MathCastle

STATE COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

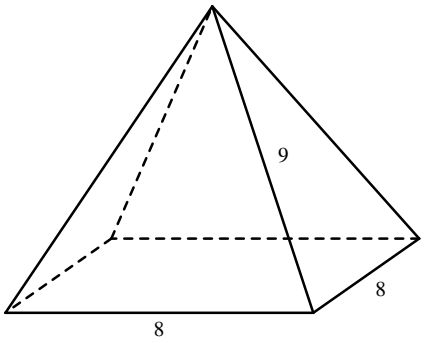
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Maya opens a savings account with a balance of \$0. At the start of every month she deposits \$250. At the end of every month the bank credits interest of 0.4% of the whole balance at that moment (including that month's deposit). At the end of which month, after that month's interest has been credited, does Maya's balance first exceed \$10,000? (The first month is month 1.)

1. _____

---

2. A pyramid has a square base with side length 8, and each of its four slanted edges (from the apex to a base corner) has length 9. A sphere touches all eight edges of the pyramid: each edge is tangent to the sphere at a point of that edge. What is the radius of the sphere? Express your answer as a common fraction in simplest radical form.



2. _____

---

3. A positive integer N has exactly 30 positive divisors. The number $\frac{N}{2}$ is an integer with exactly 25 positive divisors, and the number $3N$ has exactly 36 positive divisors. What is N?

3. _____

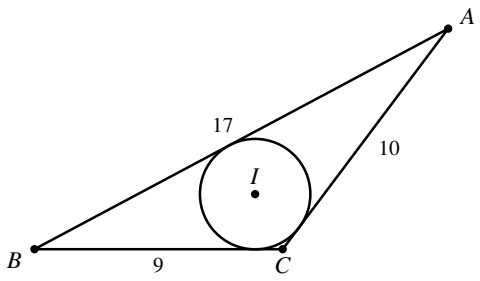
---

4. Five friends each find out the day of the week on which they were born. Assume each friend was equally likely to be born on any of the 7 days of the week, independently of the others. Two days are called neighbouring if one comes right after the other; the week wraps around, so Sunday and Monday are also neighbouring. What is the probability that no two of the five friends were born on neighbouring days? (Two or more friends may share the same day; that is allowed.) Express your answer as a common fraction.

4. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

5. Triangle ABC has side lengths $BC = 9$, $CA = 10$ and $AB = 17$. Let I be the center of the circle inscribed in the triangle. What is the value of the product $IA \cdot IB \cdot IC$?



5. _____

---

6. A rectangular box has edge lengths that are positive real numbers (not necessarily integers). The total length of all twelve of its edges is 48, and its total surface area is 90. Among all such boxes, what is the smallest possible volume?

6. _____

---

7. Twelve seats stand in a row. Five people sit down, choosing 5 of the 12 seats at random so that every set of 5 seats is equally likely. The 7 empty seats form blocks, where a block is a group of consecutive empty seats with no empty seat directly to its left or right (so a lone empty seat between two people is a block). What is the expected number of blocks? Express your answer as a common fraction.

7. _____

---

8. Some cells of a 5×5 grid are marked. A strip is a set of three consecutive cells in one row or in one column; the grid has 30 strips. The marking is called blocking if every strip contains at least one marked cell. Let m be the smallest number of marked cells in a blocking marking. In how many different ways can exactly m cells be marked so that the marking is blocking? (Markings that differ by a rotation or reflection of the grid count as different.)

8. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — MathCounts State Target Practice Contest 1

1. 38

2. $\frac{20\sqrt{2}}{7}$

3. 2592

4. $\frac{211}{2401}$

5. 170

6. 50

7. $\frac{7}{2}$

8. 5

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts State Target Practice Contest 1

1. Answer: 38**. The idea.** Follow each deposit separately. The deposit made at the start of a month earns interest for that month and every later one, so it is multiplied by 1.004 once per month it has spent in the account. The balance is therefore a geometric series, which has a closed form a calculator can handle quickly.

Work it out. After m months the newest deposit has grown to $250(1.004)$, the one before to $250(1.004)^2$, and the first to $250(1.004)^m$. So the balance is $B_m = 250\left(1.004 + 1.004^2 + \cdots + 1.004^m\right) = 250 \cdot 1.004 \cdot \frac{1.004^m - 1}{0.004} = 62750\,(1.004^m - 1)$. We need $62750(1.004^m - 1) > 10000$, that is, $1.004^m > 1 + \frac{10000}{62750} \approx 1.15936$. The deposits alone reach \$10,000 at month 40, so the answer is a little below 40; try nearby values. $1.004^{37} \approx 1.15917$, giving $B_{37} \approx$ \$9,987.96, just short. $1.004^{38} \approx 1.16381$, giving $B_{38} \approx$ \$10,278.91. So the balance first exceeds \$10,000 at the end of month 38.

Check. From month 37 to month 38 the balance should change by one deposit plus interest on everything: $(9987.96 + 250)(1.004) \approx 10278.91$, matching. At month 38 Maya has deposited $38 \cdot 250 =$ \$9,500, so about \$779 of her balance is interest, which is reasonable for about three years at roughly 5% per year.

2. Answer: $\frac{20\sqrt{2}}{7}$**. The idea.** From any point outside a sphere, all tangent segments to the sphere have the same length. Each corner of the pyramid sends tangent segments along the edges that meet there, so we can find how far along each edge the tangency points sit. Then Pythagoras relates the tangent length t, the radius r and the distance d from the sphere's center O to that corner: $d^2 = r^2 + t^2$.

Work it out. By symmetry all four base corners have the same tangent length t. A base edge of length 8 is split by its tangency point into two tangent segments, one from each end, so $2t = 8$ and $t = 4$. A slanted edge of length 9 is split into 4 (from the base corner) and $9 - 4 = 5$ (from the apex).

The half-diagonal of the base is $4\sqrt{2}$, so the height is $\sqrt{81 - 32} = 7$. Put the base center at the origin with the apex at height 7; by symmetry O is on the axis, at height z. Then from a base corner: $r^2 = OA^2 - 4^2 = (32 + z^2) - 16 = 16 + z^2$; from the apex: $r^2 = OV^2 - 5^2 = (7 - z)^2 - 25$. Setting these equal: $16 + z^2 = 49 - 14z + z^2 - 25$, so $14z = 8$ and $z = \frac{4}{7}$. Then $r^2 = 16 + \frac{16}{49} = \frac{800}{49}$ and $r = \frac{20\sqrt{2}}{7}$.

Check. The distance from $O = (0, 0, \frac{4}{7})$ to the base edge line $x = 4$, $z = 0$ is $\sqrt{16 + \frac{16}{49}}$, the same value, and its foot is the edge's midpoint, which is on the edge. On a slanted edge the tangency point is 4 from the base corner and 5 from the apex, both inside the edge. Also $r \approx 4.04$ is larger than z, as it must be: the sphere pokes out below the base, touching the base edges from above and inside.

3. Answer: 2592**. The idea.** Pull out the power of 2: write $N = 2^a m$ with m odd and $a \geq 1$. Halving N only changes the exponent of 2, so comparing $d(N)$ with $d(N/2)$ pins down both a and the number of divisors of m. Then the condition on $3N$ tells us which prime is in m.

Work it out. Divisor counts multiply over the prime powers, so $d(N) = (a + 1)\,d(m) = 30$ and $d(N/2) = a\,d(m) = 25$. Subtracting, $d(m) = 5$, and then $a = 5$. An odd number with exactly 5

divisors must be p^4 for an odd prime p (since 5 is prime, there is a single prime with exponent 4). So $N = 2^5 p^4$.

If $p \neq 3$, then $3N = 2^5 \cdot 3 \cdot p^4$ has $6 \cdot 2 \cdot 5 = 60$ divisors, not 36. If $p = 3$, then $3N = 2^5 \cdot 3^5$ has $6 \cdot 6 = 36$ divisors. So $p = 3$ and $N = 32 \cdot 81 = 2592$.

Check. $2592 = 2^5 \cdot 3^4$ has $6 \cdot 5 = 30$ divisors; $1296 = 2^4 \cdot 3^4$ has $5 \cdot 5 = 25$; $7776 = 2^5 \cdot 3^5$ has 36. All three conditions hold, and the argument shows no other N works.

4. Answer: $\frac{211}{2401}$. **The idea.** Sort the outcomes by the set D of days that actually occur. The condition only concerns D: no two days of D may be neighbours on the 7-day cycle of the week. So count the allowed sets D of each size, and for each one count the ways to hand out the five friends so that every day of D is used.

Work it out. There are $7^5 = 16807$ equally likely outcomes. Allowed sets D: of size 1, there are 7. Of size 2, there are $\binom{7}{2} = 21$ pairs of days minus the 7 neighbouring pairs, so 14. Of size 3, the three gaps between chosen days (going around the week) must each be at least 2 and add up to 7, so they are $2, 2, 3$ in some rotation; such a set is fixed by the day that starts its gap of 3, giving 7 sets (for example Monday, Wednesday, Friday). Size 4 or more would need at least $2 \cdot 4 = 8$ days, so it is impossible.

The number of ways to send the five friends onto a given set of k days using every day is 1 for $k = 1$, $2^5 - 2 = 30$ for $k = 2$, and $3^5 - 3 \cdot 2^5 + 3 = 150$ for $k = 3$ (inclusion and exclusion on the days left out). So the number of good outcomes is $7 \cdot 1 + 14 \cdot 30 + 7 \cdot 150 = 7 + 420 + 1050 = 1477$, and the probability is $\frac{1477}{16807} = \frac{211}{2401}$, since $1477 = 7 \cdot 211$ and $16807 = 7 \cdot 2401$.

Check. Count the size-2 sets another way: each of the 7 days has 4 days that are neither itself nor a neighbour, and each pair is counted twice, giving $\frac{7 \cdot 4}{2} = 14$. The size-3 sets are Mon-Wed-Fri, Tue-Thu-Sat, Wed-Fri-Sun, Thu-Sat-Mon, Fri-Sun-Tue, Sat-Mon-Wed and Sun-Tue-Thu: 7 of them. Also, all five friends share one day in 7 of the outcomes, and every such outcome is good, matching the size-1 term. A computer run over all 16807 outcomes also finds exactly 1477 good ones.

5. Answer: 170. **The idea.** The incircle touches each side, and from each vertex the two tangent segments to the incircle have the same length, the tangent length $s -$ (opposite side) where s is the semiperimeter. The radius to a tangency point is perpendicular to the side, so each IX is the hypotenuse of a right triangle with legs r and the tangent length at X.

Work it out. The semiperimeter is $s = \frac{9+10+17}{2} = 18$. By Heron's formula the area is $\sqrt{18 \cdot 9 \cdot 8 \cdot 1} = \sqrt{1296} = 36$, so the inradius is $r = \frac{36}{18} = 2$. The tangent lengths are $s - BC = 9$ at A, $s - CA = 8$ at B, and $s - AB = 1$ at C. Hence $IA = \sqrt{2^2 + 9^2} = \sqrt{85}$, $IB = \sqrt{2^2 + 8^2} = \sqrt{68}$, $IC = \sqrt{2^2 + 1^2} = \sqrt{5}$, and $IA \cdot IB \cdot IC = \sqrt{85 \cdot 68 \cdot 5} = \sqrt{28900} = 170$.

Check. Coordinates: put $A = (0, 0)$, $B = (17, 0)$. Then $C = (x, y)$ with $x^2 + y^2 = 100$ and $(x - 17)^2 + y^2 = 81$, giving $x = \frac{154}{17}$ and $y = \frac{72}{17}$ (so the area is $\frac{1}{2} \cdot 17 \cdot \frac{72}{17} = 36$, as before). The incircle touches AB at distance 9 from A and has radius 2, so $I = (9, 2)$. Then $IA^2 = 85$, $IB^2 = 64 + 4 = 68$, and $IC^2 = \left(\frac{154}{17} - 9\right)^2 + \left(\frac{72}{17} - 2\right)^2 = \frac{1+1444}{289} = 5$, matching.

6. Answer: 50. **The idea.** With edges a, b, c, the data say $a + b + c = 12$ and $ab + bc + ca = 45$. These are two of the three symmetric functions of a, b, c; the third is the volume $V = abc$. So a, b, c are the three roots of $f(x) = V$, where $f(x) = x^3 - 12x^2 + 45x$, and the question becomes:

for which V does the horizontal line $y = V$ cross the graph of f three times (counting tangency as a double root) at positive x? It is tempting to think the volume can be made tiny by squashing one edge, but the constraints forbid that.

Work it out. $f'(x) = 3x^2 - 24x + 45 = 3(x-3)(x-5)$, so f rises to a local maximum $f(3) = 27 - 108 + 135 = 54$, falls to a local minimum $f(5) = 125 - 300 + 225 = 50$, and then rises again. A horizontal line $y = V$ meets the graph in three real points (with multiplicity) exactly when $50 \le V \le 54$. All three roots are then positive, since $f(x) < 0 \le 50$ for $x < 0$. So every box has $50 \le V \le 54$, and $V = 50$ happens: $f(x) - 50 = (x-5)^2(x-2)$, the box $5 \times 5 \times 2$. The smallest volume is 50.

Check. The box $5 \times 5 \times 2$ has edge total $4(5+5+2) = 48$ and surface area $2(25+10+10) = 90$, volume 50. Squashing is impossible: if one edge c tends to 0, then $a + b \to 12$ and $ab \to 45$, but $ab \le \left(\frac{a+b}{2}\right)^2 = 36$. In fact the shortest edge can be no less than 2. The largest volume, for comparison, is 54, from $3 \times 3 \times 6$.

7. Answer: $\frac{7}{2}$. The idea. Every block has exactly one leftmost seat. So the number of blocks equals the number of empty seats that start a block: an empty seat that is either seat 1 or has an occupied seat on its left. By linearity of expectation we only need the probability that each seat starts a block, with no need to understand how the blocks interact.

Work it out. Seat 1 starts a block exactly when it is empty: probability $\frac{7}{12}$. For each seat $k = 2, \dots, 12$, it starts a block when it is empty and seat $k-1$ is occupied. Seat k is empty with probability $\frac{7}{12}$; given that, the other 11 seats hold all 5 people uniformly, so seat $k-1$ is occupied with probability $\frac{5}{11}$. That gives $\frac{7}{12} \cdot \frac{5}{11} = \frac{35}{132}$ for each of the 11 seats. The expected number of blocks is $\frac{7}{12} + 11 \cdot \frac{35}{132} = \frac{7}{12} + \frac{35}{12} = \frac{42}{12} = \frac{7}{2}$.

Check. Count by gaps instead. The five people cut the row into 6 gaps (before the first person, between neighbours, after the last), holding $g_0, \dots, g_5 \ge 0$ empty seats with total 7; the seatings correspond one-to-one to these, and there are $\binom{12}{5} = 792$ of each. A gap is empty in $\binom{11}{4} = 330$ of them (share 7 seats among the other 5 gaps), so it is nonempty with probability $1 - \frac{330}{792} = \frac{7}{12}$. The blocks are the nonempty gaps: $6 \cdot \frac{7}{12} = \frac{7}{2}$.

8. Answer: 5. **The idea.** Look at one line (row or column) with cells $1, \dots, 5$. Its strips are $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$. A single mark hits all three only if it is cell 3, the middle. So every line has at least one mark, and a line with exactly one mark (call it a *single* line) has it in the middle. A two-mark set without the middle must hit $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, which leaves only $\{1,4\}$, $\{2,4\}$, $\{2,5\}$. Number rows and columns 1 to 5, and write s and t for the numbers of single rows and single columns. Single rows put their mark in column 3; single columns put theirs in row 3.

Work it out. *Row* 3 *and column* 3 *are never single.* If row 3 had only the center marked, then a single column, whose mark lies in row 3, could only be column 3, so $t \le 1$. With M marks, counting by rows gives $M \ge s + 2(5 - s) = 10 - s$, and by columns $M \ge 10 - t$. For $M \le 8$ we get $t \ge 2$, a contradiction. The same goes for column 3. So all single rows and single columns come from lines $1, 2, 4, 5$.

If $s = 4$: rows $1, 2, 4, 5$ are marked only in column 3, so columns $1, 2, 4, 5$ get marks only from row 3, which then needs 4 marks: at least 8 marks in all. Likewise if $t = 4$.

$m = 8$. Suppose $M = 7$. Then $s, t \ge 3$, and by the last paragraph $s = t = 3$. Let R be the one non-single row among $1, 2, 4, 5$ and C the one non-single column among $1, 2, 4, 5$. In row R the

cells in single columns are unmarked, so its ≥ 2 marks are in columns 3 and C: both $(R, 3)$ and (R, C) are marked. Likewise $(3, C)$ is marked. Together with the 3 marks of the single rows and the 3 of the single columns, that is $9 > 7$ marks. So $m \geq 8$, and the marking below shows $m = 8$.

Counting the markings with 8 *marks.* Now $s, t \geq 2$. If $s = 4$ or $t = 4$, the $s = 4$ paragraph forces rows $1, 2, 4, 5$ marked only in column 3 and row 3 marked at columns $1, 2, 4, 5$: the plus sign through the center, center removed. It is blocking. If $s = 3$ (so $t \leq 3$), the argument for $M = 7$ applied to any non-single column $C \neq 3$ (there are $4 - t \geq 1$ of them) needs marks $(3, C)$ and (R, C), and row R needs 2 marks, so the total is at least $3 + (t + (4 - t)) + 2 = 9$. The same holds for $t = 3$. So otherwise $s = t = 2$.

With $s = t = 2$, the row count $8 = 2+$(three rows with ≥ 2) forces every other row, row 3 included, to have exactly 2 marks. Row 3 already holds the 2 marks of the single columns T, so those are all its marks, and T must be one of $\{1, 4\}, \{2, 4\}, \{2, 5\}$. The two remaining rows (not single, not row 3) have no marks in the T columns or in column 3 (column 3 likewise holds only the marks from the single rows S), so each is marked exactly in the other two columns $U = \{1, 2, 4, 5\} \setminus T$, and U must also be a valid pair. The complements are $\{2, 5\}, \{1, 5\}, \{1, 4\}$, and $\{1, 5\}$ fails, so $T \in \{\{1, 4\}, \{2, 5\}\}$. By the same reasoning on columns, $S \in \{\{1, 4\}, \{2, 5\}\}$. Each of the $2 \cdot 2 = 4$ choices gives a marking: column 3 at rows S, row 3 at columns T, and the four cells in rows $\{1, 2, 4, 5\} \setminus S$ and columns U. In a line of each kind the marks are a middle cell, the pair $\{1, 4\}$, or the pair $\{2, 5\}$, so every strip is hit.

So there are $1 + 4 = 5$ blocking markings with 8 cells.

Check. Take $S = T = \{2, 5\}$. Row by row the marks are: row 1 at columns $1, 4$; row 2 at column 3; row 3 at columns $2, 5$; row 4 at columns $1, 4$; row 5 at column 3. These are exactly the 8 cells (r, c) whose sum $r + c$ leaves remainder 2 when divided by 3: a diagonal stripe pattern, and any three consecutive cells in a line contain one of them. Turning the grid by 90°, 180° and 270° gives the other three choices of S and T, while the plus sign is unchanged by every rotation, so the 5 markings are 4 stripes and 1 plus. A computer search over all sets of at most 8 cells confirms that no 7 cells block every strip and that exactly 5 sets of 8 cells do.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.