MathCastle

STATE COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Two nonzero real numbers x and y have a surprising property: their sum $x + y$, their product xy and their quotient $\frac{x}{y}$ are all equal to one another. What is the value of $x+y$? Express your answer as a common fraction.

1. _____

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2. Eight consecutive integers are written in increasing order. The sum of the first five of them is equal to the sum of the last three. What is the largest of the eight integers?

2. _____

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3. At a school, 60% of the students play on a sports team and 40% of the students play a musical instrument. Among the students who play on a sports team, the fraction who also play an instrument is exactly equal to the fraction of the instrument players who do *not* play on a sports team. What percent of all the students play both on a sports team and an instrument?

3. _____

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4. The ordinary 10×10 multiplication table has 100 entries: the products $i \cdot j$ for every pair of whole numbers i and j from 1 to 10. Many values appear more than once (for example, 12 appears as $2 \cdot 6$, $3 \cdot 4$, $4 \cdot 3$ and $6 \cdot 2$). How many *different* numbers appear in the table?

4. _____

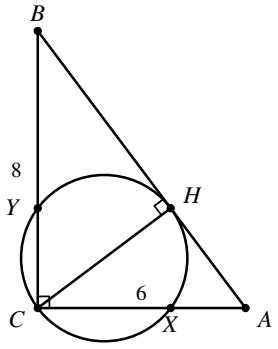
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5. Eight unit cubes, 4 white and 4 black, are glued together to form a $2 \times 2 \times 2$ cube. The colors are placed at random: every choice of which 4 of the 8 positions hold the white cubes is equally likely. What is the probability that none of the six faces of the large cube is entirely one color? Express your answer as a common fraction.

5. _____

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6. Right triangle ABC has its right angle at C, with $CA = 6$ and $CB = 8$. Point H is the foot of the altitude from C to the hypotenuse $\overline{AB}$. The circle with diameter $\overline{CH}$ meets leg $\overline{CA}$ again at a point X and leg $\overline{CB}$ again at a point Y. What is the length XY? Express your answer as a common fraction.



6. _____

7. Eight cards are numbered 1 through 8. They are shuffled and dealt into two hands of four cards each, with every way of choosing the four cards of the first hand equally likely. What is the probability that the numbers in the two hands have the same sum? Express your answer as a common fraction.

7. _____

8. Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$, where a, b, c and d are real constants. The equation $f\big(g(x)\big) = 0$ has exactly four real solutions: $x = 1$, 3, 4 and 6. What is the positive difference between the two roots of f?

8. _____

9. A function f is defined for every nonzero real number, and for every nonzero real x it satisfies $2f(x) - f\left(\frac{1}{x}\right) = 3x$. There are exactly two real numbers x for which $f(x) = 5$. What is the sum of their squares? Express your answer as a common fraction.

9. _____

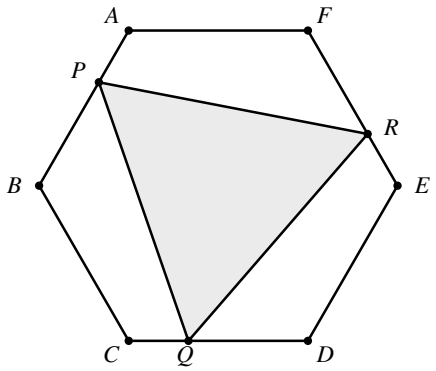
10. For each integer n from 1 to 2026, Priya adds the power 2^n to the number n itself; for example, $n = 6$ gives $2^6 + 6 = 70$. For how many of these 2026 values of n is the result $2^n + n$ divisible by 7?

10. _____

11. Rectangle $ABCD$ has $AB = 13$ and $BC = 6$, with its vertices labelled in order around it. There are exactly two points P on side $\overline{CD}$ for which $\angle APB = 90°$. How far apart are these two points?

11. _____

12. Regular hexagon $ABCDEF$ has area 36. Points P, Q and R lie on sides $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$, respectively, with $AP : PB = CQ : QD = ER : RF = 1 : 2$. What is the area of triangle PQR?



12. _____

13. The numbers r, s and t are the three roots of $x^3 - 5x^2 + 2x + 7 = 0$. What is the value of $\frac{1}{r-2} + \frac{1}{s-2} + \frac{1}{t-2}$? 13. _____

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14. A $4 \times 4 \times 4$ cube is painted red on its outside and then cut into 64 unit cubes. One of the unit cubes is chosen at random and rolled like a fair die. The face that lands on top is unpainted. What is the probability that the face touching the table (the bottom face) is painted? Express your answer as a common fraction. 14. _____

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15. The number $25! = 1 \cdot 2 \cdot 3 \cdots 25$ ends in a string of zeros. After those trailing zeros are removed, what is the units digit of the number that remains? In other words, what is the last nonzero digit of 25!? 15. _____

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16. Two two-digit numbers are chosen at random, independently, from 10 to 99 (they may be equal), and they are added in the usual column way. What is the probability that no carrying is needed at all, neither out of the ones column nor out of the tens column? Express your answer as a common fraction. 16. _____

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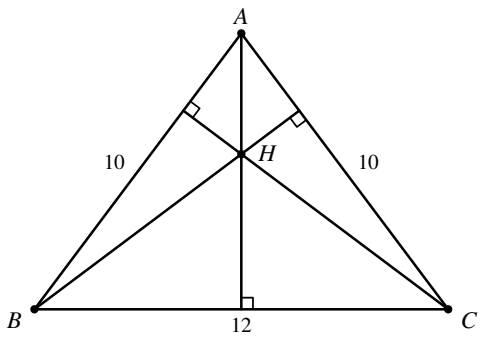
17. The whole numbers from 100 to 199 are written one after another, in increasing order and with no spaces, to form the 300-digit number $N = 100101102103 \ldots 198199$. What is the remainder when N is divided by 1001? 17. _____

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18. What is the least possible value of $x^2 + 4xy + 5y^2 - 2x - 6y + 20$, where x and y can be any real numbers? 18. _____

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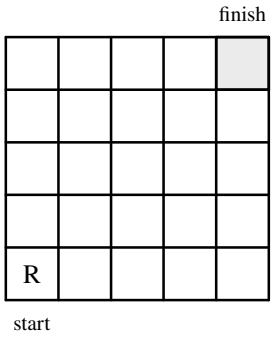
19. Triangle ABC has $AB = AC = 10$ and $BC = 12$. Its three altitudes (the segments from each vertex perpendicular to the opposite side) all pass through one point H. What is the distance AH? Express your answer as a common fraction.



19. _____

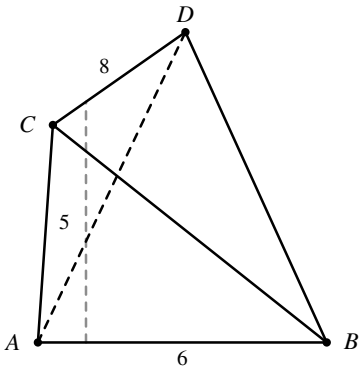
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20. A rook sits in the bottom-left square of a 5×5 board and wants to reach the top-right square. Each move takes it any positive number of squares to the right or any positive number of squares up, staying on the board; it never moves left or down. Two journeys are different if their sequences of moves are different. How many different journeys are there?



20. _____

21. In tetrahedron $ABCD$, edge $\overline{AB}$ has length 6 and the opposite edge $\overline{CD}$ has length 8. The lines AB and CD are perpendicular to each other (though they do not meet), and the distance between these two lines, measured along a segment perpendicular to both of them, is 5. What is the volume of the tetrahedron?



21. _____

22. A grid has 4 rows and 7 columns of unit squares. Some of its 28 squares are to be shaded so that no 2×2 block of four squares (two adjacent rows and two adjacent columns) is completely shaded. First find the largest number of squares that can be shaded this way. How many different shadings use exactly that largest number of shaded squares?

22. _____

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23. A sequence starts with $a_1 = 1$, and each later term is the previous term plus its reciprocal: $a_{n+1} = a_n + \frac{1}{a_n}$ for $n \geq 1$. So $a_2 = 2$, $a_3 = \frac{5}{2}$ and $a_4 = \frac{29}{10}$. What is the greatest integer that is less than or equal to a_{2026}? 23. _____

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24. A path from $(0, 0)$ to $(6, 4)$ is made of 10 unit steps, each one unit to the right or one unit up, and one of the $\binom{10}{4} = 210$ such paths is chosen at random, all equally likely. A *turn* is a place where a step is immediately followed by a step in the other direction. What is the expected (average) number of turns on the chosen path? Express your answer as a common fraction. 24. _____

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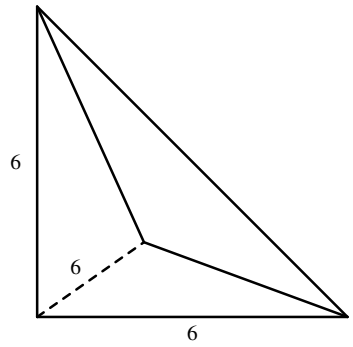
25. A repunit is a positive integer whose digits are all 1s, such as 1, 11 or 111. What is the smallest positive integer n such that the repunit made of n ones is divisible by 7749? 25. _____

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26. Seven friends each join the chess club, the drama club, or both clubs; nobody joins neither. In how many different ways can this happen if each of the two clubs ends up with at least 5 of the seven friends as members? 26. _____

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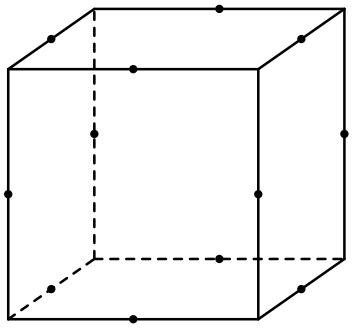
27. A cube has edge length 6. One corner is sliced off by the plane through the three vertices adjacent to that corner, giving a tetrahedron with three mutually perpendicular edges of length 6 meeting at the corner. Sphere S_1 is the largest sphere inside this tetrahedron (it touches all four faces). Sphere S_2 sits in the right-angled corner of the tetrahedron: it touches the three faces that meet at that corner, and it is externally tangent to S_1. Find the radius of S_2. Express your answer in simplest radical form.



27. _____

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28. The 12 midpoints of the edges of a cube are marked. How many equilateral triangles have all three of their vertices among these 12 marked points?



28. _____

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29. For a positive integer n, write its digits twice in a row to form a new number; for example, $n = 57$ gives 5757 and $n = 2026$ gives 20262026. For which n is this new number divisible by n^2? Find the sum of all such positive integers n that are less than 1,000,000.

29. _____

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30. Nine points are placed on a circle, and each of the 36 pairs of points is joined by a chord. The points are placed so that no three chords pass through a common point inside the circle. The chords form many triangles (each side of such a triangle lies along one of the chords). How many of these triangles have all three of their vertices strictly inside the circle?

30. _____

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Answer Key — MathCounts State Sprint Practice Contest 4

1. $-\frac{1}{2}$

2. 11

3. 24

4. 42

5. $\frac{32}{35}$

6. $\frac{24}{5}$

7. $\frac{4}{35}$

8. 6

9. $\frac{21}{4}$

10. 289

11. 5

12. 14

13. -6

14. $\frac{1}{3}$

15. 4

16. $\frac{11}{45}$

17. 50

18. 18

19. $\frac{7}{2}$

20. 838

21. 40

22. 27

23. 63

24. $\frac{24}{5}$

25. 270

26. 519

27. $9 - 5\sqrt{3}$

28. 16

29. 144

30. 84

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Solutions — MathCounts State Sprint Practice Contest 4

1. Answer: $-\frac{1}{2}$. The idea. Of the three equal quantities, the product and the quotient are the easiest pair to compare: setting $xy = \frac{x}{y}$ and cancelling the nonzero x pins y down almost completely. Only then use the sum.

Work it out. From $xy = \frac{x}{y}$, multiply by y to get $xy^2 = x$; since $x \neq 0$, $y^2 = 1$, so $y = 1$ or $y = -1$. If $y = 1$, the condition $x + y = xy$ becomes $x + 1 = x$, which is impossible. So $y = -1$, and $x + y = xy$ becomes $x - 1 = -x$, giving $x = \frac{1}{2}$. Then $x + y = \frac{1}{2} - 1 = -\frac{1}{2}$.

Check. With $x = \frac{1}{2}$ and $y = -1$: the sum is $-\frac{1}{2}$, the product is $-\frac{1}{2}$, and the quotient is $\frac{1/2}{-1} = -\frac{1}{2}$. All three agree, and this is the only pair that works.

2. Answer: 11**. The idea.** Line the last three numbers up against the first three. Each of the last three is exactly 5 more than the number five places before it, so the last three beat the first three by $3 \cdot 5 = 15$. For the sums to balance, the fourth and fifth numbers must make up exactly those 15.

Work it out. Call the numbers $n, n+1, \ldots, n+7$. The last three, $n+5$, $n+6$, $n+7$, add to 15 more than the first three, n, $n+1$, $n+2$. So the equation "first five = last three" says that the fourth and fifth numbers add to 15: $(n+3)+(n+4) = 15$, so $2n = 8$ and $n = 4$. The numbers are $4, 5, \ldots, 11$, and the largest is 11. (Directly: $5n + 10 = 3n + 18$ gives $n = 4$ as well.)

Check. $4 + 5 + 6 + 7 + 8 = 30$ and $9 + 10 + 11 = 30$.

3. Answer: 24**. The idea.** Measure everything as a percent of the whole school and let b be the percent who do both. Each of the two "fractions" in the statement is a part of a known group, so the condition becomes one linear equation in b.

Work it out. The sports players are 60% of the school, and b% of the school are sports players who also play an instrument, so that fraction is $\frac{b}{60}$. The instrument players are 40% of the school; those who play no sport are $(40 - b)$% of the school, so that fraction is $\frac{40-b}{40}$. Setting them equal: $40b = 60(40 - b)$, so $40b = 2400 - 60b$, $100b = 2400$ and $b = 24$.

Check. In a school of 100 students, 24 do both. Of the 60 athletes, $\frac{24}{60} = 40\%$ play an instrument; of the 40 musicians, $40 - 24 = 16$ play no sport, and $\frac{16}{40} = 40\%$ too. (For the record, $60 + 40 - 24 = 76\%$ do at least one activity.)

4. Answer: 42**. The idea.** Because $i \cdot j = j \cdot i$, it is enough to look at products with $i \leq j$. Go row by row, $i = 1, 2, \ldots, 10$, using only $j \geq i$, and count just the values that have not appeared in an earlier row.

Work it out. Row 1 gives $1, 2, \ldots, 10$: 10 new. Row 2 ($j \geq 2$) gives 4, 6, 8, 10 (already seen) and 12, 14, 16, 18, 20: 5 new. Row 3 gives 9, 12, 18 (seen) and 15, 21, 24, 27, 30: 5 new. Row 4 gives 16, 20, 24 (seen) and 28, 32, 36, 40: 4 new. Row 5: 30, 40 are old; 25, 35, 45, 50 are new: 4. Row 6: 36 is old; 42, 48, 54, 60: 4. Row 7: 49, 56, 63, 70: 4. Row 8: 64, 72, 80: 3. Row 9: 81, 90: 2. Row 10: 100: 1. Total: $10 + 5 + 5 + 4 + 4 + 4 + 4 + 3 + 2 + 1 = 42$.

Check. Count by tens instead. 1–10: all 10; 11–20: 12, 14, 15, 16, 18, 20 (6); 21–30: 21, 24, 25, 27, 28, 30 (6); 31–40: 32, 35, 36, 40 (4); 41–50: 42, 45, 48, 49, 50 (5); 51–60: 54, 56, 60 (3); 61–70: 63, 64, 70 (3); 71–80: 72, 80 (2); 81–90: 81, 90 (2); 91–100: 100 (1). Again 42.

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5. Answer: $\frac{32}{35}$**. The idea.** Count the bad arrangements instead. Each face of the large cube shows exactly 4 of the small cubes, and there are exactly 4 white cubes. So a face that is all white uses up every white cube, and a face that is all black leaves the white cubes no room except the opposite face. Either way the white cubes fill one whole face.

Work it out. There are $\binom{8}{4} = 70$ equally likely sets of white positions. If some face is all white, the white set is exactly that face. If some face is all black, the four white cubes sit in the other four positions, which form the opposite face. So an arrangement is bad exactly when the white cubes form one of the 6 faces: 6 bad arrangements (and in each of them two opposite faces are single-colored). The probability that no face is one color is $\frac{70-6}{70} = \frac{64}{70} = \frac{32}{35}$.

Check. A good arrangement, for instance white cubes at four corners no two of which share an edge (a regular tetrahedron's vertices), shows 2 white and 2 black on every face. And a bad one, white cubes filling the top layer, makes the top face all white and the bottom face all black, while each side face shows 2 and 2: exactly one of the 6 bad sets.

6. Answer: $\frac{24}{5}$**. The idea.** There is no need to find X and Y. An angle inscribed in a semicircle is a right angle, so X and Y see the diameter $\overline{CH}$ at right angles, and together with the right angle at C this makes $CXHY$ a rectangle. The diagonals of a rectangle are equal, so $XY = CH$.

Work it out. Since $\overline{CH}$ is a diameter, $\angle CXH = 90°$ and $\angle CYH = 90°$; also $\angle XCY = 90°$. A quadrilateral with three right angles is a rectangle, so $CXHY$ is a rectangle and its diagonals $\overline{XY}$ and $\overline{CH}$ are equal. The hypotenuse is $AB = \sqrt{6^2 + 8^2} = 10$, and computing the area two ways, $\frac{1}{2} \cdot 6 \cdot 8 = \frac{1}{2} \cdot 10 \cdot CH$, gives $CH = \frac{48}{10} = \frac{24}{5}$. So $XY = \frac{24}{5}$.

Check. Put $C = (0, 0)$, $A = (6, 0)$, $B = (0, 8)$. The foot of the altitude is $H = \left(\frac{96}{25}, \frac{72}{25}\right)$, and X, Y are the feet of the perpendiculars from H to the legs: $X = \left(\frac{96}{25}, 0\right)$ and $Y = \left(0, \frac{72}{25}\right)$. Then $XY = \frac{1}{25}\sqrt{96^2 + 72^2} = \frac{24 \cdot 5}{25} = \frac{24}{5}$.

7. Answer: $\frac{4}{35}$**. The idea.** The eight numbers add to 36, so equal sums means each hand adds to exactly 18. Count the 4-card sets with sum 18 carefully; a neat way to organize them is by the hand that holds card 1, since the other hand is then determined.

Work it out. There are $\binom{8}{4} = 70$ equally likely first hands. The hand containing 1 needs three more cards from 2–8 adding to 17: $\{2, 7, 8\}$, $\{3, 6, 8\}$, $\{4, 5, 8\}$, $\{4, 6, 7\}$ (a triple without 8 is at most $7 + 6 + 5 = 18$, and the only one reaching 17 is $\{4, 6, 7\}$; with 8 the other two add to 9: $2 + 7$, $3 + 6$, $4 + 5$). So there are 4 equal splits, i.e. 4 sets containing 1 and their 4 complements: 8 good first hands out of 70. The probability is $\frac{8}{70} = \frac{4}{35}$.

Check. The eight good hands are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, $\{1, 4, 6, 7\}$ and their complements $\{3, 4, 5, 6\}$, $\{2, 4, 5, 7\}$, $\{2, 3, 6, 7\}$, $\{2, 3, 5, 8\}$; each adds to 18, and no other set does.

8. Answer: 6. **The idea.** If f has roots p and q, then $f(g(x)) = 0$ means $g(x) = p$ or $g(x) = q$. Each of those equations has at most two solutions, and the two solutions of $g(x) = k$ are always placed symmetrically about the axis of the parabola $y = g(x)$. So the four solutions split into two pairs with the same midpoint.

Work it out. To get four solutions, $p \neq q$ and each equation $g(x) = p$, $g(x) = q$ has two of them. The two pairs must have the same midpoint (the axis of g). Among the ways to pair up $1, 3, 4, 6$, only $\{1, 6\}$ and $\{3, 4\}$ share a midpoint, namely $\frac{7}{2}$ (the pairing $\{1, 3\}$, $\{4, 6\}$ has midpoints 2 and

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5, and $\{1,4\}, \{3,6\}$ has $\frac{5}{2}$ and $\frac{9}{2}$). So $g(x) = \left(x - \frac{7}{2}\right)^2 + k$ for some constant k, and the roots of f are $g(1) = \frac{25}{4} + k$ and $g(3) = \frac{1}{4} + k$. Their difference is $\frac{25}{4} - \frac{1}{4} = 6$, whatever k is.

Check. Take $k = -\frac{1}{4}$, so $g(x) = x^2 - 7x + 12 = (x-3)(x-4)$; then $g(1) = 6$ and $g(3) = 0$, so $f(t) = t(t-6)$. Indeed $f(g(x)) = (x-3)(x-4)(x^2-7x+6) = (x-3)(x-4)(x-1)(x-6)$, with roots exactly $1, 3, 4, 6$, and the roots 0 and 6 of f differ by 6.

9. Answer: $\frac{21}{4}$. The idea. The equation links $f(x)$ with $f(1/x)$. Replace x by $1/x$ to get a second equation linking the same two unknowns, and solve the pair like any two linear equations in two unknowns. Then use Vieta's formulas instead of the ugly roots.

Work it out. The two equations are $2f(x) - f(1/x) = 3x$ and $2f(1/x) - f(x) = \frac{3}{x}$. Double the first and add the second: $3f(x) = 6x + \frac{3}{x}$, so $f(x) = 2x + \frac{1}{x}$. Now $f(x) = 5$ means $2x + \frac{1}{x} = 5$, that is, $2x^2 - 5x + 1 = 0$ (and $x = 0$ is not a root, so nothing is lost by multiplying by x). Its discriminant is $25 - 8 = 17 > 0$, so there are two real roots, with sum $\frac{5}{2}$ and product $\frac{1}{2}$. The sum of their squares is $\left(\frac{5}{2}\right)^2 - 2 \cdot \frac{1}{2} = \frac{25}{4} - 1 = \frac{21}{4}$.

Check. With $f(x) = 2x + \frac{1}{x}$ we get $f(1/x) = \frac{2}{x} + x$, and $2f(x) - f(1/x) = 4x + \frac{2}{x} - \frac{2}{x} - x = 3x$. The roots are $\frac{5 \pm \sqrt{17}}{4} \approx 2.28$ and 0.22, whose squares add to about $5.20 + 0.05 = 5.25 = \frac{21}{4}$.

10. Answer: 289**. The idea.** The remainder of 2^n on division by 7 repeats every 3 steps $(2, 4, 1, 2, 4, 1, \ldots)$, and the remainder of n repeats every 7 steps. So the remainder of $2^n + n$ repeats every $\operatorname{lcm}(3, 7) = 21$ steps: check one block of 21 and multiply.

Work it out. We need $n \equiv -2^n \pmod 7$. If $n \equiv 1 \pmod 3$, $2^n \equiv 2$, so $n \equiv 5 \pmod 7$; combined with $n \equiv 1 \pmod 3$ this gives $n \equiv 19 \pmod{21}$. If $n \equiv 2 \pmod 3$, $2^n \equiv 4$, so $n \equiv 3 \pmod 7$: $n \equiv 17 \pmod{21}$. If $n \equiv 0 \pmod 3$, $2^n \equiv 1$, so $n \equiv 6 \pmod 7$: $n \equiv 6 \pmod{21}$. So exactly 3 of every 21 consecutive integers work. Since $2026 = 96 \cdot 21 + 10$, the numbers 1 to 2016 contain 96 full blocks and 288 solutions; among the last ten, 2017 to 2026 (remainders 1 to 10 mod 21), only remainder 6 works, namely $n = 2022$. The total is $288 + 1 = 289$.

Check. The smallest solutions are 6, 17, 19: $2^6 + 6 = 70 = 7 \cdot 10$, $2^{17} + 17 = 131089 = 7 \cdot 18727$, $2^{19} + 19 = 524307 = 7 \cdot 74901$, and a direct check of $n = 1, \ldots, 21$ finds no others.

11. Answer: 5**. The idea.** Put P at distance x from D along $\overline{CD}$. The right angle at P is a Pythagorean condition on triangle APB: $PA^2 + PB^2 = AB^2$. Both PA and PB are hypotenuses of right triangles with one leg 6, so this is a quadratic in x whose two roots are the two points.

Work it out. From right triangles ADP and BCP, $PA^2 = 6^2 + x^2$ and $PB^2 = 6^2 + (13-x)^2$. The condition $PA^2 + PB^2 = 13^2$ becomes $72 + x^2 + (13-x)^2 = 169$, i.e. $2x^2 - 26x + 72 = 0$, i.e. $x^2 - 13x + 36 = 0$, so $(x-4)(x-9) = 0$. The two points are 4 and 9 units from D, so they are $9 - 4 = 5$ units apart. (Equivalently, triangles ADP and PCB are similar, so $DP \cdot PC = AD \cdot BC = 36$.)

Check. The points with $\angle APB = 90^\circ$ lie on the circle with diameter $\overline{AB}$: centre at the midpoint of $\overline{AB}$, radius $\frac{13}{2}$. Side $\overline{CD}$ is 6 from that centre, so it cuts the circle in a chord of length $2\sqrt{\left(\frac{13}{2}\right)^2 - 6^2} = 2\sqrt{\frac{25}{4}} = 5$. And at $x = 4$: $PA^2 + PB^2 = 52 + 117 = 169$.

12. Answer: 14**. The idea.** Turning the hexagon 120° about its centre O sends $A \to C \to E$ and $B \to D \to F$, so it sends $P \to Q \to R \to P$. Hence $OP = OQ = OR$ and the angles POQ,

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QOR, ROP are all 120°: triangle PQR is equilateral with centre O. Its size is set by the single distance OP, which the law of cosines gives. Since only a ratio of areas matters, work with side length 3 and scale at the end.

Work it out. Take side length 3, so $AP = 1$. The triangle OAB is equilateral, so $OA = 3$ and $\angle OAP = 60°$. By the law of cosines, $OP^2 = 3^2 + 1^2 - 2 \cdot 3 \cdot 1 \cdot \frac{1}{2} = 7$. An equilateral triangle with centre-to-vertex distance ρ is made of three triangles with two sides ρ and a 120° angle between them, so its area is $3 \cdot \frac{1}{2}\rho^2 \sin 120° = \frac{3\sqrt{3}}{4}\rho^2 = \frac{21\sqrt{3}}{4}$. The hexagon is six equilateral triangles of side 3, with area $6 \cdot \frac{9\sqrt{3}}{4} = \frac{27\sqrt{3}}{2}$. So the triangle is $\frac{21/4}{27/2} = \frac{42}{108} = \frac{7}{18}$ of the hexagon, and the answer is $\frac{7}{18} \cdot 36 = 14$.

Check. Coordinates with O at the origin and $A = (3, 0)$, $B = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$, $C = \left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$, $D = (-3, 0)$: then $P = \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ and $Q = \left(-2, \sqrt{3}\right)$, so $PQ^2 = \left(\frac{9}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{81+3}{4} = 21$. An equilateral triangle of side $\sqrt{21}$ has area $\frac{\sqrt{3}}{4} \cdot 21$, matching the value above.

13. Answer: -6. The idea. The roots themselves are ugly, but $r - 2$, $s - 2$ and $t - 2$ are the roots of a new cubic: the one obtained by substituting $x = y + 2$. For that cubic, the sum of the reciprocals of its roots is (sum of pairwise products) ÷ (product), and both come straight from its coefficients.

Work it out. Put $x = y + 2$: $(y+2)^3 - 5(y+2)^2 + 2(y+2) + 7 = (y^3 + 6y^2 + 12y + 8) - (5y^2 + 20y + 20) + (2y + 4) + 7 = y^3 + y^2 - 6y - 1$. Its roots are $a = r - 2$, $b = s - 2$, $c = t - 2$, and by Vieta's formulas $ab + bc + ca = -6$ and $abc = 1$. Therefore $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ca+ab}{abc} = \frac{-6}{1} = -6$.

Check. Another route: if $P(x) = (x-r)(x-s)(x-t)$ then $\frac{P'(x)}{P(x)} = \frac{1}{x-r} + \frac{1}{x-s} + \frac{1}{x-t}$. Here $P(2) = 8 - 20 + 4 + 7 = -1$ and $P'(2) = 12 - 20 + 2 = -6$, so $\sum \frac{1}{2-r} = 6$ and $\sum \frac{1}{r-2} = -6$. Numerically the roots are about -0.931, 1.837 and 4.094, and $\frac{1}{-2.931} + \frac{1}{-0.163} + \frac{1}{2.094} \approx -0.341 - 6.136 + 0.478 \approx -6$.

14. Answer: $\frac{1}{3}$. The idea. Don't split into corner, edge and face cubes. Every (unit cube, face on top) pair is equally likely, so count pairs. The bottom face is the face opposite the top one. So we are asking how many unpainted faces have a painted face opposite them.

Work it out. There are $64 \cdot 6 = 384$ unit faces. The painted ones are the big cube's surface, $6 \cdot 16 = 96$ of them, so 288 are unpainted. On a unit cube, two opposite faces can't both be painted (that would need the big cube to be only 1 unit wide). So the face opposite each painted face is unpainted, and these pairs don't overlap: exactly 96 unpainted faces have a painted face opposite them. The probability is $\frac{96}{288} = \frac{1}{3}$.

Check. Count by type. Each of the 8 corner cubes has 3 top choices with painted bottom, each of the 24 edge cubes has 2, and each of the 24 face cubes has 1: $24 + 48 + 24 = 96$. The unpainted faces are $8 \cdot 3 + 24 \cdot 4 + 24 \cdot 5 + 8 \cdot 6 = 24 + 96 + 120 + 48 = 288$. That gives $\frac{96}{288} = \frac{1}{3}$ again.

15. Answer: 4. The idea. Every trailing zero is a factor $10 = 2 \cdot 5$. Strip exactly those out and study $N = 25!/10^k$. Working mod 10 directly is awkward because we must divide by powers of 2, so find N mod 2 and N mod 5 separately.

Work it out. Factors of 5 in 25!: $5, 10, 15, 20$ give one each and 25 gives two, so $k = 6$ (there are far more 2s: $12 + 6 + 3 + 1 = 22$). N still has $22 - 6 = 16$ factors of 2, so N is even. Mod 5: remove all 5s from 25!. The multiples of 5 become $1, 2, 3, 4, 1$, whose product is $24 \equiv -1$. The other 20 numbers form five blocks like $1 \cdot 2 \cdot 3 \cdot 4, 6 \cdot 7 \cdot 8 \cdot 9, \ldots$, each $\equiv 4! \equiv -1 \pmod 5$, giving

$(-1)^5 = -1$. So $25!/5^6 \equiv (-1)(-1) = 1 \pmod 5$. Then $N = \frac{25!/5^6}{2^6}$, and $2^6 = 64 \equiv 4 \pmod 5$; since $4 \cdot 4 \equiv 1$, dividing by 4 is multiplying by 4, so $N \equiv 4 \pmod 5$. An even number that is 4 more than a multiple of 5 ends in 4.

Check. In fact 25! = 15511210043330985984000000: six zeros, and the last nonzero digit is 4.

16. Answer: $\frac{11}{45}$**. The idea.** The four digits are chosen independently: each ones digit is equally likely to be any of 0 to 9, and each tens digit any of 1 to 9. No carrying means the two ones digits add to at most 9 and the two tens digits add to at most 9, so the probability is a product of two separate counts.

Work it out. Ones digits: for first digit a there are $10 - a$ choices of the second with sum at most 9, giving $10 + 9 + \cdots + 1 = 55$ good pairs out of 100. Tens digits (from 1 to 9): first digit c allows $9 - c$ second digits from 1 to $9 - c$, giving $8 + 7 + \cdots + 1 + 0 = 36$ good pairs out of 81. The probability is $\frac{55}{100} \cdot \frac{36}{81} = \frac{11}{20} \cdot \frac{4}{9} = \frac{11}{45}$.

Check. In counts: $55 \cdot 36 = 1980$ of the $90 \cdot 90 = 8100$ ordered pairs, and $\frac{1980}{8100} = \frac{11}{45}$. For example $34 + 52 = 86$ needs no carrying, while $38 + 52$ carries out of the ones column and $64 + 52$ carries out of the tens column.

17. Answer: 50**. The idea.** $1001 = 1000 + 1$, so $1000 \equiv -1 \pmod{1001}$. Cutting a number into blocks of three digits from the right, the blocks are multiplied by $1, 1000, 1000^2, \ldots$, which are $1, -1, 1, \ldots$ mod 1001. So N has the same remainder as the alternating sum of its three-digit blocks, and here the blocks are exactly $100, 101, \ldots, 199$.

Work it out. The last block, 199, gets sign +; then 198 gets −, 197 gets +, and so on back to 100, which gets − (it is 99 places from the end). So $N \equiv (199 - 198) + (197 - 196) + \cdots + (101 - 100) \pmod{1001}$. There are 50 pairs, each equal to 1, so $N \equiv 50$, and since $0 \le 50 < 1001$ the remainder is 50.

Check. Test the rule on a small case: the six-digit number 100101 should leave $101 - 100 = 1$. Indeed $100101 = 100 \cdot 1001 + 1$. A computer check of the full 300-digit number also gives 50.

18. Answer: 18**. The idea.** Complete the square twice. First gather every term containing x into one perfect square, which leaves an expression in y alone; then complete the square in y. A sum of two squares plus a constant is smallest when both squares are 0, provided both can be 0 at once.

Work it out. The terms with x are $x^2 + 4xy - 2x = x^2 + 2x(2y - 1)$, which is the start of $(x + 2y - 1)^2 = x^2 + 4xy - 2x + 4y^2 - 4y + 1$. So $x^2 + 4xy + 5y^2 - 2x - 6y + 20 = (x + 2y - 1)^2 + (5y^2 - 4y^2) + (-6y + 4y) + (20 - 1) = (x + 2y - 1)^2 + y^2 - 2y + 19$. Next, $y^2 - 2y + 19 = (y - 1)^2 + 18$. Altogether the expression equals $(x + 2y - 1)^2 + (y - 1)^2 + 18$, which is at least 18. Both squares are 0 when $y = 1$ and $x = 1 - 2y = -1$, so the value 18 is reached and is the least value.

Check. At $x = -1$, $y = 1$: $1 - 4 + 5 + 2 - 6 + 20 = 18$. A nearby point such as $x = 0$, $y = 1$ gives $0 + 0 + 5 - 0 - 6 + 20 = 19$, and the formula agrees: $(0 + 2 - 1)^2 + 0 + 18 = 19$. A common slip is to complete the squares in x and y separately and ignore the $4xy$ term; the cross term is exactly why the first square must be $(x + 2y - 1)^2$ rather than $(x - 1)^2$.

19. Answer: $\frac{7}{2}$**. The idea.** By symmetry H lies on the altitude from A, which is also the median to $\overline{BC}$. To locate it, intersect that line with one more altitude, say the one from B; similar right triangles do this without coordinates.

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Work it out. Let M be the midpoint of $\overline{BC}$, so $BM = 6$ and $AM = \sqrt{10^2 - 6^2} = 8$. The altitude from B is perpendicular to $\overline{AC}$, so in right triangle HMB the angle at B equals $90° - \angle C$, which is the angle $\angle MAC$. Hence triangle HMB is similar to triangle CMA, and $\frac{HM}{BM} = \frac{CM}{AM}$, so $HM = \frac{6\cdot 6}{8} = \frac{9}{2}$. Since the triangle is acute, H lies between A and M, and $AH = 8 - \frac{9}{2} = \frac{7}{2}$.

Check. Coordinates: $A = (0, 8)$, $B = (-6, 0)$, $C = (6, 0)$. The line through B perpendicular to $\overline{AC}$ (direction $(6, -8)$) has direction $(8, 6)$ and reaches $x = 0$ at $y = 6 \cdot \frac{6}{8} = \frac{9}{2}$, so $H = \left(0, \frac{9}{2}\right)$ and $AH = \frac{7}{2}$. Another route: $AH = 2R\cos A$ with $R = \frac{10\cdot 10\cdot 12}{4\cdot 48} = \frac{25}{4}$ and $\cos A = \frac{100+100-144}{200} = \frac{7}{25}$, giving $2 \cdot \frac{25}{4} \cdot \frac{7}{25} = \frac{7}{2}$.

20. Answer: 838**. The idea.** The last move into a square comes either from some square to its left in the same row or from some square below it in the same column, and it can come from any of them in one leap. So the number of journeys to a square is the sum of the numbers for all squares to its left in its row plus all squares below it in its column. Fill in a table from the start.

Work it out. Write the counts row by row from the bottom, starting with 1 in the starting corner. Bottom row: 1, 1, 2, 4, 8 (each is the sum of everything to its left). Second row: 1, 2, 5, 12, 28. For example, $12 = (1 + 2 + 5) + (4)$ from the left and below. Third row: 2, 5, 14, 37, 94; for example, $37 = (2+5+14)+(4+12)$. Fourth row: 4, 12, 37, 106, 289; for example, $106 = (4+12+37)+(4+12+37)$. Top row: 8, 28, 94, 289, 838, where $838 = (8+28+94+289)+(8+28+94+289) = 2\cdot 419$.

Check. The table is symmetric about its diagonal, as it must be (swap right and up). A small case by hand: on a 2×2 board the rook can go right then up or up then right, and the table gives $1 + 1 = 2$ in the far corner; on a 3×3 board the table gives 14, and listing journeys by number of moves (2 moves: 2; 3 moves: 6; 4 moves: 6) also gives 14.

21. Answer: 40**. The idea.** Slice the tetrahedron by the plane through $\overline{CD}$ that is perpendicular to line AB (it exists because $CD \perp AB$). It meets line AB at one point M, and it cuts the solid into two pyramids with the common base MCD and apexes A and B; their heights are MA and MB, measured along AB.

Work it out. The segment realising the distance between the lines is perpendicular to AB, so it lies in a plane perpendicular to AB; hence the distance from M to line CD inside the slicing plane is exactly 5, and triangle MCD has area $\frac{1}{2} \cdot 8 \cdot 5 = 20$. If M lies between A and B, the volume is $\frac{1}{3}\cdot 20\cdot MA + \frac{1}{3}\cdot 20\cdot MB = \frac{1}{3}\cdot 20\cdot AB = \frac{1}{3}\cdot 20\cdot 6 = 40$. (If M lay outside $\overline{AB}$, the tetrahedron would be the difference of the two pyramids, with heights differing by AB, and the same 40 comes out.)

Check. Coordinates: $A = (0, 0, 0)$, $B = (6, 0, 0)$, $C = (2, -3, 5)$, $D = (2, 5, 5)$ satisfy everything ($CD = 8$ along the y-direction, AB along the x-direction, and the lines are 5 apart). The volume is $\frac{1}{6}\left|\det\begin{pmatrix} 6 & 0 & 0 \\ 2 & -3 & 5 \\ 2 & 5 & 5 \end{pmatrix}\right| = \frac{1}{6} \cdot 6 \cdot |-15 - 25| = 40$. The general formula is $\frac{1}{6} \cdot AB \cdot CD \cdot (\text{distance}) \cdot \sin(\text{angle between the lines})$.

22. Answer: 27**. The idea.** Think about the unshaded squares ("holes"): every 2×2 block needs a hole. Disjoint blocks each need their own hole, which gives a lower bound on the number of holes. When the bound is exact, there is no slack at all, and that rigidity pins down where the holes can be.

Work it out. *The maximum.* Rows 1–2 and rows 3–4 crossed with columns 1–2, 3–4, 5–6 give six disjoint blocks, so there are at least 6 holes and at most 22 shaded squares. Holes in rows 2

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and 3 of columns 2, 4, 6 work: every pair of adjacent columns contains an even column 2, 4 or 6, and every pair of adjacent rows contains row 2 or row 3. So the maximum is 22, with exactly 6 holes.

Where the holes go. Let h_j be the number of holes in column j. The strip of columns 1–2 contains two disjoint blocks, so $h_1 + h_2 \geq 2$; likewise $h_3 + h_4 \geq 2$ and $h_5 + h_6 \geq 2$. These add to at least 6, so all three are exactly 2 and $h_7 = 0$. Using the strips 2–3, 4–5, 6–7 the same way gives $h_1 = 0$. Now $h_1 = 0$ forces $h_2 = 2$, then $h_2 + h_3 = 2$ forces $h_3 = 0$, so $h_4 = 2$, $h_5 = 0$, $h_6 = 2$. So columns 2, 4, 6 each hold two holes and the other columns none.

Which rows. Column 1 is fully shaded, so the three blocks in columns 1–2 (rows 1–2, 2–3, 3–4) must each contain one of column 2's two holes. The pairs of rows that meet all three of $\{1,2\}, \{2,3\}, \{3,4\}$ are $\{1,3\}$, $\{2,3\}$ and $\{2,4\}$ (not $\{1,2\}$, $\{3,4\}$ or $\{1,4\}$). Any of these three also covers the blocks in columns 2–3, and in the same way columns 4 and 6 each independently pick one of the three pairs. That makes $3 \cdot 3 \cdot 3 = 27$ shadings.

Check. The same reasoning on a 4×3 grid (holes only in column 2) predicts 3 maximal shadings with $12 - 2 = 10$ squares, and on a 4×5 grid $3^2 = 9$ with 16 squares; a direct search confirms both, and confirms 27 for 4×7.

23. Answer: 63**. The idea.** The terms themselves are messy fractions, but their squares grow in a very regular way: squaring the rule gives $a_{n+1}^2 = a_n^2 + 2 + \frac{1}{a_n^2}$, so each step adds a little more than 2 to the square. A lower bound and a crude upper bound on a_{2026}^2 then pin it between two consecutive perfect squares.

Work it out. Adding the identity $a_{k+1}^2 = a_k^2 + 2 + \frac{1}{a_k^2}$ for $k = 1, 2, \ldots, 2025$ gives $a_{2026}^2 = 1 + 2 \cdot 2025 + S = 4051 + S$, where $S = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{2025}^2}$.

Lower bound. $S > 0$, so $a_{2026}^2 > 4051 > 3969 = 63^2$, and $a_{2026} > 63$.

Upper bound. The same sum (dropping the positive extra terms) shows $a_k^2 \geq 1 + 2(k-1) = 2k - 1$, so $S \leq 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{4049}$. Group the odd numbers by powers of 2: the odd numbers from 2^j up to 2^{j+1} are 2^{j-1} in number and each is at least 2^j, so each group adds at most $\frac{1}{2}$. The odd numbers up to $4049 < 4096 = 2^{12}$ fill the groups $j = 1, \ldots, 11$, so $S \leq 1 + 11 \cdot \frac{1}{2} = 6.5$. Hence $a_{2026}^2 \leq 4057.5 < 4096 = 64^2$, and $a_{2026} < 64$.

So $63 < a_{2026} < 64$ and the answer is 63.

Check. Small cases obey the pattern: $a_4^2 = \frac{841}{100} = 8.41$, and the formula gives $1 + 2 \cdot 3 + \left(1 + \frac{1}{4} + \frac{4}{25}\right) = 7 + 1.41 = 8.41$. Computing a_{2026} directly (to many decimal places) gives about 63.683, whose square 4055.53 sits well inside the window found above.

24. Answer: $\frac{24}{5}$**. The idea.** Do not sort the paths by how many turns they have. A path has 9 places between consecutive steps, and the number of turns is the number of those places where the two steps differ. The expected number of turns is the sum, over the 9 places, of the probability that the steps there differ, and by symmetry that probability is the same at every place.

Work it out. A random path is a random arrangement of 6 R's and 4 U's. For steps 1 and 2: $P(\text{R then U}) = \frac{6}{10} \cdot \frac{4}{9}$ and $P(\text{U then R}) = \frac{4}{10} \cdot \frac{6}{9}$, so they differ with probability $\frac{48}{90} = \frac{8}{15}$. Every other pair of neighbouring positions behaves exactly the same way. So the expected number of turns is $9 \cdot \frac{8}{15} = \frac{24}{5}$.

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Check. Sorted by turns, the 210 paths split as $2, 8, 30, 45, 60, 40, 20, 5$ paths with $1, 2, \ldots, 8$ turns (for instance only RRRRRRUUUU and UUUURRRRRR have one turn). The total number of turns is $2 + 16 + 90 + 180 + 300 + 240 + 140 + 40 = 1008$, and $\frac{1008}{210} = \frac{24}{5}$.

25. Answer: 270**. The idea.** Factor $7749 = 27 \cdot 7 \cdot 41$ and handle each prime power separately; the repunit with n ones is $R_n = \frac{10^n-1}{9}$. For 7 and 41 what matters is the first power of 10 that is 1 more than a multiple of the prime. For 27 the digit-sum test is not enough (it only sees 9), and we need one more idea: split R_{9m} into R_9 times a sum of m powers of 10^9.

Work it out. The factor 27: the digit sum of R_n is n, so $9 \mid R_n$ exactly when $9 \mid n$; so we need $n = 9m$. Then $R_{9m} = R_9 \cdot (1 + 10^9 + 10^{18} + \cdots + 10^{9(m-1)})$. Now $R_9 = 111111111 = 9 \cdot 12345679$, and 12345679 has digit sum 37, so it is not a multiple of 3: R_9 holds exactly two factors of 3. The second factor is a sum of m numbers that are each 1 more than a multiple of 3 (every power of 10 is), so it is a multiple of 3 exactly when $3 \mid m$. Hence $27 \mid R_n$ exactly when $27 \mid n$. The factor 7: since 7 does not divide 9, $7 \mid R_n$ exactly when $7 \mid 10^n - 1$; the powers of 10 mod 7 run $3, 2, 6, 4, 5, 1$, so this happens exactly when $6 \mid n$. The factor 41: $10^5 - 1 = 99999 = 41 \cdot 2439$, while $9, 99, 999, 9999$ are not multiples of 41 ($41 \cdot 243 = 9963$, $41 \cdot 244 = 10004$), so $41 \mid R_n$ exactly when $5 \mid n$. All together, n must be a multiple of $\operatorname{lcm}(27, 6, 5) = 270$, and $n = 270$ works.

Check. The tempting slip is to treat 27 like 9 with the digit-sum rule, which would stop at $\operatorname{lcm}(9, 6, 5) = 90$. But $R_{90} = R_9 \cdot (1 + 10^9 + \cdots + 10^{81})$ with 10 terms in the second factor, and $3 \nmid 10$, so R_{90} has only two factors of 3 and $27 \nmid R_{90}$. A direct computer check confirms 270 is the first n that works.

26. Answer: 519**. The idea.** Each friend is in one of three states: chess only, drama only, or both. A friend is missing from the chess club exactly when he or she is in the drama club only. So "the chess club has at least 5 members" simply says that at most 2 friends are drama-only, and likewise at most 2 are chess-only.

Work it out. Let a friends be chess-only and b drama-only, with $a, b \in \{0, 1, 2\}$; the other $7-a-b$ are in both. For each pair (a, b) the number of ways is $\frac{7!}{a!\,b!\,(7-a-b)!}$: $(0, 0)$: 1; $(1, 0)$ and $(0, 1)$: 7 each; $(2, 0)$ and $(0, 2)$: 21 each; $(1, 1)$: 42; $(2, 1)$ and $(1, 2)$: 105 each; $(2, 2)$: 210. The total is $1 + 14 + 42 + 42 + 210 + 210 = 519$.

Check. Count by the number k of friends in both clubs. $k = 7$: 1; $k = 6$: the other one is chess-only or drama-only, 14; $k = 5$: the other two are split 2-0, 1-1 or 0-2: $21 \cdot (1 + 2 + 1) = 84$; $k = 4$: three others split 2-1 or 1-2: $35 \cdot 6 = 210$; $k = 3$: four others split 2-2: $35 \cdot 6 = 210$; $k \le 2$ is impossible. Again $1 + 14 + 84 + 210 + 210 = 519$.

27. Answer: $9 - 5\sqrt{3}$**. The idea.** Put the right-angled corner at the origin. Any sphere touching the three perpendicular faces has its center on the diagonal (t, t, t), and its radius is t. The insphere is then fixed by also touching the slanted face. The small sphere is fixed by touching the insphere.

Work it out. The slanted face is the plane $x + y + z = 6$. The distance from (r, r, r) to it is $\frac{6-3r}{\sqrt{3}}$, and setting this equal to r gives $6 - 3r = \sqrt{3}r$, so $r = \frac{6}{3+\sqrt{3}} = 3 - \sqrt{3}$. (The same comes from $r = \frac{3V}{\text{surface area}} = \frac{108}{54+18\sqrt{3}}$.) Now let S_2 have center (t, t, t) and radius t, with $t < r$. The distance between the centers is $\sqrt{3}(r - t)$, and external tangency makes this $r + t$. So $t(\sqrt{3}+1) = r(\sqrt{3}-1)$, and $t = r \cdot \frac{\sqrt{3}-1}{\sqrt{3}+1} = r(2 - \sqrt{3})$. Therefore $t = (3 - \sqrt{3})(2 - \sqrt{3}) = 6 - 3\sqrt{3} - 2\sqrt{3} + 3 = 9 - 5\sqrt{3}$.

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Check. $9 - 5\sqrt{3} \approx 0.34$ and $r \approx 1.27$. The ratio $2 - \sqrt{3} \approx 0.27$ is the same for any two tangent spheres stacked in a cube corner, since shrinking toward the corner keeps the picture similar. The distance from $(0.34, 0.34, 0.34)$ to $(1.27, 1.27, 1.27)$ is $\sqrt{3} \cdot 0.93 \approx 1.61 \approx 0.34 + 1.27$.

28. Answer: 16. **The idea.** Put the cube at $-1 \le x, y, z \le 1$ (edge 2). Then the marked points are exactly the points with one coordinate 0 and the other two equal to ± 1, like $(1, 1, 0)$. Every distance between two marked points is one of just four values, and an equilateral triangle uses a single value. So sort the pairs by distance and, for each distance, count triangles through one fixed point.

Work it out. Fix $P = (1, 1, 0)$; by symmetry every marked point looks the same. The squared distances from P to the other 11 points:

2: the four points $(1, 0, \pm 1)$ and $(0, 1, \pm 1)$;

4: the two points $(1, -1, 0)$ and $(-1, 1, 0)$;

6: the four points $(-1, 0, \pm 1)$ and $(0, -1, \pm 1)$;

8: the one point $(-1, -1, 0)$.

An equilateral triangle through P with squared side s needs two points at squared distance s from P that are also at squared distance s from each other.

$s = 2$: among $(1, 0, 1)$, $(1, 0, -1)$, $(0, 1, 1)$, $(0, 1, -1)$, the pairs at squared distance 2 are $(1, 0, 1)$ with $(0, 1, 1)$ and $(1, 0, -1)$ with $(0, 1, -1)$: 2 triangles. (These are the triangles cut off at a corner of the cube, by the three midpoints next to that corner.)

$s = 4$: the two points are at squared distance 8 from each other: none.

$s = 6$: among $(-1, 0, 1)$, $(-1, 0, -1)$, $(0, -1, 1)$, $(0, -1, -1)$, the pairs at squared distance 6 are $(-1, 0, 1)$ with $(0, -1, -1)$ and $(-1, 0, -1)$ with $(0, -1, 1)$: 2 triangles.

$s = 8$: only one point: none.

So each marked point lies on 2 small and 2 large triangles. Counting over all 12 points counts each triangle 3 times: $\frac{12 \cdot 2}{3} = 8$ small ones and $\frac{12 \cdot 2}{3} = 8$ large ones, 16 in all.

Check. The 8 small triangles match the 8 corners of the cube, one each. Each large triangle, like $(1, 1, 0)$, $(-1, 0, 1)$, $(0, -1, -1)$, has coordinates adding to $(0, 0, 0)$, so its centre is the centre of the cube; its side is $\sqrt{6}$, and $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$ is the ratio of large side to small side. A computer check of all $\binom{12}{3} = 220$ triples finds exactly 16 equilateral ones.

29. Answer: 144. **The idea.** If n has k digits, writing it twice gives $n \cdot 10^k + n = n(10^k + 1)$. So the condition is that n divides $10^k + 1$. But n has k digits, so it is at least 10^{k-1}, and the quotient $\frac{10^k+1}{n}$ is therefore tiny: at most 10 (or 11 when $k = 1$). Test the few possible quotients.

Work it out. Let $m = \frac{10^k+1}{n}$, so $m \le \frac{10^k+1}{10^{k-1}} = 10 + \frac{1}{10^{k-1}}$. Now $10^k + 1$ is odd, ends in 1 (so is not a multiple of 5), and leaves remainder 2 on division by 3; that rules out $m = 2, 3, 4, 5, 6, 8, 9, 10$. The value $m = 1$ would make $n = 10^k + 1$, which has $k + 1$ digits. So $m = 7$, or $m = 11$ with $k = 1$, which gives $n = 1$. For $m = 7$ we need $7 \mid 10^k + 1$. The powers $10, 10^2, 10^3, \ldots$ leave remainders $3, 2, 6, 4, 5, 1$ on division by 7 and then repeat, so $10^k \equiv -1 \equiv 6$ exactly when $k = 3, 9, 15, \ldots$. Below 1,000,000 only $k = 3$ is available, giving $n = \frac{1001}{7} = 143$, which does have 3 digits. The answer is $1 + 143 = 144$.

Check. $143143 = 143 \cdot 1001 = 143 \cdot 7 \cdot 143 = 7 \cdot 143^2$, and $11 = 11 \cdot 1^2$. The next solution is the 9-digit number $\frac{10^9+1}{7} = 142857143$, above the limit.

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30. Answer: 84**. The idea.** A triangle with all three vertices inside the circle is cut out by three chords that cross one another pairwise inside the circle. Look at the endpoints of those three chords: they are six different points (two chords meeting at an endpoint meet on the circle, not inside), and for six points on a circle there is exactly one way to join them in pairs so that every two chords cross. So such triangles correspond exactly to sets of six of the nine points.

Work it out. Label six chosen points $1, 2, \ldots, 6$ in order around the circle. Chords $\overline{ab}$ and $\overline{cd}$ with four different endpoints cross exactly when their endpoints alternate around the circle. If point 1 were joined to 2 (or to 6), that chord would have all the other endpoints on one side and would cross nothing. If 1 were joined to 3 (or to 5), the chord would have only one other point on one side, so it could cross only the one chord through that point, not both of the others. So 1 is joined to 4, and the same argument for points 2 and 3 shows that the only pairing with all three chords crossing is $\overline{14}, \overline{25}, \overline{36}$ (the three long diagonals of the hexagon), and they do form a triangle since no three chords are concurrent. Each set of six points gives exactly one triangle, and different sets give different triangles, so the count is $\binom{9}{6} = 84$.

Check. With only six points on the circle the formula gives $\binom{6}{6} = 1$: the three long diagonals of a (non-regular) convex hexagon bound a single small triangle, and every other triangle in the figure has a vertex on the circle. A direct computer check over all $\binom{36}{3}$ triples of chords among nine points also finds exactly 84 pairwise-crossing triples.

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