MathCastle

STATE COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Two squares are drawn on a sheet of paper. The perimeter of the larger square is 16 cm more than the perimeter of the smaller square, and the area of the larger square is 64 square centimetres more than the area of the smaller square. What is the area of the larger square, in square centimetres?

1. _____

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2. A jar holds only dimes and quarters, worth \$6.00 in all. If every dime in the jar were replaced by a quarter and every quarter by a dime, the coins would be worth \$7.65. How many coins are in the jar?

2. _____

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3. Maya rides her bike along the same 30-kilometre route every week at a steady speed. One week she rides 3 kilometres per hour faster than her usual speed and finishes the route 20 minutes sooner than usual. What is her usual speed, in kilometres per hour?

3. _____

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4. Three fair six-sided dice are rolled. What is the probability that the three numbers rolled can be put in some order to form an arithmetic sequence? (A constant sequence such as $4, 4, 4$ counts as arithmetic.) Express your answer as a common fraction.

4. _____

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5. When a positive integer n is divided by 7, the quotient is q and the remainder is r (so $n = 7q + r$ with $0 \le r \le 6$). When the same n is divided by 9, the quotient and remainder trade places: the quotient is r and the remainder is q. What is the sum of all possible values of n?

5. _____

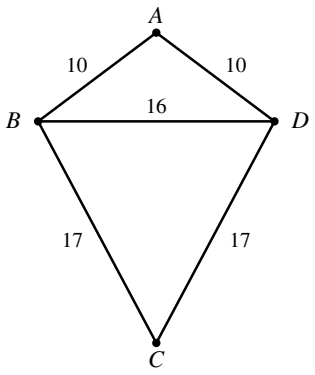
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6. Ravi's bedside clock displays hours and minutes as four digits, using the 24-hour system: midnight is 00:00, seven minutes past nine in the morning is 09:07, and the last display of the day is 23:59. Over one whole day the clock shows 1440 different displays. How many of them use four digits that are all different from one another?

6. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

7. Convex kite $ABCD$ has $AB = AD = 10$ and $CB = CD = 17$, and its diagonal $\overline{BD}$ has length 16. What is the area of the kite?



7. _____

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8. Five different cards are drawn at random from eleven cards numbered 1 through 11, with every set of five cards equally likely. What is the probability that the median of the five numbers drawn (the middle number when the five are listed from smallest to largest) is at least 7? Express your answer as a common fraction.

8. _____

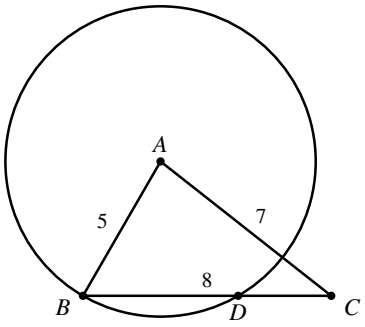
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9. A polynomial $p(x)$ of degree 3 agrees with the identity function at three points, $p(1) = 1$, $p(2) = 2$ and $p(3) = 3$, but then $p(4) = 5$. What is the value of $p(6)$?

9. _____

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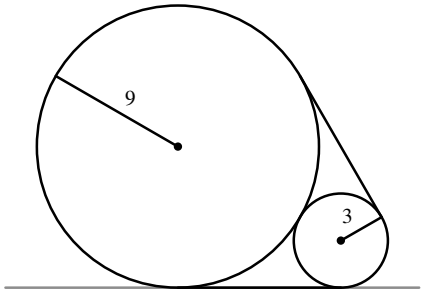
10. Triangle ABC has $AB = 5$, $AC = 7$ and $BC = 8$. The circle with center A that passes through B crosses side $\overline{BC}$ again at a point D different from B. What is the length CD?



10. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

11. Two long round pipes lie side by side on the ground, touching each other along their whole length. Their cross-sections are circles of radius 9 cm and 3 cm. A strap is pulled tight once around both pipes, perpendicular to their length, so it hugs part of each circle and runs straight across the two gaps between them. What is the length of the strap, in centimetres? Express your answer in terms of π and in simplest radical form.



11. _____

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12. Two whole numbers m and n are each chosen from $1, 2, 3, \ldots, 30$; they may be equal, and the pair is ordered, so $(5, 10)$ and $(10, 5)$ count as different pairs. For how many of the 900 ordered pairs is the greatest common divisor of m and n exactly 5?

12. _____

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13. An arithmetic sequence has first term 5 and a common difference that is not 0. Its 2nd, 5th and 14th terms, in that order, form a geometric sequence. What is the sum of the first 20 terms of the arithmetic sequence?

13. _____

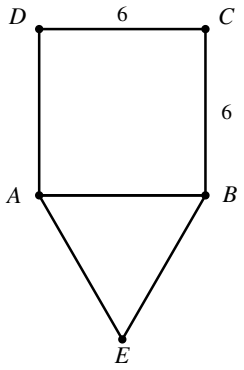
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14. Compute the sum $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \cdots + \frac{1}{47\cdot49\cdot51}$, in which each term is the reciprocal of the product of three consecutive odd numbers. Express your answer as a common fraction.

14. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

15. Square $ABCD$ has side length 6, with its vertices labelled in order around the square. Equilateral triangle ABE is drawn on side $\overline{AB}$, on the outside of the square. What is the radius of the circle that passes through the three points C, D and E?



15. _____

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16. A bag holds four cards. Two are red on both sides, one is red on one side and blue on the other, and one is blue on both sides. You draw a card at random, look at a random side of it, and see red. You set that card aside, draw one of the three remaining cards at random, and look at a random side of it. What is the probability that this second side you see is also red? Express your answer as a common fraction.

16. _____

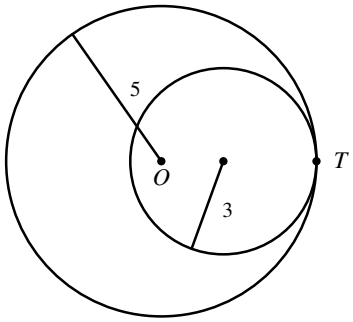
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17. A token starts on square 0 of a long straight track whose squares are numbered $0, 1, 2, 3, \ldots$. On each turn a fair coin is tossed: heads moves the token forward 1 square, and tails moves it forward 2 squares. The token keeps moving forever. What is the probability that the token lands on square 10 at some point? Express your answer as a common fraction.

17. _____

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18. A circle with centre O has radius 5. A smaller circle of radius 3 lies inside it and touches it at exactly one point T. Among all chords of the larger circle that are tangent to the smaller circle, what is the length of the longest one? Express your answer in simplest radical form.



18. _____

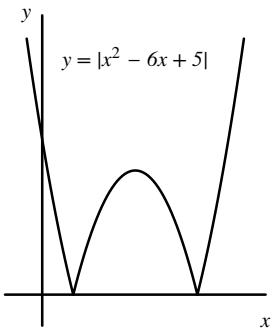
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19. For how many integers n with $1 \le n \le 50$ is the sum of three factorials $n! + (n+1)! + (n+2)!$ divisible by 49? (Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$.)

19. _____

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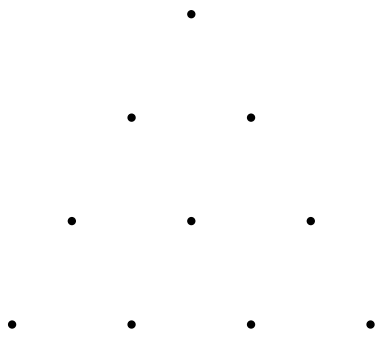
20. The line $y = mx$, where m is a real number, meets the graph of $y = |x^2 - 6x + 5|$ in exactly three points. What is m? Express your answer in simplest radical form.



20. _____

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21. Ten dots form a triangular array with 4 rows of 1, 2, 3 and 4 dots, so that neighbouring dots are 1 unit apart and the ten dots fill an equilateral triangle of side 3 (three of the dots are its corners). How many equilateral triangles, of any size and in any orientation, have all three of their vertices among these ten dots?



21. _____

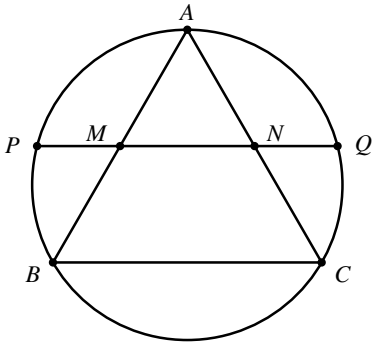
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22. The cubic polynomial $x^3 + ax^2 + bx + c$ has three roots that are consecutive integers, and its coefficients satisfy $a + b + c = 23$. What is the value of c?

22. _____

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23. Equilateral triangle ABC has all three of its vertices on a circle of radius 6. Points M and N are the midpoints of sides $\overline{AB}$ and $\overline{AC}$. The line MN is extended in both directions until it meets the circle at two points P and Q. What is the length PQ? Express your answer in simplest radical form.



23. _____

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24. Six fair six-sided dice are rolled. Call the roll "sociable" if no number appears exactly once: every number that shows up on at least one die shows up on at least two dice. What is the probability that the roll is sociable? Express your answer as a common fraction.

24. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

25. Four players each secretly choose a whole number from 1 to 12, independently and at random, with all twelve numbers equally likely for each player (two players may choose the same number). A player wins if her number is smaller than the numbers of all three other players. What is the probability that one of the players wins? Express your answer as a common fraction.

25. _____

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26. A standard deck of 52 cards, which includes 4 aces and 4 kings, is shuffled so that every order is equally likely. The cards are turned face up one at a time until at least one ace and at least one king have appeared. What is the expected number of cards turned face up? Express your answer as a common fraction.

26. _____

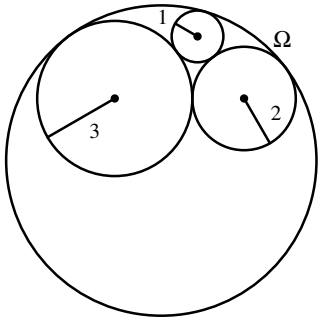
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27. For a positive integer N, let $T(N)$ be the number of ordered triples (a, b, c) of positive integers with $\text{lcm}(a, b) = \text{lcm}(b, c) = \text{lcm}(c, a) = N$. For example, $T(2) = 4$, from the triples $(2, 2, 2)$, $(1, 2, 2)$, $(2, 1, 2)$ and $(2, 2, 1)$. What is the smallest positive integer N with $T(N) = 280$?

27. _____

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28. Circles with radii 1, 2 and 3 are each externally tangent to the other two. A large circle Ω encloses all three and is internally tangent to each of them. A fifth circle lies inside Ω, does not overlap any of the three small circles, and is tangent to Ω, to the circle of radius 1, and to the circle of radius 3. What is the radius of the fifth circle? Express your answer as a common fraction.



28. _____

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29. For a real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. Compute $\left\lfloor \frac{1^2}{101} \right\rfloor + \left\lfloor \frac{2^2}{101} \right\rfloor + \left\lfloor \frac{3^2}{101} \right\rfloor + \cdots + \left\lfloor \frac{100^2}{101} \right\rfloor$.

29. _____

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30. Twelve students sit at the twelve desks of a classroom, which form 3 rows of 4 desks. At a signal, every student moves to a desk directly in front of, behind, to the left of or to the right of his or her own desk (a desk sharing a side with it), and afterwards every desk is again occupied by exactly one student. Two students may swap desks. In how many different ways can the students end up seated?

30. _____

Answer Key — MathCounts State Sprint Practice Contest 5

1. 100

2. 39

3. 15

4. $\frac{7}{36}$

5. 93

6. 644

7. 168

8. $\frac{181}{462}$

9. 16

10. 3

11. $12\sqrt{3} + 14\pi$

12. 23

13. 2000

14. $\frac{208}{2499}$

15. 6

16. $\frac{8}{15}$

17. $\frac{683}{1024}$

18. $4\sqrt{6}$

19. 39

20. $6 - 2\sqrt{5}$

21. 15

22. 6

23. $3\sqrt{15}$

24. $\frac{71}{1296}$

25. $\frac{121}{144}$

26. $\frac{689}{45}$

27. 360

28. $\frac{6}{11}$

29. 3300

30. 121

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts State Sprint Practice Contest 5

1. Answer: 100. **The idea.** Call the side lengths S and s. The perimeters give the *difference* $S - s$, and the areas give the difference of squares $S^2 - s^2 = (S - s)(S + s)$. Dividing one by the other gives the *sum* $S + s$, and a sum and a difference pin down both sides.

Work it out. $4S - 4s = 16$, so $S - s = 4$. Then $S^2 - s^2 = (S - s)(S + s) = 4(S + s) = 64$, so $S + s = 16$. Adding, $2S = 20$, so $S = 10$ (and $s = 6$). The larger square has area $10^2 = 100$ square centimetres.

Check. Perimeters 40 and 24 differ by 16; areas 100 and 36 differ by 64.

2. Answer: 39. **The idea.** We want the total number of coins, not each kind separately. Adding the two value equations makes every coin count as one dime plus one quarter, which is exactly 35 cents per coin.

Work it out. With d dimes and q quarters, in cents: $10d + 25q = 600$ and $25d + 10q = 765$. Adding, $35d + 35q = 1365$, so $d + q = \frac{1365}{35} = 39$.

Check. Subtracting the equations instead gives $15(d - q) = 165$, so $d - q = 11$; with $d + q = 39$ this is $d = 25$, $q = 14$. Indeed $25 \cdot 10 + 14 \cdot 25 = 250 + 350 = 600$ cents and $25 \cdot 25 + 14 \cdot 10 = 625 + 140 = 765$ cents.

3. Answer: 15. **The idea.** The information is about *times*, so write each trip's time as distance over speed and set their difference equal to 20 minutes, which is $\frac{1}{3}$ of an hour. Clearing fractions leaves a product of two numbers 3 apart.

Work it out. With usual speed v: $\frac{30}{v} - \frac{30}{v+3} = \frac{1}{3}$. The left side is $\frac{30(v+3)-30v}{v(v+3)} = \frac{90}{v(v+3)}$, so $v(v+3) = 270$. We need two positive numbers 3 apart with product 270: $15 \cdot 18 = 270$, so $v = 15$. (The other root of $v^2 + 3v - 270 = 0$ is -18, which is not a speed.)

Check. At 15 km/h the route takes $\frac{30}{15} = 2$ hours; at 18 km/h it takes $\frac{30}{18} = 1\frac{2}{3}$ hours. The difference is $\frac{1}{3}$ hour, which is 20 minutes.

4. Answer: $\frac{7}{36}$. **The idea.** Sort the three numbers as $a \le b \le c$. They can be ordered into an arithmetic sequence exactly when $b - a = c - b$. So count the *sets* of values that work, then count the rolls that produce each set.

Work it out. If two of the numbers are equal but not all three, say $a = b < c$, then $b - a = 0 \neq c - b$, so this never works. All three equal: 6 rolls. Three different numbers $m - d, m, m + d$ with $d \ge 1$, all between 1 and 6: for $d = 1$ the sets $\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}$ (4 sets), for $d = 2$ the sets $\{1, 3, 5\}, \{2, 4, 6\}$ (2 sets), and $d \ge 3$ is impossible since $m + 3 \le 6$ and $m - 3 \ge 1$ cannot both hold. Each of these 6 sets appears in $3! = 6$ orders, giving 36 rolls. In all $6 + 36 = 42$ of the $6^3 = 216$ rolls work, and $\frac{42}{216} = \frac{7}{36}$.

Check. Count by the middle value m instead: the number of $d \ge 1$ with $1 \le m - d$ and $m + d \le 6$ is $0, 1, 2, 2, 1, 0$ for $m = 1, \ldots, 6$, a total of 6 sets of different numbers, agreeing with the list.

5. Answer: 93. **The idea.** Write n both ways and set the two expressions equal. That gives one equation in q and r, and the size limits on remainders leave only a few possibilities.

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Work it out. We need $7q + r = 9r + q$, so $6q = 8r$, that is $3q = 4r$. Hence $q = 4t$ and $r = 3t$ for some whole number t. Remainders are limited: $r \le 6$ (remainder on division by 7) and $q \le 8$ (remainder on division by 9). So $t \le 2$. The value $t = 0$ gives $n = 0$, which is not positive. $t = 1$: $q = 4$, $r = 3$, $n = 7 \cdot 4 + 3 = 31$. $t = 2$: $q = 8$, $r = 6$, $n = 7 \cdot 8 + 6 = 62$. The sum is $31 + 62 = 93$.

Check. $31 = 7 \cdot 4 + 3 = 9 \cdot 3 + 4$ and $62 = 7 \cdot 8 + 6 = 9 \cdot 6 + 8$, with every remainder smaller than its divisor.

6. Answer: 644**. The idea.** Choose the hour first. Once its two digits are fixed (and different), the tens digit of the minutes must be one of 0–5 that is not already used, and the units digit of the minutes only has to avoid the three digits already used. So the number of good displays for an hour depends only on how many of its two digits are 5 or less.

Work it out. The hours whose two digits differ are 01–09 (9 hours), 10 and 12–19 (9 hours), and 20, 21, 23 (3 hours): 21 hours in all (00, 11 and 22 repeat a digit). For every one of them the units digit of the minutes has $10 - 3 = 7$ choices. If both hour digits are at most 5 (the hours 01–05, 10, 12–15, 20, 21 and 23, which is 13 hours), the tens digit of the minutes has $6 - 2 = 4$ choices. If one hour digit is 6 or more (06–09 and 16–19, which is 8 hours), it has $6 - 1 = 5$ choices. The total is $(13 \cdot 4 + 8 \cdot 5) \cdot 7 = 92 \cdot 7 = 644$.

Check. Hour 12: the minutes' tens digit is 0, 3, 4 or 5, and the units digit is any of the 7 digits not yet used, so $4 \cdot 7 = 28$ displays; 12:34 and 12:09 are among them, while 12:31 and 12:52 are not. Hour 18: the tens digit is 0, 2, 3, 4 or 5, giving $5 \cdot 7 = 35$. And $13 + 8 = 21$ accounts for every usable hour.

7. Answer: 168**. The idea.** In a kite the diagonal $\overline{AC}$ joining the two "tips" is the perpendicular bisector of the other diagonal $\overline{BD}$ (both A and C are equally far from B and D). So the kite is made of right triangles, and its area is half the product of its diagonals.

Work it out. Let M be the midpoint of $\overline{BD}$, so $BM = 8$ and $\overline{AC} \perp \overline{BD}$ at M. Then $AM = \sqrt{10^2 - 8^2} = 6$ and $CM = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$. The kite is convex, so A and C are on opposite sides of $\overline{BD}$ and $AC = 6 + 15 = 21$. The area is $\frac{1}{2} \cdot BD \cdot AC = \frac{1}{2} \cdot 16 \cdot 21 = 168$.

Check. Split along $\overline{BD}$ instead: triangle ABD has area $\frac{1}{2} \cdot 16 \cdot 6 = 48$ and triangle CBD has area $\frac{1}{2} \cdot 16 \cdot 15 = 120$, and $48 + 120 = 168$. (A non-convex "dart" with the same sides would have $AC = 15 - 6 = 9$ and area 72.)

8. Answer: $\frac{181}{462}$**. The idea.** A hand has median m exactly when it contains m, two cards below m and two cards above m. Rather than add up several such counts, notice the symmetry $k \leftrightarrow 12 - k$: it turns a hand with median m into a hand with median $12 - m$. So hands with median at least 7 are exactly as common as hands with median at most 5, and only median 6 needs counting.

Work it out. There are $\binom{11}{5} = 462$ hands. Median 6: choose 2 of the 5 cards below 6 and 2 of the 5 cards above it, $\binom{5}{2} \cdot \binom{5}{2} = 100$ hands. The other 362 hands split evenly between median ≤ 5 and median ≥ 7, so 181 hands have median at least 7. The probability is $\frac{181}{462}$, in lowest terms because 181 is prime and $462 = 2 \cdot 3 \cdot 7 \cdot 11$.

Check. Count directly. Median 7: $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$. Median 8: $\binom{7}{2}\binom{3}{2} = 21 \cdot 3 = 63$. Median 9: $\binom{8}{2}\binom{2}{2} = 28$. A median of 10 or more is impossible, since two cards must lie above it. Total $90 + 63 + 28 = 181$.

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9. Answer: 16. **The idea.** Don't solve for four coefficients. Look at $q(x) = p(x) - x$ instead: it is a cubic that vanishes at $x = 1, 2, 3$, so it must be a constant multiple of $(x-1)(x-2)(x-3)$.

Work it out. Write $p(x) = x + c(x-1)(x-2)(x-3)$. Then $p(4) = 4 + c \cdot 3 \cdot 2 \cdot 1 = 4 + 6c = 5$, so $c = \frac{1}{6}$. Therefore $p(6) = 6 + \frac{1}{6} \cdot 5 \cdot 4 \cdot 3 = 6 + 10 = 16$.

Check. Use finite differences. $p(x) = x + \frac{1}{6}(x-1)(x-2)(x-3)$ gives $p(5) = 5 + 4 = 9$, so the values at $1, 2, \ldots, 6$ are $1, 2, 3, 5, 9, 16$. First differences: $1, 1, 2, 4, 7$; second: $0, 1, 2, 3$; third: $1, 1, 1$, constant, as a cubic must be.

10. Answer: 3. **The idea.** AB and AD are both radii, so triangle ABD is isosceles and the foot H of the altitude from A to $\overline{BC}$ is the midpoint of $\overline{BD}$. So we only need BH, and the altitude AH is shared by the two right triangles ABH and ACH.

Work it out. Let $BH = x$, so $HC = 8 - x$. Then $AH^2 = 5^2 - x^2 = 7^2 - (8-x)^2$. Expanding, $25 - x^2 = 49 - 64 + 16x - x^2$, so $16x = 40$ and $x = \frac{5}{2}$. Hence $BD = 2x = 5$ and $CD = 8 - 5 = 3$.

Check. $BD = 5 = AB = AD$, so triangle ABD is equilateral and angle B is $60°$. The law of cosines agrees: $AC^2 = 25 + 64 - 2 \cdot 5 \cdot 8 \cdot \frac{1}{2} = 49$. Also $AD = 5$ with $DC = 3$ and angle $ADC = 120°$ gives $AC^2 = 25 + 9 + 15 = 49$.

11. Answer: $12\sqrt{3} + 14\pi$. **The idea.** The strap is two straight pieces along the common outer tangent lines plus two arcs, one on each circle. A radius to a point of tangency is perpendicular to the tangent line, so the two centres and two tangent points form a right trapezoid. That trapezoid gives both the straight length and the angles, which tell us how much of each circle the strap covers.

Work it out. Let the centres be O (radius 9) and P (radius 3), so $OP = 9 + 3 = 12$. Let one straight piece touch the circles at T and U. Both $\overline{OT}$ and $\overline{PU}$ are perpendicular to $\overline{TU}$, so drop a perpendicular from P to $\overline{OT}$ at Q: then $QT = PU = 3$, $OQ = 6$ and $PQ = TU$. From right triangle OQP, $TU = \sqrt{12^2 - 6^2} = 6\sqrt{3}$. Since $OQ = \frac{1}{2}OP$, angle QOP is $60°$. So each tangent radius on the big circle makes a $60°$ angle with $\overline{OP}$, and the strap covers the big circle's arc on the far side from P: $360° - 2 \cdot 60° = 240°$, of length $\frac{240}{360} \cdot 2\pi \cdot 9 = 12\pi$. On the small circle $\overline{PU}$ is parallel to $\overline{OT}$, so it makes a $180° - 60° = 120°$ angle with $\overline{PO}$, and the strap covers $360° - 2 \cdot 120° = 120°$ of it: $\frac{120}{360} \cdot 2\pi \cdot 3 = 2\pi$. The length is $2 \cdot 6\sqrt{3} + 12\pi + 2\pi = 12\sqrt{3} + 14\pi$.

Check. The two arcs cover $240° + 120° = 360°$ in all, as they must: going once around, the strap turns through one full turn, and it turns only along the arcs. The length, about 64.8 cm, is more than the big circle's circumference $18\pi \approx 56.5$ cm and less than the two circumferences together, $24\pi \approx 75.4$ cm.

12. Answer: 23. **The idea.** If $\gcd(m, n) = 5$, both numbers are multiples of 5, say $m = 5a$ and $n = 5b$, and then $\gcd(a, b)$ must be 1 (any common factor of a and b would make the gcd bigger than 5). So we are counting ordered pairs of relatively prime numbers a, b from 1 to 6.

Work it out. Since $5a \le 30$, a and b run from 1 to 6. For each a, count the b from 1 to 6 sharing no factor with it: $a = 1$: all 6; $a = 2$: $1, 3, 5$ (3); $a = 3$: $1, 2, 4, 5$ (4); $a = 4$: $1, 3, 5$ (3); $a = 5$: $1, 2, 3, 4, 6$ (5); $a = 6$: $1, 5$ (2). The total is $6 + 3 + 4 + 3 + 5 + 2 = 23$.

Check. Count the 36 pairs (a, b) that are *not* relatively prime: both even, $3 \cdot 3 = 9$; both multiples of 3, $2 \cdot 2 = 4$; both multiples of 6 (counted twice), 1; both multiples of 5, just $(5, 5)$. That is $9 + 4 - 1 + 1 = 13$, and $36 - 13 = 23$.

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13. Answer: 2000**. The idea.** Three numbers x, y, z form a geometric sequence exactly when $y^2 = xz$ (the middle term squared equals the product of its neighbours). Writing the three terms with the common difference d turns this into one equation for d.

Work it out. The terms are $5 + d$, $5 + 4d$ and $5 + 13d$. So $(5 + 4d)^2 = (5 + d)(5 + 13d)$, that is $25 + 40d + 16d^2 = 25 + 70d + 13d^2$, which simplifies to $3d^2 = 30d$. Since $d \neq 0$, $d = 10$. The sum of the first 20 terms is $20 \cdot 5 + (0 + 1 + \cdots + 19) \cdot 10 = 100 + 190 \cdot 10 = 2000$.

Check. With $d = 10$ the three terms are 15, 45 and 135, a geometric sequence with ratio 3. The sum is also $20 \cdot \frac{5+195}{2} = 2000$, since the 20th term is $5 + 19 \cdot 10 = 195$.

14. Answer: $\frac{208}{2499}$**. The idea.** Split each term into a difference of two terms of the same kind with one fewer factor, so that the sum collapses. The two-factor pieces $\frac{1}{(2k-1)(2k+1)}$ and $\frac{1}{(2k+1)(2k+3)}$ differ by $\frac{(2k+3)-(2k-1)}{(2k-1)(2k+1)(2k+3)} = \frac{4}{(2k-1)(2k+1)(2k+3)}$.

Work it out. So $\frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{1}{4}\left(\frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)}\right)$. The terms run from $k = 1$ $(1 \cdot 3 \cdot 5)$ to $k = 24$ $(47 \cdot 49 \cdot 51)$. Adding, every inner piece cancels, leaving $\frac{1}{4}\left(\frac{1}{1\cdot 3} - \frac{1}{49\cdot 51}\right) = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{2499}\right) = \frac{1}{4} \cdot \frac{833-1}{2499} = \frac{832}{4\cdot 2499} = \frac{208}{2499}$. Since $2499 = 3 \cdot 7^2 \cdot 17$ and $208 = 2^4 \cdot 13$, the fraction is in lowest terms.

Check. The formula for the first n terms is $\frac{1}{4}\left(\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right)$. For $n = 1$ it gives $\frac{1}{4}\left(\frac{1}{3} - \frac{1}{15}\right) = \frac{1}{15}$, which is indeed $\frac{1}{1\cdot 3\cdot 5}$. For $n = 2$: $\frac{1}{4}\left(\frac{1}{3} - \frac{1}{35}\right) = \frac{8}{105}$, and $\frac{1}{15} + \frac{1}{105} = \frac{8}{105}$.

15. Answer: 6**. The idea.** Find the angle $\angle DEC$, which is an inscribed angle of the circle standing on the chord $\overline{CD}$ of length 6. The central angle on the same chord is twice as big, and a nice angle there makes the radius obvious.

Work it out. $EA = AB = AD = 6$, and $\angle EAD = 60° + 90° = 150°$. So triangle EAD is isosceles with base angles $\frac{180°-150°}{2} = 15°$; in particular $\angle AED = 15°$. By symmetry $\angle BEC = 15°$. Both D and C lie inside the $60°$ angle AEB, so $\angle DEC = 60° - 15° - 15° = 30°$. The central angle $\angle DOC$ (O the centre) is therefore $60°$, so triangle DOC is isosceles with a $60°$ apex angle, which makes it equilateral. The radius equals $CD = 6$.

Check. Coordinates: $A = (0, 0)$, $B = (6, 0)$, $C = (6, 6)$, $D = (0, 6)$, $E = (3, -3\sqrt{3})$. The centre lies on $x = 3$, say at $(3, k)$. Equal distances to D and E: $9 + (6 - k)^2 = (k + 3\sqrt{3})^2$, which gives $18 = k(12 + 6\sqrt{3})$, so $k = \frac{3}{2+\sqrt{3}} = 6 - 3\sqrt{3}$. The radius squared is $3^2 + (6 - k)^2 = 9 + 27 = 36$, so the radius is 6.

16. Answer: $\frac{8}{15}$**. The idea.** Seeing red tells you something about which card you drew first, and that changes what is left in the bag. Count red *faces*, not cards: each of the 8 faces is equally likely to be the one you see.

Work it out. There are 5 red faces: 4 on the red-red cards and 1 on the mixed card. Given that you saw red, the first card is red-red with probability $\frac{4}{5}$ and the mixed card with probability $\frac{1}{5}$. If a red-red card was removed, the bag holds red-red, mixed and blue-blue: 3 red faces out of 6, probability $\frac{1}{2}$. If the mixed card was removed, the bag holds red-red, red-red and blue-blue: 4 red faces out of 6, probability $\frac{2}{3}$. In total, $\frac{4}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{2}{3} = \frac{2}{5} + \frac{2}{15} = \frac{8}{15}$.

Check. Count equally likely outcomes directly. The first look can land on any of 5 red faces, and the second look on any of the 6 faces of the other three cards, for 30 outcomes. For each of the 4

red faces on red-red cards, 3 of the other 6 faces are red (12 outcomes), and for the mixed card's red face, 4 are (4 outcomes). That gives $\frac{16}{30} = \frac{8}{15}$.

17. Answer: $\frac{683}{1024}$. **The idea.** Don't list the routes to square 10. Ask instead how the token could *miss* a square n. It moves forward at most 2 at a time, so the only way to get past n without stopping on it is to stand on $n-1$ and then toss tails. That links the chance of hitting n to the chance of hitting $n-1$.

Work it out. Let p_n be the probability that the token ever lands on square n, so $p_0 = 1$. The token misses n exactly when it lands on $n-1$ and the next toss is tails, so $1 - p_n = \frac{1}{2}p_{n-1}$, that is $p_n = 1 - \frac{1}{2}p_{n-1}$. Now step along: $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$, $p_3 = \frac{5}{8}$, $p_4 = \frac{11}{16}$, $p_5 = \frac{21}{32}$, $p_6 = \frac{43}{64}$, $p_7 = \frac{85}{128}$, $p_8 = \frac{171}{256}$, $p_9 = \frac{341}{512}$ and $p_{10} = 1 - \frac{341}{1024} = \frac{683}{1024}$.

Check. The average move is $\frac{3}{2}$ squares, so in the long run the token stops on about $\frac{2}{3}$ of all squares, and the values above do settle towards $\frac{2}{3}$, alternating above and below it. In fact $p_n = \frac{2+(-\frac{1}{2})^n}{3}$ satisfies $p_0 = 1$ and the rule $p_n = 1 - \frac{1}{2}p_{n-1}$, and $n = 10$ gives $\frac{2+\frac{1}{1024}}{3} = \frac{2049}{3072} = \frac{683}{1024}$.

18. Answer: $4\sqrt{6}$. **The idea.** A chord of the larger circle is longer the closer its line passes to the centre O: a chord at distance d from O has length $2\sqrt{25 - d^2}$. So we want the tangent line to the small circle that passes as close to O as possible.

Work it out. The small circle touches the big one from inside at T, so its centre Q lies on $\overline{OT}$ with $OQ = 5 - 3 = 2$. Every tangent line to the small circle stays outside the small disc, while O is inside that disc (since $OQ = 2 < 3$). The distance from O to such a line is at least $3 - OQ = 1$, and it equals 1 for the tangent line at the point S of the small circle farthest from T (on ray TO, beyond O), which is perpendicular to OT and has $OS = 3 - 2 = 1$. So the longest chord has length $2\sqrt{25-1} = 2\sqrt{24} = 4\sqrt{6}$.

Check. Coordinates: $O = (0,0)$, $T = (5,0)$, $Q = (2,0)$, and the far point of the small circle is $S = (-1,0)$. The chord is the line $x = -1$, meeting the big circle at $(-1, \pm\sqrt{24})$: length $2\sqrt{24} = 4\sqrt{6} \approx 9.80$, just short of the diameter 10, as it should be for a chord only 1 unit from the centre.

19. Answer: 39. **The idea.** Factor out $n!$: the sum is $n!\,\big(1+(n+1)+(n+1)(n+2)\big) = n!\,(n^2+4n+4) = n!\,(n+2)^2$. Now count factors of 7: we need at least two of them in $n!\,(n+2)^2$.

Work it out. If $n \geq 14$, then $n!$ contains both 7 and 14, so it already has two factors of 7: that is $n = 14, \ldots, 50$, which is 37 values. If $7 \leq n \leq 13$, $n!$ has exactly one factor 7, and we need $7 \mid n+2$ as well; since then $(n+2)^2$ supplies two more, this works exactly for $n = 12$. If $n \leq 6$, $n!$ has no factor 7 and we need $7 \mid n+2$, which gives $n = 5$ (and then $(n+2)^2 = 49$). The total is $37 + 1 + 1 = 39$.

Check. $n = 5$: $120 + 720 + 5040 = 5880 = 49 \cdot 120$. $n = 13$ fails: $13! \cdot 15^2$ has only one factor 7. And $n = 6$ fails: $6! \cdot 64 = 46080$ is not even a multiple of 7.

20. Answer: $6 - 2\sqrt{5}$. **The idea.** Sketch the graph. The parabola $y = x^2 - 6x + 5 = (x-1)(x-5)$ dips below the axis between $x = 1$ and $x = 5$, and the absolute value flips that piece up into a hump $y = -x^2 + 6x - 5$ with top $(3,4)$. A line through the origin with positive slope always crosses the two outer arms, once each. So the only way to get exactly three points is for the line to touch the hump: it must be tangent to it.

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Work it out. If $m \le 0$, the line has $y \le 0$ for $x > 0$, so it can meet the graph (which is never negative) at most at $(1,0)$ and $(5,0)$ when $m = 0$, and otherwise only at points with $x < 0$, of which the equation $x^2 - (6+m)x + 5 = 0$ gives at most two. So $m > 0$, and then every meeting point has $x > 0$. *Outer arms* ($x \le 1$ or $x \ge 5$): $x^2 - (6+m)x + 5 = 0$. The left side equals $-m < 0$ at $x = 1$ and $-5m < 0$ at $x = 5$, so there is exactly one root below 1 and one above 5: always two points. *Hump* ($1 < x < 5$): $-x^2 + 6x - 5 = mx$, i.e. $x^2 + (m-6)x + 5 = 0$. We need exactly one root in $(1,5)$. At $x = 1$ and $x = 5$ the left side is $m > 0$ and $5m > 0$, so the roots are either both in $(1,5)$ or both outside; one point means a double root: $(m-6)^2 = 20$, so $m = 6 \pm 2\sqrt{5}$. The double root is $x = \frac{6-m}{2} = \pm\sqrt{5}$, and it lies in $(1,5)$ only for $m = 6 - 2\sqrt{5}$, touching at $x = \sqrt{5}$. (For $0 < m < 6 - 2\sqrt{5}$ the line cuts the hump twice, giving four points; for larger m it misses the hump, giving two.) So $m = 6 - 2\sqrt{5}$.

Check. At $x = \sqrt{5}$ the hump has height $-5 + 6\sqrt{5} - 5 = 6\sqrt{5} - 10$, and the line has height $(6 - 2\sqrt{5})\sqrt{5} = 6\sqrt{5} - 10$, the same. The hump's slope there is $-2\sqrt{5} + 6 = m$, so the line really is tangent. Also $m \approx 1.53$ is a sensible slope: the line from the origin to the top of the hump, $(3,4)$, has slope $\frac{4}{3} \approx 1.33$, and the tangent must be a bit steeper.

21. Answer: 15**. The idea.** Upright triangles are easy to count, but there are also upside-down and tilted ones. The key is that every equilateral triangle on the dots fits snugly inside exactly one upright triangle of the array, with one vertex on each side of it, each the same distance t along its side (going around the same way). An upright triangle of side s holds exactly s such inscribed triangles, one for each $t = 0, 1, \ldots, s-1$ ($t = 0$ is the upright triangle itself).

Work it out. The array contains 6 upright triangles of side 1, 3 of side 2 and 1 of side 3. Each contributes as many equilateral triangles as its side length: $6 \cdot 1 + 3 \cdot 2 + 1 \cdot 3 = 15$.

Check. List them by type: $6 + 3 + 1 = 10$ upright ones; 3 upside-down triangles of side 1 (the middle triangles of the three upright triangles of side 2); and 2 tilted triangles of side $\sqrt{7}$ inside the whole array, with vertices 1 unit along each side (clockwise for one, counterclockwise for the other). That is $10 + 3 + 2 = 15$.

22. Answer: 6**. The idea.** The sum $1+a+b+c$ is just the value of the polynomial at $x = 1$. Writing the polynomial in factored form turns that value into a product of three consecutive integers, which can equal only one thing.

Work it out. Let the roots be m, $m+1$, $m+2$, so the polynomial is $(x-m)(x-m-1)(x-m-2)$. At $x = 1$ it equals $1 + a + b + c = 24$, so $(1-m)(-m)(-m-1) = 24$, i.e. $(m-1)\,m\,(m+1) = -24$. A product of three consecutive integers equal to -24 must be $(-4)(-3)(-2)$, so $m = -3$ and the roots are $-3, -2, -1$. The polynomial is $(x+1)(x+2)(x+3) = x^3 + 6x^2 + 11x + 6$, and $c = 6$.

Check. $a + b + c = 6 + 11 + 6 = 23$. Also, c is minus the product of the roots: $-(-3)(-2)(-1) = 6$. No other m works, since $(m-1)m(m+1)$ is 0 for $m = -1, 0, 1$, is positive for $m \ge 2$, and is $-6, -24, -60, \ldots$ for $m = -2, -3, -4, \ldots$.

23. Answer: $3\sqrt{15}$**. The idea.** PQ is a chord of the circle, so all we need is its distance from the centre O. The centre of the circle is also the centroid of the triangle, which pins down where line MN sits.

Work it out. Put O at the origin with A straight above it, $OA = 6$. The centroid divides the altitude from A in the ratio $2:1$, so side $\overline{BC}$ is 3 below O. The midline MN is parallel to $\overline{BC}$ and

halfway between A and $\overline{BC}$, at height $\frac{6+(-3)}{2} = \frac{3}{2}$ above O. A chord at distance $\frac{3}{2}$ from the centre of a circle of radius 6 has half-length $\sqrt{36 - \frac{9}{4}} = \sqrt{\frac{135}{4}} = \frac{3\sqrt{15}}{2}$, so $PQ = 3\sqrt{15}$.

Check. Use intersecting chords at M. The side is $6\sqrt{3}$, so $AM = MB = 3\sqrt{3}$ and $MN = 3\sqrt{3}$. By symmetry $PM = NQ = x$, and chords $\overline{AB}$ and $\overline{PQ}$ cross at M, so $x\,(x + 3\sqrt{3}) = AM \cdot MB = 27$. Then $x = \frac{-3\sqrt{3}+\sqrt{27+108}}{2} = \frac{3\sqrt{15}-3\sqrt{3}}{2}$ and $PQ = 2x + 3\sqrt{3} = 3\sqrt{15}$.

24. Answer: $\frac{71}{1296}$. The idea. Sort sociable rolls by their pattern of repeats: how many dice show each number that appears. Those counts are at least 2 and add up to 6, so there are only four patterns. Count the ordered outcomes (the dice are different) for each pattern.

Work it out. The multiplicities form a partition of 6 into parts of size at least 2: 6, 4 + 2, 3 + 3 and 2 + 2 + 2. Pattern 6 (all dice equal): 6 outcomes. Pattern 4 + 2: choose the number shown four times (6 ways), the number shown twice (5 ways) and which two dice show it ($\binom{6}{2} = 15$): 450. Pattern 3 + 3: choose the two numbers ($\binom{6}{2} = 15$) and which three dice show the smaller one ($\binom{6}{3} = 20$): 300. Pattern 2 + 2 + 2: choose the three numbers ($\binom{6}{3} = 20$) and split the dice into a pair for each ($\frac{6!}{2!\,2!\,2!} = 90$): 1800. The total is $6 + 450 + 300 + 1800 = 2556$ out of $6^6 = 46656$ outcomes, and $\frac{2556}{46656} = \frac{71}{1296}$.

Check. Try the same method on four dice, where the patterns are 4 and 2 + 2: $6 + \binom{6}{2} \cdot \binom{4}{2} = 6 + 90 = 96$ of 1296. Listing directly, a sociable roll of four dice is either four of a kind (6) or two pairs of different numbers (15 choices of numbers times 6 placements), 96 again. For six dice the answer is about 5.5%, which is believable: most rolls of six dice contain some number exactly once.

25. Answer: $\frac{121}{144}$. The idea. Someone wins exactly when the smallest number chosen is chosen by only one player. Split by who the winner is and what the winning number is: then the other three players just have to pick larger numbers. The sum that appears is a sum of cubes, and sums of cubes are perfect squares, which makes the answer collapse.

Work it out. No two players can both win, so the probability is the sum over the 4 players and the 12 possible winning numbers m of the probability that this player picks m and the other three pick numbers larger than m: $4 \sum_{m=1}^{12} \frac{1}{12} \left(\frac{12-m}{12}\right)^3 = \frac{4}{12^4} \sum_{j=0}^{11} j^3$. Now $0^3 + 1^3 + \cdots + 11^3 = (0 + 1 + \cdots + 11)^2 = 66^2 = 4356$, so the probability is $\frac{4 \cdot 4356}{20736} = \frac{17424}{20736} = \frac{121}{144}$.

Check. The same computation with numbers 1 to N gives $\frac{4}{N^4} \left(\frac{N(N-1)}{2}\right)^2 = \left(\frac{N-1}{N}\right)^2$, and $N = 12$ gives $\left(\frac{11}{12}\right)^2 = \frac{121}{144}$. Test $N = 2$ directly: someone wins only when exactly one player picks 1 and the other three pick 2, which is 4 of the 16 outcomes, $\frac{1}{4} = \left(\frac{1}{2}\right)^2$. Also sensible: with many numbers to choose from, ties are rare and someone almost always wins.

26. Answer: $\frac{689}{45}$. The idea. Let X be the position of the first ace and Y the position of the first king. We stop at $\max(X, Y)$. A maximum is awkward, but $\max(X, Y) + \min(X, Y) = X + Y$ always, and $\min(X, Y)$ is just the position of the first card that is an ace *or* a king. So $E[\max] = E[X] + E[Y] - E[\min]$, and each of these is a "first special card" expectation, which the gap trick gives instantly.

Work it out. Gap trick: if k special cards are in a deck of n, they cut the $n - k$ other cards into

$k+1$ gaps (before the first special card, between consecutive ones, after the last), and by symmetry every gap has the same expected size $\frac{n-k}{k+1}$. The first special card comes right after the first gap, so its expected position is $\frac{n-k}{k+1}+1=\frac{n+1}{k+1}$. For the aces, $E[X]=\frac{53}{5}$; likewise $E[Y]=\frac{53}{5}$. For the 8 aces and kings together, $E[\min(X,Y)]=\frac{53}{9}$. So the answer is $\frac{53}{5}+\frac{53}{5}-\frac{53}{9}=\frac{954-265}{45}=\frac{689}{45}$.

Check. Try a 3-card deck of one ace, one king and one other card. The formula gives $\frac{4}{2}+\frac{4}{2}-\frac{4}{3}=\frac{8}{3}$. Directly, the six orders AKO, AOK, KAO, KOA, OAK, OKA stop after $2,3,2,3,3,3$ cards, averaging $\frac{16}{6}=\frac{8}{3}$. Also $\frac{689}{45}\approx 15.3$ is more than $\frac{53}{5}=10.6$, as it must be, since waiting for both takes longer than waiting for one.

27. Answer: 360**. The idea.** Least common multiples work prime by prime: lcm takes the larger exponent of each prime. So the conditions split into independent conditions on the exponents of each prime in a, b, c, and $T(N)$ is a product of one factor per prime. Find that factor, then design the cheapest N whose factors multiply to 280.

Work it out. Let p^e exactly divide N, and let x, y, z be the exponents of p in a, b, c (all at most e, since each divides N). Every pair must have maximum e, which means at most one of x, y, z can be less than e. Either all three equal e (1 way), or exactly one is smaller: 3 positions times e values $0, \ldots, e-1$. That is $3e+1$ choices, so $T(N)=\prod(3e_i+1)$ over the prime powers $p_i^{e_i}$ in N (and $T(2)=3\cdot 1+1=4$ matches the example). Now $280=2^3\cdot 5\cdot 7$ must be written as a product of factors that are each 1 more than a multiple of 3 and at least 4. Its divisors of that form are $4, 7, 10, 28, 40, 70, 280$, and the possible factorizations are 280, $4\cdot 70$, $7\cdot 40$, $10\cdot 28$ and $4\cdot 7\cdot 10$, with exponent lists $\{93\}$, $\{1,23\}$, $\{2,13\}$, $\{3,9\}$ and $\{1,2,3\}$. For each list, the smallest N gives the largest exponent to the smallest prime. The candidates are 2^{93}, $2^{23}\cdot 3$, $2^{13}\cdot 3^2$, $2^9\cdot 3^3=13824$ and $2^3\cdot 3^2\cdot 5=360$. The smallest is 360.

Check. $T(360)=(3\cdot 3+1)(3\cdot 2+1)(3\cdot 1+1)=10\cdot 7\cdot 4=280$. A computer that counts the triples directly for every N from 1 up finds that 360 is the first with exactly 280.

28. Answer: $\frac{6}{11}$**. The idea.** The centers of the three circles are $1+2=3$, $1+3=4$ and $2+3=5$ apart: a 3-4-5 right triangle, which invites coordinates. Every tangency becomes a distance condition ("distance between centers = sum of radii" outside, "difference of radii" inside). Subtracting two such equations kills the squares and leaves a line.

Work it out. Put the radius-1 center at $O_1=(0,0)$, radius-2 at $O_2=(3,0)$, radius-3 at $O_3=(0,4)$. *Find* Ω. Its center (x,y) and radius R satisfy $x^2+y^2=(R-1)^2$, $(x-3)^2+y^2=(R-2)^2$, $x^2+(y-4)^2=(R-3)^2$. Subtracting the first from the others: $x=\frac{R+3}{3}$ and $y=\frac{R+2}{2}$. Substituting: $4(R+3)^2+9(R+2)^2=36(R-1)^2$, i.e. $23R^2-132R-36=0=(R-6)(23R+6)$. So $R=6$ and Ω has center $(3,4)$. *The fifth circle.* Its center (x,y) and radius r satisfy $x^2+y^2=(1+r)^2$, $x^2+(y-4)^2=(3+r)^2$ and $(x-3)^2+(y-4)^2=(6-r)^2$. The first two give $y=1-\frac{r}{2}$; the first and third give $6x+8y=14r-10$, so $x=3r-3$. Then $9(r-1)^2+\left(1-\frac{r}{2}\right)^2=(1+r)^2$ simplifies to $11r^2-28r+12=0=(r-2)(11r-6)$. The root $r=2$ is the circle of radius 2 itself (center $(3,0)$), which is also tangent to all three; the other root is the fifth circle, $r=\frac{6}{11}$, with center $\left(-\frac{15}{11},\frac{8}{11}\right)$.

Check. Descartes' circle theorem: with curvatures $k=\frac{1}{r}$, and $-\frac{1}{6}$ for the enclosing circle, a circle tangent to three mutually tangent circles has $k_4=k_1+k_2+k_3\pm 2\sqrt{k_1k_2+k_2k_3+k_3k_1}$. For $1, \frac{1}{2}, \frac{1}{3}$: $\frac{11}{6}\pm 2\cdot 1$, and $-\frac{1}{6}$ confirms $R=6$. For $-\frac{1}{6}, 1, \frac{1}{3}$: $\frac{7}{6}\pm 2\sqrt{\frac{1}{9}}=\frac{7}{6}\pm\frac{2}{3}$, which is $\frac{1}{2}$ (the radius-2 circle) or $\frac{11}{6}$, radius $\frac{6}{11}$.

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29. Answer: 3300**. The idea.** Write $k^2 = 101\lfloor k^2/101 \rfloor + r_k$, where r_k is the remainder. Summing, the floors add to $\frac{1}{101}\left(\sum k^2 - \sum r_k\right)$, and $\sum k^2$ is easy. The whole problem is the sum of the remainders, and the trick is to pair them: $100 = 10^2$ is a perfect square with $100 \equiv -1 \pmod{101}$, so whenever r is a square remainder, so is $101 - r$.

Work it out. $\sum_{k=1}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} = 338350$. The remainders: k and $101 - k$ have the same square mod 101, and 101 is prime, so the 100 values r_k consist of 50 different nonzero remainders, each appearing exactly twice. If $r \equiv m^2$, then $101 - r \equiv -m^2 \equiv 10^2 m^2 = (10m)^2$, so $101 - r$ is also among the 50 remainders, and $r \neq 101 - r$ because 101 is odd. So the 50 square remainders split into 25 pairs each adding to 101: they total $25 \cdot 101 = 2525$, and $\sum r_k = 2 \cdot 2525 = 5050$. Therefore the sum of floors is $\frac{338350 - 5050}{101} = \frac{333300}{101} = 3300$.

Check. The same argument works for any prime p that leaves remainder 1 when divided by 4 and gives $\frac{(p-1)(p-2)}{3}$; for $p = 101$ that is $\frac{100 \cdot 99}{3} = 3300$. Try $p = 5$: $\lfloor \frac{1}{5} \rfloor + \lfloor \frac{4}{5} \rfloor + \lfloor \frac{9}{5} \rfloor + \lfloor \frac{16}{5} \rfloor = 0 + 0 + 1 + 3 = 4 = \frac{4 \cdot 3}{3}$. For $p = 13$: the formula gives 44, and direct addition of $\lfloor k^2/13 \rfloor$ for $k = 1, \ldots, 12$ (0, 0, 0, 1, 1, 2, 3, 4, 6, 7, 9, 11) gives 44 too.

30. Answer: 121**. The idea.** Colour the desks like a checkerboard: 6 black and 6 white. Every move goes to a desk of the other colour. So the 6 students from black desks fill all 6 white desks, each moving to a neighbour: that pairs every black desk with a neighbouring white desk, which is exactly a tiling of the 3×4 board by dominoes. The students from white desks likewise give a second domino tiling, chosen independently of the first. So the answer is T^2, where T is the number of domino tilings of a 3×4 board.

Work it out. Cut the board down the middle into two 3×2 halves. A 3×2 block has 3 tilings (three horizontal dominoes, or one horizontal domino at the top or at the bottom with two vertical ones beside it), so there are $3 \cdot 3 = 9$ tilings with no domino crossing the middle line. A crossing domino is horizontal; each half has 6 cells, so an even number of dominoes cross, and 2 crossing in rows 1 and 2 or in rows 2 and 3 each leave just one way to finish, while rows 1 and 3 leave the middle-row cell next to the cut stranded. So $T = 9 + 2 = 11$, and the answer is $11^2 = 121$.

Check. Conversely, any two tilings give a valid move: each black student slides along his domino of the first tiling and each white student along her domino of the second, and every desk receives exactly one student. When the same domino appears in both tilings, its two students simply swap desks.

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