is a scaled copy of the bottom rim, with scale factor $\frac{6}{15} = \frac{2}{5}$. So if L is the slant height of the full cone, the removed tip cone has slant height $\frac{2}{5}L$, and $L - \frac{2}{5}L = 15$ gives $L = 25$ (and 10 for the tip). Laid flat, the bottom rim, of length $2\pi \cdot 15 = 30\pi$, becomes an arc of a circle of radius 25, whose full circumference is 50π. The angle is $\frac{30\pi}{50\pi} \cdot 360^\circ = 216^\circ$.

Check. The top rim, of length 12π, lies on the circle of radius 10 (circumference 20π), and $\frac{12\pi}{20\pi} \cdot 360^\circ = 216^\circ$ as well. Areas agree too: the flat piece has area $\frac{216}{360}\pi(25^2 - 10^2) = \frac{3}{5} \cdot 525\pi = 315\pi$, and the frustum's curved area is $\pi(15 + 6) \cdot 15 = 315\pi$.

8. Answer: 80**. The idea.** Divisibility by 99 means divisibility by both 11 and 9. For 11, group the alternating digit sum of a climbing number into consecutive differences: they are never negative and telescope to at most 9, so the alternating sum is trapped in a range whose only multiple of 11 is 0. That forces the digits to come in equal pairs. The condition for 9 then becomes a question about digit multisets modulo 9, which a symmetry (add 1 to every digit, with $9 \to 1$ wrapping around) answers almost for free.

Work it out. A climbing number has no digit 0 (its first digit is at least 1 and later ones are no smaller). Let its digits be $1 \le d_1 \le d_2 \le \cdots \le d_k \le 9$.

The factor 11. If $k = 2m+1$ is odd, the alternating sum is $S = d_1 + (d_3 - d_2) + \cdots + (d_{2m+1} - d_{2m})$, and $1 \le d_1 \le S \le d_1 + (d_2 - d_1) + (d_3 - d_2) + \cdots + (d_{2m+1} - d_{2m}) = d_{2m+1} \le 9$ (we only added non-negative gaps). No multiple of 11 lies in $[1, 9]$, so odd lengths are impossible. If $k = 2m$, then $S = (d_2 - d_1) + (d_4 - d_3) + \cdots + (d_{2m} - d_{2m-1})$ satisfies $0 \le S \le d_{2m} - d_1 \le 8$ in the same way, so $11 \mid S$ forces $S = 0$: $d_1 = d_2$, $d_3 = d_4$, and so on. So the climbing multiples of 11 are exactly the numbers made by writing each digit of a climbing string $e_1 \le \cdots \le e_m$ twice, like 337799; equivalently, a multiset of m digits from 1 to 9.

The factor 9. Such a number has digit sum $2(e_1 + \cdots + e_m)$, divisible by 9 exactly when $e_1 + \cdots + e_m$ is. Below 10^8 we have $k \le 8$, so $m = 1, 2, 3, 4$. Think of the digits $1, \ldots, 9$ as the nine residues modulo 9. Adding 1 to every element of a multiset (with $9 \to 1$) is a reversible shift that changes the sum by m modulo 9. When m is 1, 2 or 4 (coprime to 9), no multiset is fixed by a nonzero power of the shift, so the $\binom{8+m}{m}$ multisets split into groups of 9, and the sums within each group run through all nine residues once. So exactly one ninth have sum divisible by 9: $\frac{9}{9} = 1$ for $m = 1$, $\frac{45}{9} = 5$ for $m = 2$ and $\frac{495}{9} = 55$ for $m = 4$.

For $m = 3$ the shift moves the sum by 3, so it does not mix all residues; count directly the multisets $a \le b \le c$ of digits 1 to 9 with sum 9, 18 or 27. Sum 9: $\{1,1,7\}, \{1,2,6\}, \{1,3,5\}, \{1,4,4\}, \{2,2,5\}, \{2,3,4\}, \{3,$ which is 7. Sum 27: only $\{9,9,9\}$. Sum 18: with $c = 9$, $a + b = 9$ gives 4; $c = 8$, $a + b = 10$ with $b \le 8$ gives 4; $c = 7$, $a + b = 11$ with $b \le 7$ gives 2; $c = 6$ gives only $\{6,6,6\}$; $c \le 5$ is too small. That is 11. So $m = 3$ contributes $7 + 11 + 1 = 19$.

Total: $1 + 5 + 19 + 55 = 80$.

Check. $m = 1$: the only one is 99. $m = 2$: $1188, 2277, 3366, 4455, 9999$, and $1188 = 99 \cdot 12$. $m = 3$: $\{1,3,7\}$ has sum 11, so 113377 fails, while $\{1,3,5\}$ gives $113355 = 99 \cdot 1145$. The $m = 3$ count is consistent with the shift argument: the 165 multisets consist of the one group $\{1,4,7\}, \{2,5,8\}, \{3,6,9\}$ of size 3 plus 18 groups of 9, and residues come in steps of 3, so the answer must be 1 plus a multiple of 3 (indeed $19 = 1 + 6 \cdot 3$). A computer scan of every multiple of 99 below 10^8 finds exactly 80 climbing ones.

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