MathCastle

STATE COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The sum of four consecutive odd positive integers is a perfect cube. What is the smallest possible value of the smallest of these four integers?

1. _____

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2. In the addition $\overline{ABC} + \overline{ABC} + \overline{ABC} = \overline{BBB}$, each letter stands for a digit, different letters stand for different digits, and $A \neq 0$. (Here $\overline{ABC}$ is a three-digit number and $\overline{BBB}$ is a three-digit number with three equal digits.) What is the three-digit number $\overline{ABC}$?

2. _____

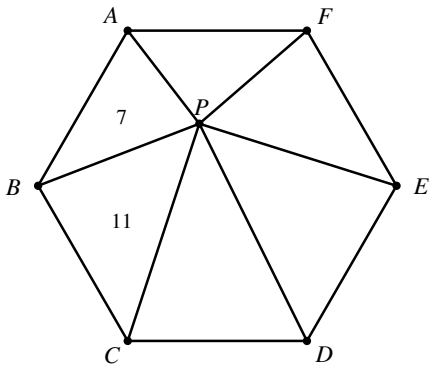
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3. Seven identical apples are shared among Ann, Ben and Cal so that each of them gets at least one apple and all seven apples are given out. In how many ways can this be done if Ann must get more apples than Ben?

3. _____

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4. Regular hexagon $ABCDEF$ has area 60. A point P inside the hexagon is joined to all six vertices, cutting the hexagon into the six triangles PAB, PBC, PCD, PDE, PEF and PFA. Triangle PAB has area 7 and triangle PBC has area 11. What is the sum of the areas of triangles PDE and PEF?



4. _____

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5. Two runners start together from the same point of a circular track that is 400 m around. They run in opposite directions, one at a steady 7 m per second and the other at a steady 3 m per second, and they never stop. At how many different points of the track do the two runners ever meet (counting the starting point)?

5. _____

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6. When a positive integer n is divided by 7 and also divided by 8, the two whole-number quotients (ignoring the remainders) may or may not be equal. For example, $n = 20$ gives $20 = 7 \cdot 2 + 6$ and $20 = 8 \cdot 2 + 4$, so both quotients are 2, while $n = 24$ gives quotients 3 and 3 too, but $n = 7$ gives 1 and 0. For how many positive integers n are the two quotients equal?

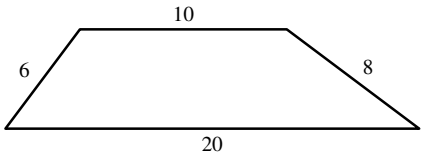
6. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

7. Alex and Bo play a game, taking turns with Alex going first. On each of his turns Alex tosses one fair coin; on each of her turns Bo tosses two fair coins. The first player to toss a head wins, so Bo wins on her turn if at least one of her two coins lands heads. What is the probability that Alex wins the game? Express your answer as a common fraction.

7. _____

8. A trapezoid has parallel sides of lengths 10 and 20, and its other two sides have lengths 6 and 8. What is the area of the trapezoid?

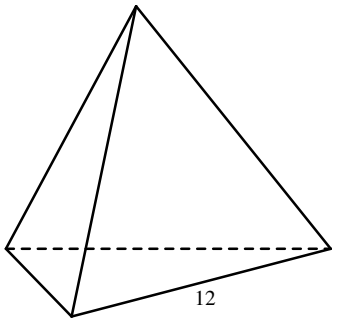


8. _____

9. The equation $x^2 - 20x + c = 0$ has two different real solutions, and one of the solutions is the square of the other. What is the sum of all possible values of c?

9. _____

10. A regular tetrahedron has all six edges of length 12. Two of its edges are called opposite if they share no vertex. What is the shortest distance between two opposite edges of this tetrahedron? Express your answer in simplest radical form.



10. _____

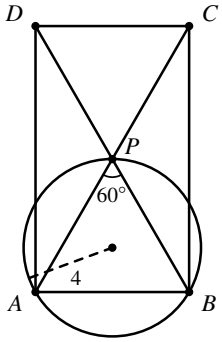
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

11. Find the sum of the squares of all real numbers x that satisfy $\sqrt{x^2+7x+30}+\sqrt{x^2+7x+6}=12$. (Each square root means the non-negative root.)

11. _____

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12. The diagonals of rectangle $ABCD$ meet at point P, and $\angle APB = 60°$, where $\overline{AB}$ is one of the shorter sides of the rectangle. The circle that passes through A, B and P has radius 4. What is the area of the rectangle? Express your answer in simplest radical form.



12. _____

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13. Real numbers x and y satisfy both $x+y+xy=11$ and $x^2y+xy^2=18$. What is the value of x^2+y^2?

13. _____

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14. Four children, Ann, Ben, Cal and Dee, pass a ball among themselves, and Ann starts with it. Each pass goes from the child holding the ball to one of the other three children, except that nobody ever passes the ball straight back to the child who has just passed it to them. In how many different ways can exactly six passes be made so that the ball is with Ann after the sixth pass? (Two ways are different if the ball is held by different children at some moment.)

14. _____

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15. Six people are at a meeting. Some pairs of them shake hands (no pair shakes hands more than once), and in the end each person has shaken hands with exactly three of the other five. How many different sets of handshakes are possible? (Two sets are different if some pair shakes hands in one of them but not in the other.)

15. _____

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16. The three edge lengths of a rectangular box, listed from shortest to longest, form a geometric sequence. The twelve edges of the box have a total length of 84 cm, and the six faces have a total area of 252 square cm. What is the volume of the box, in cubic centimeters?

16. _____

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17. A square frame is made of four rods, each 8 units long, joined by hinges at the corners. The frame is pushed out of shape into a rhombus (still with all four sides 8) whose area is 32 square units, exactly half the area of the original square. What is the length of the shorter diagonal of the rhombus? Express your answer in simplest radical form.

17. _____

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18. Real numbers x and y satisfy $x^2 + y^2 = 4x + 2y$. What is the largest possible value of $3x + 4y$? Express your answer in simplest radical form.

18. _____

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19. Each of the 99 fractions $\frac{1}{100}, \frac{2}{100}, \frac{3}{100}, \ldots, \frac{99}{100}$ is rewritten in lowest terms; for example, $\frac{30}{100}$ becomes $\frac{3}{10}$ and $\frac{7}{100}$ stays $\frac{7}{100}$. What is the sum of the 99 numerators of the rewritten fractions?

19. _____

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20. Each cell of a grid with 4 rows and 5 columns is filled with a 0 or a 1. The filling must have an odd number of 1s in every row and an even number of 1s in every column. How many such fillings are there?

20. _____

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21. How many ordered pairs (b, c) of integers with $1 \le b \le 20$ and $-20 \le c \le 20$ have the property that the equation $x^2 + bx + c = 0$ has at least one real solution and every real solution of it is an integer?

21. _____

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22. A calculator display shows the number 1. The calculator has only two keys: key A adds 1 to the number on the display, and key B doubles the number on the display. For example, pressing A, then B, then A changes the display from 1 to 2, then to 4, then to 5. How many different sequences of key presses change the display from 1 to 30?

22. _____

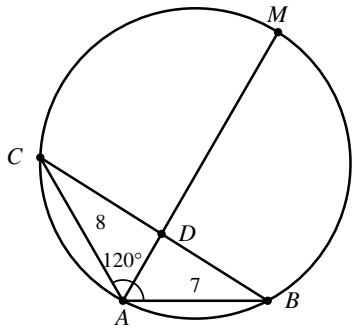
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23. For each positive integer n, let g_n be the greatest common divisor of n^2+5 and n^3+3. As n runs through all positive integers, g_n takes only finitely many different values. What is the sum of all the different values that g_n takes?

23. _____

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24. Triangle ABC has $AB = 7$, $AC = 8$ and $\angle BAC = 120°$. The bisector of $\angle BAC$ meets side $\overline{BC}$ at D and meets the circle through A, B and C again at M (so D lies between A and M). What is the length DM? Express your answer as a common fraction.



24. _____

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25. Ten people stand in line to buy a \$5 ticket each. Five of them pay with a single \$5 bill and five pay with a single \$10 bill. At the start the cashier's till holds exactly one \$5 bill and nothing else, and the only change she can give is the \$5 bills she has on hand. Counting only which positions in line hold \$5 payers and which hold \$10 payers, how many of the orders of the line let the cashier give correct change to every \$10 payer?

25. _____

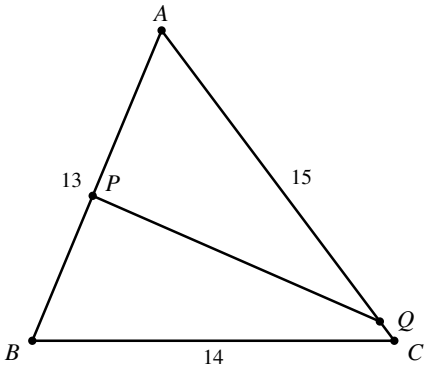
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26. Pascal's triangle begins with row 0, which is the single entry 1; row 1 is 1 1; row 2 is 1 2 1; and in general row n lists $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. How many of the entries in rows 0 through 99 (inclusive) are odd?

26. _____

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27. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. Point P lies on side $\overline{AB}$ and point Q lies on side $\overline{AC}$, and segment $\overline{PQ}$ cuts the triangle into two pieces, triangle APQ and quadrilateral $PBCQ$, that have equal areas and also equal perimeters. What is the length PQ? Express your answer in simplest radical form.



27. _____

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28. How many subsets of $\{1, 2, 3, \ldots, 16\}$, including the empty set, have the property that every two different numbers in the subset are relatively prime (that is, have no common factor greater than 1)?

28. _____

29. Four balls, each of radius 1, are stacked: three rest on a flat floor, each touching the other two, and the fourth rests on top of them, touching all three. A right circular cone stands on the floor with its circular base on the floor and its axis the vertical line through the center of the top ball. The cone covers the stack, and its curved surface touches all four balls. What is the height of the cone? Express your answer in simplest radical form.

29. _____

30. A deck of 10 cards is numbered 1 through 10 from top to bottom. It is given one riffle shuffle: the deck is cut into a top pile, made of the top k cards for some k with $0 \le k \le 10$, and a bottom pile made of the other $10 - k$ cards; then the two piles are merged into a single deck of 10 cards in any way that keeps the cards of each pile in the same order as before. Different cuts and merges can lead to the same final order of the deck. How many different final orders have exactly two cards in their original positions?

30. _____

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Answer Key — MathCounts State Sprint Practice Contest 6

1. 13

2. 148

3. 6

4. 22

5. 10

6. 27

7. $\frac{4}{7}$

8. 72

9. -61

10. $6\sqrt{2}$

11. 87

12. $48\sqrt{3}$

13. 77

14. 30

15. 70

16. 216

17. $4\sqrt{6} - 4\sqrt{2}$

18. $10 + 5\sqrt{5}$

19. 2865

20. 4096

21. 85

22. 166

23. 204

24. $\frac{169}{15}$

25. 132

26. 1251

27. $7\sqrt{3}$

28. 728

29. $1 + \sqrt{3} + \frac{2\sqrt{6}}{3}$

30. 192

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Solutions — MathCounts State Sprint Practice Contest 6

1. Answer: 13**. The idea.** Don't try cubes one by one blindly: first find out what kind of number the sum always is. Four consecutive odd numbers add up to a multiple of 8, so the cube must be a multiple of 8, which rules out 1, 27, 125, ... at once.

Work it out. Call the numbers a, $a+2$, $a+4$, $a+6$ with a odd. Their sum is $4a+12 = 4(a+3)$, and $a+3$ is even, so the sum is $8 \cdot \frac{a+3}{2}$. A cube that is a multiple of 8 has an even cube root, so it is $(2k)^3 = 8k^3$, and then $\frac{a+3}{2} = k^3$. Since $a \geq 1$, $\frac{a+3}{2} \geq 2$, so $k^3 = 1$ is too small and the least choice is $k^3 = 8$. That gives $a+3 = 16$, so $a = 13$.

Check. $13 + 15 + 17 + 19 = 64 = 4^3$. The smaller cube 8 would need $a = -1$, and the odd cubes 27 and 125 can never occur, because the sum is always even.

2. Answer: 148**. The idea.** A number with three equal digits B is $111 \cdot B$, and $111 = 3 \cdot 37$. So the equation $3 \cdot \overline{ABC} = 111B$ says $\overline{ABC} = 37B$: the unknown number is a multiple of 37 that must also carry its own multiplier B as its tens digit.

Work it out. B is a digit from 1 to 9, and $37B$ must be a three-digit number, so $B \geq 3$. The candidates are $37 \cdot 3 = 111$, 148, 185, 222, 259, 296, 333 for $B = 3, 4, 5, 6, 7, 8, 9$. The tens digit must equal B: 111 has tens digit $1 \neq 3$; 148 has tens digit $4 = B$; 185 has $8 \neq 5$; 222 has $2 \neq 6$; 259 has $5 \neq 7$; 296 has $9 \neq 8$; 333 has $3 \neq 9$. Only 148 works, and its digits $1, 4, 8$ are all different.

Check. $148 + 148 + 148 = 444$, and indeed $B = 4$ is the tens digit of 148.

3. Answer: 6**. The idea.** First count all the allowed shares, then use the symmetry between Ann and Ben: swapping their amounts turns a share with Ann ahead into one with Ben ahead. So after removing the ties, exactly half of what is left has Ann ahead.

Work it out. Shares (a, b, c) of 7 with every part at least 1: give everyone one apple, then share the other 4 freely, which is $\binom{4+2}{2} = 15$ ways. Ties $a = b$: $(1, 1, 5)$, $(2, 2, 3)$, $(3, 3, 1)$, that is 3 shares ($a = b = 4$ would leave Cal -1). The other 12 shares split evenly between "Ann ahead" and "Ben ahead", so the answer is $\frac{12}{2} = 6$.

Check. List them by Cal's share: $c = 1$: $(5, 1)$, $(4, 2)$; $c = 2$: $(4, 1)$, $(3, 2)$; $c = 3$: $(3, 1)$; $c = 4$: $(2, 1)$; $c = 5$: none with $a > b$. That is $2 + 2 + 1 + 1 = 6$.

4. Answer: 22**. The idea.** In a regular hexagon, opposite sides are parallel and equal. Side $\overline{AB}$ is opposite $\overline{DE}$, and $\overline{BC}$ is opposite $\overline{EF}$. For two opposite sides, the heights of the two triangles from P add up to the fixed distance between the parallel sides, so the two areas add up to the same total wherever P is.

Work it out. Let the side be s and the distance between opposite sides be h. Then $[PAB] + [PDE] = \frac{1}{2}s \cdot h_1 + \frac{1}{2}s \cdot h_2 = \frac{1}{2}sh$, a constant. Taking P at the centre, all six triangles are equal with area $\frac{60}{6} = 10$, so the constant is 20. Hence $[PDE] = 20 - 7 = 13$ and, in the same way, $[PEF] = 20 - 11 = 9$. The sum is $13 + 9 = 22$.

Check. The third pair of opposite sides gives $[PCD] + [PFA] = 20$, and all six areas then add to $7 + 11 + 13 + 9 + 20 = 60$, the whole hexagon. Such a point exists: solving the two area conditions in coordinates gives a single point, and all six of its triangle areas come out positive, so it lies inside.

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5. Answer: 10**. The idea.** Running towards each other, the two runners together cover one full lap between consecutive meetings. So the meetings come at regular times, and each meeting point is the previous one shifted by a fixed distance. The question becomes: which points does "keep adding the same distance around a 400 m loop" reach?

Work it out. Together they cover $7+3 = 10$ m each second, so they meet every $\frac{400}{10} = 40$ seconds. In 40 seconds the faster runner goes 280 m, so the k-th meeting is $280k$ m from the start, measured along the faster runner's direction and reduced modulo 400. The multiples of 280 modulo 400 are exactly the multiples of $\gcd(280, 400) = 40$: $0, 280, 160, 40, 320, 200, 80, 360, 240, 120$, and then back to 0 at the 10th meeting. So there are 10 meeting points.

Check. From the slower runner's side: in 40 seconds he covers 120 m in the other direction, and going back 120 m is the same as going forward 280 m on a 400 m loop. The multiples of 120 modulo 400 are again the multiples of $\gcd(120, 400) = 40$, giving the same 10 points.

6. Answer: 27**. The idea.** Write n both ways with the same quotient q: $n = 7q + r$ with $0 \le r \le 6$ and $n = 8q + s$ with $0 \le s \le 7$. Comparing, $r = q + s$. Since r can be at most 6, the quotient q cannot exceed 6, so there are only finitely many such n, and for each q we can count the allowed remainders.

Work it out. For a fixed q, the choices are $s = 0, 1, \ldots, 6 - q$ (then $r = q + s \le 6$ automatically), which is $7 - q$ values of n, namely $n = 8q, 8q + 1, \ldots, 7q + 6$. For $q = 0$ that is $n = 0, 1, \ldots, 6$, but $n = 0$ is not positive, leaving 6. For $q = 1, 2, 3, 4, 5, 6$ we get $6, 5, 4, 3, 2, 1$. The total is $6 + 6 + 5 + 4 + 3 + 2 + 1 = 27$.

Check. The runs are 1 to 6, 8 to 13, 16 to 20, 24 to 27, 32 to 34, 40 to 41, and 48. For instance $48 = 7 \cdot 6 + 6 = 8 \cdot 6 + 0$ works, while from $q = 7$ on, $8q > 7q + 6$ and the two ranges no longer overlap (e.g. $56 = 7 \cdot 8 = 8 \cdot 7$).

7. Answer: $\frac{4}{7}$**. The idea.** If nobody wins in the first round (Alex's turn and then Bo's turn), the game is exactly back at the start. So Alex's winning chance p satisfies an equation: he wins right away, or the round is wasted and he then wins with probability p again.

Work it out. Alex tosses a head on his first turn with probability $\frac{1}{2}$. The first round is wasted when Alex tosses a tail ($\frac{1}{2}$) and then both of Bo's coins land tails ($\frac{1}{4}$), with probability $\frac{1}{8}$. So $p = \frac{1}{2} + \frac{1}{8}p$, which gives $\frac{7}{8}p = \frac{1}{2}$ and $p = \frac{4}{7}$.

Check. Bo wins in the first round with probability $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$, so the ratio of Alex's first-round chance to Bo's is $\frac{1}{2} : \frac{3}{8} = 4 : 3$. Every round has the same ratio, so the whole game is won $4 : 3$, that is Alex $\frac{4}{7}$ and Bo $\frac{3}{7}$. Going first is worth more than Bo's second coin.

8. Answer: 72**. The idea.** The height is the missing piece. Slide the leg of length 8 parallel to itself along the long base, by 10 units towards the other leg, so that its top end lands on the other end of the short base. It then meets the leg of length 6 at the top, and together with the leftover $20 - 10 = 10$ of the long base the two legs form a triangle with sides 6, 8, 10. That triangle has the same height as the trapezoid.

Work it out. Since $6^2 + 8^2 = 10^2$, the triangle has a right angle between the sides 6 and 8. Its area is $\frac{1}{2} \cdot 6 \cdot 8 = 24$, and its height to the side of length 10 is $\frac{2 \cdot 24}{10} = \frac{24}{5}$. So the trapezoid's height is $\frac{24}{5}$, and its area is $\frac{10+20}{2} \cdot \frac{24}{5} = 15 \cdot \frac{24}{5} = 72$.

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Check. With coordinates: long base from $(0,0)$ to $(20,0)$, short base from (p,h) to $(p+10,h)$, legs $p^2+h^2=36$ and $(10-p)^2+h^2=64$. Subtracting, $100-20p=28$, so $p=\frac{18}{5}$ and $h^2=36-\frac{324}{25}=\frac{576}{25}$, so $h=\frac{24}{5}$ again.

9. Answer: -61. **The idea.** Don't touch the quadratic formula. Call the roots r and r^2 and use Vieta: their sum is 20 and their product is c. The sum condition is a quadratic in r alone, and each of its solutions gives one value of $c=r\cdot r^2=r^3$.

Work it out. $r+r^2=20$ means $r^2+r-20=(r+5)(r-4)=0$, so $r=4$ or $r=-5$. For $r=4$ the roots are 4 and 16, and $c=64$. For $r=-5$ the roots are -5 and 25, and $c=-125$. In both cases the two roots are different real numbers, as required. The sum of the possible values is $64+(-125)=-61$.

Check. $x^2-20x+64=(x-4)(x-16)$ and $16=4^2$; $x^2-20x-125=(x+5)(x-25)$ and $25=(-5)^2$. No other c is possible, because $r+r^2=20$ has only these two solutions.

10. Answer: $6\sqrt{2}$. **The idea.** A regular tetrahedron fits inside a cube: pick four vertices of a cube, no two on a common edge. Each edge of the tetrahedron is then a diagonal of a face of the cube, and two opposite edges are diagonals of two opposite faces of the cube, crossing at right angles.

Work it out. The face diagonal is 12, so the cube's edge is $s=\frac{12}{\sqrt{2}}=6\sqrt{2}$. Two opposite edges lie in two parallel faces, and the segment joining the two face centers is perpendicular to both faces, so it is perpendicular to both edges and passes through both of them (each diagonal passes through its face's center). That common perpendicular has length s, so the distance is $6\sqrt{2}$.

Check. Without the cube: let M and N be the midpoints of opposite edges $\overline{AB}$ and $\overline{CD}$. N is equidistant from A and B, so $\overline{MN}\perp\overline{AB}$, and likewise $\overline{MN}\perp\overline{CD}$. CM is a median of an equilateral triangle, $CM=6\sqrt{3}$, and $CN=6$, so $MN^2=CM^2-CN^2=108-36=72$ and $MN=6\sqrt{2}$.

11. Answer: 87. **The idea.** Both radicands contain x^2+7x, and they differ by the constant 24. So if a and b are the two roots, we know $a+b$ and we know a^2-b^2; dividing gives $a-b$, and the equation collapses without any squaring.

Work it out. Let $a=\sqrt{x^2+7x+30}$ and $b=\sqrt{x^2+7x+6}$. Then $a+b=12$ and $a^2-b^2=24$, so $a-b=\frac{24}{12}=2$. Adding, $a=7$ (and $b=5$). So $x^2+7x+30=49$, that is $x^2+7x-19=0$. Its discriminant is $49+76=125>0$, so it has two real roots x_1,x_2, and both are genuine solutions because each gives $a=7$, $b=5$ exactly. By Vieta, $x_1+x_2=-7$ and $x_1x_2=-19$, so $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=49+38=87$.

Check. The roots are $\frac{-7\pm5\sqrt{5}}{2}$, and their squares are $\frac{49\mp70\sqrt{5}+125}{4}$, which add to $\frac{348}{4}=87$.

12. Answer: $48\sqrt{3}$. **The idea.** The diagonals of a rectangle are equal and bisect each other, so $PA=PB$. With a 60° angle between them, triangle APB is equilateral, and the circle through its vertices is its circumcircle. That gives AB, then the diagonals, then the other side.

Work it out. Triangle APB is isosceles with apex angle 60°, so it is equilateral; call its side s. The circumradius of an equilateral triangle is $\frac{s}{\sqrt{3}}$ (two thirds of its altitude $\frac{\sqrt{3}}{2}s$), so $\frac{s}{\sqrt{3}}=4$ and $s=4\sqrt{3}$. Thus $AB=4\sqrt{3}$ and each diagonal is $2\cdot PA=8\sqrt{3}$. Then $BC=\sqrt{(8\sqrt{3})^2-(4\sqrt{3})^2}=\sqrt{192-48}=12$, and the area is $4\sqrt{3}\cdot12=48\sqrt{3}$.

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Check. Triangle BPC has $PB = PC = 4\sqrt{3}$ and $\angle BPC = 180° - 60° = 120°$, so $BC^2 = 48 + 48 + 48 = 144$ by the law of cosines, and $BC = 12$ again. Also $\frac{BC}{AB} = \sqrt{3}$, the familiar shape of a rectangle whose diagonals cross at 60°.

13. Answer: 77**. The idea.** Both equations are symmetric, so use $s = x + y$ and $p = xy$. The system becomes "$s + p = 11$ and $sp = 18$," so s and p are the two roots of one quadratic, in some order. The catch is that x and y must be real, and that decides which order is allowed.

Work it out. $x^2y + xy^2 = xy(x + y) = sp$. So $s + p = 11$, $sp = 18$, and $\{s, p\}$ are the roots of $t^2 - 11t + 18 = (t - 2)(t - 9) = 0$. Real x, y with sum s and product p exist exactly when x, y are the real roots of $u^2 - su + p$, that is, when $s^2 - 4p \geq 0$. For $(s, p) = (2, 9)$: $4 - 36 < 0$, so no real x, y. For $(s, p) = (9, 2)$: $81 - 8 = 73 > 0$, fine. Then $x^2 + y^2 = s^2 - 2p = 81 - 4 = 77$.

Check. The rejected case would give $x^2 + y^2 = 4 - 18 = -14$, which is impossible for real numbers anyway. Explicitly $x, y = \frac{9 \pm \sqrt{73}}{2}$: $x^2 + y^2 = \frac{2(81+73)}{4} = 77$, $x + y + xy = 9 + 2 = 11$ and $xy(x + y) = 18$.

14. Answer: 30**. The idea.** What a holder may do depends only on two things: whether the holder is Ann, and whether the ball just came from Ann. So sort every moment into one of three states and count how many pass sequences reach each state, pass by pass.

Work it out. Call the state A if Ann holds the ball, F if another child holds it and got it from Ann, and O if another child holds it and got it from another non-Ann child. From A, Ann may pass to any child except the one who passed to her, so there are 2 choices, both leading to F (on the very first pass nobody passed to her, so there are 3). From F, the holder may not pass back to Ann, so the ball goes to one of the 2 other non-Ann children, giving O. From O, the holder may not pass back to the child who threw to them, so the ball goes either to Ann (1 way, giving A) or to the fourth child (1 way, giving O). Writing the counts as (A, F, O) after each pass: after pass 1, $(0, 3, 0)$; after pass 2, $(0, 0, 6)$; after pass 3, $(6, 0, 6)$; after pass 4, $(6, 12, 6)$; after pass 5, $(6, 12, 30)$; after pass 6, Ann holds the ball in 30 ways (only state O can pass to Ann).

Check. Every pass after the first has exactly 2 allowed receivers, so the totals after passes 1 to 5 must be $3, 6, 12, 24, 48$, and indeed $0 + 3 + 0$, $0 + 0 + 6$, $6 + 0 + 6$, $6 + 12 + 6$ and $6 + 12 + 30$ give exactly these.

15. Answer: 70**. The idea.** Look at the handshakes that do not happen. Each person shakes hands with 3 of the 5 others, so each person misses exactly 2 of them. "Everyone misses exactly two people" is a much easier pattern to count than "everyone meets exactly three".

Work it out. Join two people by a line when they do not shake hands. Every person is on exactly 2 lines, so following the lines from anyone leads around a closed loop, and the six people split into loops of at least 3 people. With 6 people the loops are either one loop of 6 or two loops of 3. One loop of 6: arrange the 6 people in a circle; there are $\frac{5!}{2} = 60$ different circles, since rotations and reflections give the same loop. Two loops of 3: choose the group of 3 containing a fixed person, $\binom{5}{2} = 10$ ways, and each triangle of 3 is a single loop. Total $60 + 10 = 70$, and each missing-pattern gives exactly one set of handshakes.

Check. Count the two-triangle case another way: $\binom{6}{3} = 20$ ways to choose one group, divided by 2 because the two groups can be swapped, is 10. And each person in a 6-loop misses 2 people and shakes 3 hands, as required. So 70.

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16. Answer: 216**. The idea.** Call the edges $a \le b \le c$. The geometric sequence gives $b^2 = ac$, and that turns the sum of the pairwise products into a multiple of the plain sum: $ab + bc + ca = ab + bc + b^2 = b(a + b + c)$. So the two given totals determine the middle edge at once, and the volume is $abc = b \cdot ac = b^3$.

Work it out. The edges total $4(a+b+c) = 84$, so $a+b+c = 21$. The faces total $2(ab+bc+ca) = 252$, so $ab + bc + ca = 126$. Since $ac = b^2$, $ab + bc + ca = b(a + b + c) = 21b$, so $21b = 126$ and $b = 6$. The volume is $abc = b \cdot b^2 = 6^3 = 216$.

Check. Such a box really exists: $a + c = 21 - 6 = 15$ and $ac = 36$ give $a = 3$, $c = 12$. The edges $3, 6, 12$ form a geometric sequence with ratio 2; the edges total $4 \cdot 21 = 84$; the faces total $2(18 + 72 + 36) = 252$; and the volume is $3 \cdot 6 \cdot 12 = 216$.

17. Answer: $4\sqrt{6} - 4\sqrt{2}$**. The idea.** A rhombus is a parallelogram, so its area is side times height. That pins down the height, and a height of half the side means the acute angle is 30°. Put the rhombus on coordinates, compute the short diagonal with the Pythagorean Theorem, and then undo the nested radical $\sqrt{a - b\sqrt{3}}$.

Work it out. Call the rhombus $ABCD$ with AB along the bottom. Area $= 8 \cdot h = 32$, so the height is $h = 4$. Put $A = (0, 0)$ and $B = (8, 0)$; then D is 8 from A at height 4, so $D = (\sqrt{64 - 16}, 4) = (4\sqrt{3}, 4)$ (this is a 30-60-90 triangle, so $\angle DAB = 30°$). The shorter diagonal joins B and D, the ends of the sides at the acute angle: $BD^2 = (8 - 4\sqrt{3})^2 + 4^2 = 64 - 64\sqrt{3} + 48 + 16 = 128 - 64\sqrt{3}$. To simplify $\sqrt{128 - 64\sqrt{3}}$, look for $(\sqrt{m} - \sqrt{n})^2 = m + n - 2\sqrt{mn}$ with $m + n = 128$ and $2\sqrt{mn} = 64\sqrt{3}$, i.e. $mn = 3072$. The numbers 96 and 32 work ($96 + 32 = 128$, $96 \cdot 32 = 3072$), so $BD = \sqrt{96} - \sqrt{32} = 4\sqrt{6} - 4\sqrt{2}$.

Check. $4\sqrt{6} - 4\sqrt{2} \approx 9.80 - 5.66 = 4.14$. The long diagonal is AC with $C = B + D = (8 + 4\sqrt{3}, 4)$: $AC^2 = 128 + 64\sqrt{3}$, so $AC = 4\sqrt{6} + 4\sqrt{2} \approx 15.45$. The diagonals of a rhombus give its area as half their product: $\frac{1}{2}(4\sqrt{6} - 4\sqrt{2})(4\sqrt{6} + 4\sqrt{2}) = \frac{1}{2}(96 - 32) = 32$. Correct.

18. Answer: $10 + 5\sqrt{5}$**. The idea.** Completing the square shows that the points (x, y) form a circle. Each value k of $3x + 4y$ is a line $3x + 4y = k$; as k grows the line slides across the plane, and the largest k that still meets the circle belongs to the line tangent to it, whose distance from the centre equals the radius.

Work it out. $x^2 - 4x + y^2 - 2y = 0$ becomes $(x - 2)^2 + (y - 1)^2 = 5$: a circle with centre $(2, 1)$ and radius $\sqrt{5}$. The distance from $(2, 1)$ to the line $3x + 4y = k$ is $\frac{|3 \cdot 2 + 4 \cdot 1 - k|}{\sqrt{3^2 + 4^2}} = \frac{|10 - k|}{5}$. The line meets the circle exactly when this is at most $\sqrt{5}$, that is $|k - 10| \le 5\sqrt{5}$. The largest value is $k = 10 + 5\sqrt{5}$.

Check. The tangency point is the centre moved $\sqrt{5}$ in the direction $(3, 4)/5$: $(x, y) = \left(2 + \frac{3\sqrt{5}}{5}, 1 + \frac{4\sqrt{5}}{5}\right)$. It lies on the circle, since $\left(\frac{3\sqrt{5}}{5}\right)^2 + \left(\frac{4\sqrt{5}}{5}\right)^2 = \frac{9+16}{5} = 5$, and $3x + 4y = 6 + \frac{9\sqrt{5}}{5} + 4 + \frac{16\sqrt{5}}{5} = 10 + 5\sqrt{5} \approx 21.2$.

19. Answer: 2865**. The idea.** Sort the reduced fractions by their denominator. A fraction $\frac{k}{100}$ reduces to $\frac{a}{d}$ where d divides 100, $d > 1$, $1 \le a < d$ and a shares no factor with d; and conversely every such $\frac{a}{d}$ equals exactly one $\frac{k}{100}$ (namely $k = a \cdot \frac{100}{d}$). So for each divisor d we need the sum of the numbers below d that are relatively prime to d. Those numbers pair up as a and $d - a$, so their sum is $\frac{d}{2}$ times how many there are.

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Work it out. The divisors of 100 greater than 1 are $2, 4, 5, 10, 20, 25, 50, 100$. Let $\varphi(d)$ count the numbers from 1 to d relatively prime to d: $\varphi(2) = 1$, $\varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(10) = 4$, $\varphi(20) = 8$, $\varphi(25) = 20$, $\varphi(50) = 20$, $\varphi(100) = 40$. The numerator sums $\frac{d\varphi(d)}{2}$ are $1, 4, 10, 20, 80, 250, 500, 2000$ (for $d = 2$ the only numerator is $1 = \frac{2\cdot 1}{2}$). Their total is $1+4+10+20+80+250+500+2000 = 2865$.

Check. Count: $1 + 2 + 4 + 4 + 8 + 20 + 20 + 40 = 99$ fractions, one for each k, as it must be. The same method for $\frac{1}{10}, \ldots, \frac{9}{10}$ predicts $1 + 10 + 20 = 31$, and directly the reduced numerators are $1, 1, 3, 2, 1, 3, 7, 4, 9$, which add to 31.

20. Answer: 4096**. The idea.** Fill the top-left 3×4 block freely. Then every other cell is forced by a parity rule, and the only danger is that the last cell (bottom-right) gets two conflicting demands. Counting the total number of 1s in two ways shows the demands always agree.

Work it out. Choose the 12 cells in rows 1-3, columns 1-4 arbitrarily. In each of rows 1-3, the entry in column 5 is forced (to make that row odd). In each of columns 1-4, the entry in row 4 is forced (to make that column even). The corner cell (row 4, column 5) is chosen to make column 5 even. Is row 4 then odd? Now all 5 columns are even, so the total number of 1s is even. Rows 1-3 are odd, adding to an odd number, so row 4 must be odd too. Every free choice therefore extends in exactly one way, and every valid filling arises this way: $2^{12} = 4096$.

Check. The parity bookkeeping matters: with 5 rows and 4 columns the same rules would be impossible (five odd rows make an odd total, but even columns make an even total). Here 4 odd rows give an even total, consistent with even columns. A computer check over all 16^4 choices of four odd rows finds exactly 4096 with every column even.

21. Answer: 85**. The idea.** Count the roots instead of the coefficients. If one solution r is an integer, the other is $-b - r$, also an integer; call the two roots r and s (possibly equal). Then $b = -(r + s)$ and $c = rs$, and different unordered pairs $\{r, s\}$ give different equations. So we count unordered pairs of integers with $1 \le -(r + s) \le 20$ and $-20 \le rs \le 20$, split by the sign of $c = rs$.

Work it out. $c = 0$*:* the roots are 0 and $-b$, one for each b: 20 pairs. $c > 0$*:* the roots have the same sign and a negative sum, so they are $-p$ and $-q$ with $1 \le p \le q$, $p + q \le 20$ and $pq \le 20$. For $p = 1$: $q = 1, \ldots, 19$ (19); $p = 2$: $q = 2, \ldots, 10$ (9); $p = 3$: $q = 3, \ldots, 6$ (4); $p = 4$: $q = 4, 5$ (2); $p \ge 5$ is impossible. That is 34. $c < 0$*:* the roots are $p > 0$ and $-q < 0$, with a negative sum, so $q > p$, and $pq \le 20$ (then $b = q - p \le 19$ automatically). For $p = 1$: $q = 2, \ldots, 20$ (19); $p = 2$: $q = 3, \ldots, 10$ (8); $p = 3$: $q = 4, 5, 6$ (3); $p = 4$: $q = 5$ (1). That is 31. In total $20 + 34 + 31 = 85$.

Check. Spot checks: $(b, c) = (19, -20)$ is $x^2 + 19x - 20 = (x + 20)(x - 1)$, roots $-20, 1$ (counted with $p = 1$, $q = 20$); $(b, c) = (9, 20)$ is $(x + 4)(x + 5)$; $(b, c) = (2, 1)$ is $(x + 1)^2$, a double root -1. A computer test of all $20 \cdot 41 = 820$ pairs finds exactly 85.

22. Answer: 166**. The idea.** Look at the *last* key pressed. A sequence ending at n either ends with A, and before that it was a sequence ending at $n - 1$, or ends with B, and before that it was a sequence ending at $\frac{n}{2}$ (possible only when n is even). So if $f(n)$ counts the sequences that end at n, then $f(n) = f(n - 1)$ for odd n and $f(n) = f(n - 1) + f\left(\frac{n}{2}\right)$ for even n, starting from $f(1) = 1$ (pressing nothing).

Work it out. Since each odd value repeats the one before it, only the even values need work: $f(2) = 1 + 1 = 2$, $f(4) = 2 + 2 = 4$, $f(6) = 4 + 2 = 6$, $f(8) = 6 + 4 = 10$, $f(10) = 10 + 4 = 14$, $f(12) = 14 + 6 = 20$, $f(14) = 20 + 6 = 26$, $f(16) = 26 + 10 = 36$, $f(18) = 36 + 10 = 46$,

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$f(20) = 46 + 14 = 60$, $f(22) = 60 + 14 = 74$, $f(24) = 74 + 20 = 94$, $f(26) = 94 + 20 = 114$, $f(28) = 114 + 26 = 140$, and $f(30) = f(29) + f(15) = 140 + 26 = 166$. (Each time, $f(n-1)$ equals $f(n-2)$ because $n-1$ is odd.)

Check. A small case by hand: the sequences that turn 1 into 4 are AAA (through $2, 3, 4$), BAA (also through $2, 3, 4$, since doubling 1 gives 2 just as adding 1 does), AB and BB (both through $2, 4$). That is 4, matching $f(4) = 4$. Note that AB and BB count separately: they pass through the same numbers but are different sequences of presses. A computer search through every sequence that reaches 30 also finds 166.

23. Answer: 204**. The idea.** Anything that divides two numbers divides every combination of them. Combine the two expressions to lower the power of n, the way the Euclidean algorithm does, until only a constant is left. That constant limits g_n to its divisors; then decide which divisors really occur.

Work it out. Let $g = g_n$. It divides $(n^3+3) - n(n^2+5) = 3 - 5n$, so it divides $5n - 3$, hence also $(5n-3)(5n+3) = 25n^2 - 9$. It divides $25(n^2+5) = 25n^2 + 125$ too, and subtracting, g divides $134 = 2 \cdot 67$. So g_n is one of $1, 2, 67, 134$. Now test each. *1:* $n = 2$ gives $\gcd(9, 11) = 1$. *Parity:* if n is odd, both n^2+5 and n^3+3 are even; if n is even, both are odd. So $2 \mid g_n$ exactly when n is odd; $n = 1$ gives $\gcd(6, 4) = 2$. *67:* we need $67 \mid 5n-3$. Since $5 \cdot 27 = 135 = 2 \cdot 67 + 1$, this means $n \equiv 3 \cdot 27 = 81 \equiv 14 \pmod{67}$. Conversely, for such n, $25(n^2+5) = (5n-3)(5n+3) + 134$ is a multiple of 67, so $67 \mid n^2+5$, and $n^3+3 = n(n^2+5) - (5n-3)$ is a multiple of 67 too. So $67 \mid g_n$ exactly when $n \equiv 14 \pmod{67}$. $n = 14$ is even, so $g_{14} = 67$ (indeed $14^2 + 5 = 201 = 3 \cdot 67$ and $14^3 + 3 = 2747 = 41 \cdot 67$). $n = 14 + 67 = 81$ is odd, so $g_{81} = 134$. All four values occur, and their sum is $1 + 2 + 67 + 134 = 204$.

Check. $n = 81$: $81^2 + 5 = 6566 = 49 \cdot 134$ and $81^3 + 3 = 531444 = 3966 \cdot 134$. A common slip is to stop at "g divides 134" and answer 134, or to add only $1 + 2 + 67$ forgetting that the conditions for 2 and for 67 can hold at the same time. A computer search over $n \le 10^5$ finds exactly the values $1, 2, 67, 134$.

24. Answer: $\frac{169}{15}$**. The idea.** Three facts chain together. The Law of Cosines gives BC; the Angle Bisector Theorem splits BC into BD and DC; and because D is where two chords of the circle, $\overline{AM}$ and $\overline{BC}$, cross, the Power of a Point gives $AD \cdot DM = BD \cdot DC$. The only missing piece is AD, and the 120° angle makes it easy: the bisector splits it into two 60° angles, and areas do the rest.

Work it out. By the Law of Cosines, $BC^2 = 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cos 120° = 49 + 64 + 56 = 169$, so $BC = 13$. The bisector divides BC in the ratio $AB : AC = 7 : 8$, so $BD = \frac{91}{15}$ and $DC = \frac{104}{15}$. Write $AD = t$. Triangle ABC is made of triangles ABD and ADC, each with a 60° angle at A, so $\frac{1}{2} \cdot 7 \cdot 8 \sin 120° = \frac{1}{2} \cdot 7t \sin 60° + \frac{1}{2} \cdot 8t \sin 60°$. Since $\sin 120° = \sin 60°$, this says $56 = 15t$, so $t = \frac{56}{15}$. Finally, the chords $\overline{AM}$ and $\overline{BC}$ cross at D, so $AD \cdot DM = BD \cdot DC$: $\frac{56}{15} \cdot DM = \frac{91}{15} \cdot \frac{104}{15} = \frac{9464}{225}$, and $DM = \frac{9464}{225} \cdot \frac{15}{56} = \frac{169}{15}$.

Check. Then $AM = \frac{56}{15} + \frac{169}{15} = 15$. Here is an independent confirmation. M is the midpoint of arc BC not containing A, so $MB = MC$, and $\angle BMC = 180° - \angle BAC = 60°$ (opposite angles of the cyclic quadrilateral $ABMC$); hence triangle BMC is equilateral with side 13. Ptolemy's theorem in $ABMC$ gives $AM \cdot BC = AB \cdot MC + AC \cdot BM$, that is $13 \cdot AM = 13(7 + 8)$, so $AM = 15$. It agrees, and $DM = 15 - \frac{56}{15} = \frac{169}{15}$.

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25. Answer: 132**. The idea.** Track the number of \$5 bills in the till: it starts at 1, goes up 1 for each \$5 payer and down 1 for each \$10 payer, and the order works exactly when this walk never drops to -1. Count the bad walks with the reflection trick.

Work it out. There are $\binom{10}{5} = 252$ orders in all, each a walk from height 1 to height 1 with 5 up-steps and 5 down-steps. Take a bad walk and the first moment it reaches -1. Reflect everything before that moment in the line at height -1: the new walk starts at -3 instead of 1, still ends at 1, and still touches -1. Conversely, every walk from -3 to 1 must cross height -1, and reflecting its part before the first visit to -1 gives back a bad walk. So bad walks match the walks from -3 to 1 in 10 steps, which need 7 up-steps and 3 down-steps: $\binom{10}{3} = 120$ of them. The good orders number $252 - 120 = 132$.

Check. Small case: one \$5 payer and one \$10 payer. The formula gives $\binom{2}{1} - \binom{2}{-1} = 2 - 0 = 2$, and indeed both orders work since the till already holds a \$5. With two of each, the formula gives $\binom{4}{2} - \binom{4}{0} = 6 - 1 = 5$: the only failure is 10, 10, 5, 5, as you can check by listing all six orders.

26. Answer: 1251**. The idea.** Work mod 2. The first 2^k rows form a triangle T_k. The first 2^{k+1} rows are made of three copies of T_k (top, bottom left, bottom right) with an upside-down all-even triangle in the middle. That is because row 2^k is $1, 0, \ldots, 0, 1 \bmod 2$, so each of its two 1s starts a fresh copy. So the first 2^k rows hold 3^k odd entries. For 100 rows, split 100 into powers of two.

Work it out. Rows 0 to 63 are T_6: $3^6 = 729$ odd entries. Rows 64 to 127 consist of two side-by-side copies of rows 0 to 63. So rows 64 to 99 hold twice as many odd entries as rows 0 to 35. Rows 0 to 31 are T_5: $3^5 = 243$. Rows 32 to 35 are two copies of rows 0 to 3, and those have $1 + 2 + 2 + 4 = 9$ odd entries, so 18. Rows 0 to 35 therefore hold $243 + 18 = 261$, and rows 64 to 99 hold $2 \cdot 261 = 522$. The total is $729 + 522 = 1251$.

Check. Equivalently, row n has $2^{s(n)}$ odd entries, where $s(n)$ is the number of 1s in the binary form of n (Lucas' theorem). Summing $2^{s(n)}$ for $n = 0, \ldots, 99$ gives 1251, and building the rows directly by computer and counting odd entries gives 1251 as well.

27. Answer: $7\sqrt{3}$**. The idea.** Let $AP = x$ and $AQ = y$. Each condition becomes a simple equation: equal areas fixes the product xy, and equal perimeters fixes the sum $x + y$ (the cut PQ belongs to both pieces, so it cancels). Then PQ^2, computed with coordinates, only involves $x^2 + y^2$ and xy, so we never need x and y themselves.

Work it out. Triangles APQ and ABC share the angle at A, so their areas are in the ratio $\frac{xy}{13 \cdot 15}$; this must be $\frac{1}{2}$, so $xy = \frac{195}{2}$. The perimeters are $x + y + PQ$ and $(13 - x) + 14 + (15 - y) + PQ$; equal means $x + y = 21$. Now place $A = (0, 0)$ and $B = (13, 0)$. With $C = (u, v)$, $u^2 + v^2 = 225$ and $(u - 13)^2 + v^2 = 196$ give $26u = 198$, so $C = \left(\frac{99}{13}, \frac{168}{13}\right)$, and the point of $\overline{AC}$ at distance y from A is $Q = \left(\frac{33}{65}y, \frac{56}{65}y\right)$. With $P = (x, 0)$: $PQ^2 = \left(x - \frac{33}{65}y\right)^2 + \left(\frac{56}{65}y\right)^2 = x^2 + y^2 - \frac{66}{65}xy$. Here $x^2 + y^2 = (x + y)^2 - 2xy = 441 - 195 = 246$ and $\frac{66}{65}xy = \frac{66}{65} \cdot \frac{195}{2} = 99$. So $PQ^2 = 246 - 99 = 147$ and $PQ = 7\sqrt{3}$.

Check. The cut really fits on the sides: x and y are the roots of $t^2 - 21t + \frac{195}{2} = 0$, namely $\frac{21 \pm \sqrt{51}}{2} \approx 6.93$ and 14.07. The smaller must be $AP \le 13$ and the larger $AQ \le 15$, which works (the other way round, $AP \approx 14.07 > 13$, is impossible). A second route to the length: a line that halves both the area and the perimeter always passes through the incenter, and a coordinate check confirms that the incenter $(7, 4)$ lies on this $\overline{PQ}$.

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28. Answer: 728**. The idea.** Two numbers clash exactly when they share a prime, so think of each number as the set of primes it uses: a good subset is one in which no prime is used twice. The numbers 1, 11 and 13 clash with nothing else in the set and can be added or left out freely. Every other number uses only the primes $2, 3, 5, 7$, so sort them by the primes they use and count the ways to hand out those four primes.

Work it out. The number 1 and the primes 11 and 13 give a factor $2 \cdot 2 \cdot 2 = 8$. The rest are: using only 2: $2, 4, 8, 16$; only 3: $3, 9$; only 5: 5; only 7: 7; using two primes: 6 and 12 (primes $2, 3$), 10 $(2, 5)$, 14 $(2, 7)$, 15 $(3, 5)$. No number up to 16 uses three of these primes. If no two-prime number is chosen, each prime independently is unused or used by one of its own numbers: $(1 + 4)(1 + 2)(1 + 1)(1 + 1) = 60$ ways. With 6 or 12 (two choices), the primes 5 and 7 remain: $2 \cdot 2 \cdot 2 = 8$. With 10: primes 3 and 7 remain: $3 \cdot 2 = 6$. With 14: primes 3 and 5 remain: $3 \cdot 2 = 6$. With 15: primes 2 and 7 remain: $5 \cdot 2 = 10$. Two two-prime numbers need disjoint prime pairs, and the only such pair is 14 and 15: 1 way. That makes $60 + 8 + 6 + 6 + 10 + 1 = 91$, and the answer is $8 \cdot 91 = 728$.

Check. The same method for $\{1, \ldots, 6\}$: 1 and 5 are free (factor 4); without 6 there are $(1 + 2)(1 + 1) = 6$ choices from $2, 4$ and 3, and with 6 just 1, so $4 \cdot 7 = 28$, which matches a direct listing. For $\{1, \ldots, 16\}$ a computer check of all 2^{16} subsets finds 728.

29. Answer: $1 + \sqrt{3} + \frac{2\sqrt{6}}{3}$**. The idea.** Slice the picture with the vertical half-plane that contains the axis and the center L of one bottom ball. It cuts the cone's surface in a straight line g from the apex down to the floor, and a ball inside the cone touches the curved surface exactly when its center is at distance 1 from g (for a point inside a cone, the nearest point of the surface lies in the half-plane through the axis and that point). The top center T lies on the axis, so it is in this half-plane too. Two points on the same side of a line, both at distance 1 from it, lie on a parallel to it: $\overline{TL}$ is parallel to g. That gives the cone's slope for free.

Work it out. The four centers form a regular tetrahedron of edge 2. The bottom centers are at height 1 and at distance $\frac{2}{\sqrt{3}}$ from the axis (the circumradius of an equilateral triangle of side 2). The top center is higher by the tetrahedron's height $\sqrt{2^2 - \frac{4}{3}} = \frac{2\sqrt{6}}{3}$, so T is at height $1 + \frac{2\sqrt{6}}{3}$. Let θ be the angle between g and the axis. Since $g \parallel \overline{TL}$, $\tan\theta = \frac{2/\sqrt{3}}{2\sqrt{6}/3} = \frac{1}{\sqrt{2}}$, so $\sin\theta = \frac{1}{\sqrt{3}}$. The apex V is on the axis above T, and the distance from T to g is $VT \sin\theta = 1$, so $VT = \sqrt{3}$. The height of the cone is $\sqrt{3} + 1 + \frac{2\sqrt{6}}{3}$.

Check. Put V at height $h = 1 + \sqrt{3} + \frac{2\sqrt{6}}{3}$. The distance from L (height 1, distance $\frac{2}{\sqrt{3}}$ from the axis) to g is $(h - 1)\sin\theta - \frac{2}{\sqrt{3}}\cos\theta = \left(\sqrt{3} + \frac{2\sqrt{6}}{3}\right)\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} = 1 + \frac{2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = 1$, as needed. The bottom ball touches the cone at height $1 + \sin\theta \approx 1.58$, above the floor, so the contact really is on the curved surface.

30. Answer: 192**. The idea.** Describe a shuffle by the cut k and the set S of positions that receive the top pile. A top-pile card can only move down: card i lands at position i plus the number of bottom-pile cards above it. So it stays put exactly when no bottom-pile card is above it, which happens only for the top-pile cards in an unbroken run at the very top of the new deck. In the same way, a bottom-pile card stays put exactly when it is in the unbroken run of bottom-pile cards at the very bottom. So the fixed cards are the first a cards and the last b cards, where a is the length of the opening run of top-pile positions and b the length of the closing run of bottom-pile

positions.

Work it out. First, different shuffles give different orders, except for the unshuffled deck. In the final order, cards $1, \ldots, k$ appear in increasing order and so do cards $k+1, \ldots, 10$. If the deck is not back in order $1, 2, \ldots, 10$, then k is the only number whose successor appears above it, so the order reveals k, and then S is the set of positions of cards $1, \ldots, k$. The unshuffled deck has all 10 cards fixed, so it doesn't matter here. Now count choices of S with $a + b = 2$. For $(a, b) = (2, 0)$: positions $1, 2$ are in S, position 3 is not, and position 10 is (the deck must not end with a bottom-pile card). For $(1, 1)$: position 1 is in S, 2 is not, 10 is not, 9 is. For $(0, 2)$: position 1 is not in S, 9 and 10 are not, 8 is. In each case 4 positions are forced and the other 6 are free, giving $2^6 = 64$ sets, none of them the unshuffled deck. The total is $3 \cdot 64 = 192$.

Check. The same reasoning says the orders with no card in place come from sets S with position 1 not in S and position 10 in S: $2^8 = 256$ of them, and the orders with exactly one fixed card number $2 \cdot 2^7 = 256$ as well. A computer run over every cut and every merge collects $2^{10} - 10 = 1014$ distinct orders, and exactly 192 of them have two cards in their original positions.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.