MathCastle

STATE COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A club password is a string of 4 letters, each chosen from A, B, C, D and E. Letters may be repeated, but no two neighbouring letters in the string may be the same (so ABAE is allowed, but ABBE is not). The password must also contain at least one vowel, that is, at least one A or E. How many different passwords are possible?

1. _____

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2. A positive integer is called a climber if it has at least two digits and each of its digits is larger than the digit just before it (for example, 69 and 1257 are climbers, but 77 and 132 are not). How many climbers have digits that add up to 15?

2. _____

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3. A rectangle that is not a square is cut by two straight cuts, both parallel to its shorter sides, into three congruent rectangles. Each of the three small rectangles is similar to the original rectangle. What is the ratio of the length of the original rectangle's longer side to the length of its shorter side? Express your answer in simplest radical form.

3. _____

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4. Some even numbers can be written as the sum of two different primes in more than one way; for example, $16 = 3 + 13 = 5 + 11$. Two ways that differ only in the order of the two primes count as the same way. What is the smallest even number that can be written as the sum of two different primes in exactly three different ways?

4. _____

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5. A courtyard is marked with a square grid, and a small post stands at every point (x, y) where x and y are both integers. A circular fountain has its centre at the post $(0, 0)$ and radius 5. Posts lying exactly on the edge of the fountain do not count as inside it. How many posts lie strictly inside the fountain, that is, satisfy $x^2 + y^2 < 25$?

5. _____

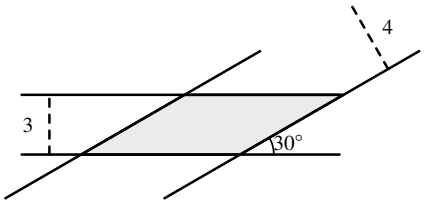
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6. A positive integer N has the property that when all of its positive divisors (including 1 and N itself) are multiplied together, the product is 6^{24}. What is N?

6. _____

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7. Two long straight strips of paper, each with parallel edges, lie flat on a table, one on top of the other. One strip is 3 cm wide and the other is 4 cm wide, and the edges of one strip make an angle of 30° with the edges of the other. What is the area, in square centimetres, of the region where the two strips overlap?



7. _____

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8. What is the smallest positive integer that is divisible by 4 and whose digits add up to 30?

8. _____

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9. A shopkeeper has a two-pan balance and exactly four weights, of 1, 4, 12 and 36 grams. An object can be weighed if, with the object on one pan, some of the weights can be placed on the two pans (weights may go on either pan, and not all of them need to be used) so that the pans balance exactly. How many different positive whole numbers of grams can be weighed this way?

9. _____

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10. Two different squares of a 4×4 board of 16 unit squares are chosen at random, with all pairs of different squares equally likely. What is the probability that the two squares touch, meaning they share a side or share just a corner? Express your answer as a common fraction.

10. _____

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11. For how many integers k do the lines $x + ky = 20$ and $kx + y = 4$ meet in exactly one point whose coordinates are both integers?

11. _____

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12. Positive real numbers x, y and z satisfy $x + y + xy = 9$, $y + z + yz = 14$ and $z + x + zx = 23$. What is the value of $x + y + z$? Express your answer as a common fraction.

12. _____

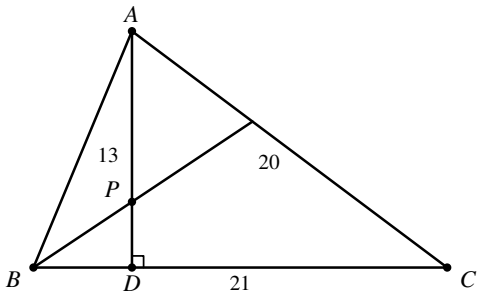
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13. A line with positive slope passes through the point $(0, 8)$ and meets the parabola $y = x^2$ at two points A and B. Let O be the origin. The area of triangle OAB is 24. What is the slope of the line?

13. _____

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14. Triangle ABC has side lengths $AB = 13$, $BC = 21$ and $CA = 20$. The altitude from A meets side $\overline{BC}$ at D. The bisector of angle ABC crosses the segment $\overline{AD}$ at the point P. What is the length BP? Express your answer in simplest radical form.



14. _____

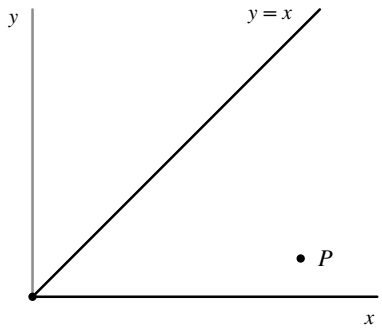
15. Mia plays a game with a fair six-sided die. She rolls it, and after seeing the number she either stops or rolls again; she may roll at most three times in all. Her score is the number on her last roll (earlier rolls are thrown away). Mia always chooses whether to stop so as to make her expected score as large as possible. What is her expected score? Express your answer as a common fraction.

15. _____

16. Eleven coins lie on a table, all showing heads. A move consists of choosing any three of the coins and turning each of those three over. What is the least number of moves needed to make all eleven coins show tails?

16. _____

17. Let $P = (7, 1)$. Point Q is chosen on the positive x-axis, and point R is chosen on the part of the line $y = x$ that lies in the first quadrant. What is the least possible perimeter of triangle PQR?



17. _____

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18. The fraction $\frac{1}{13}$ is written as a decimal. What is the sum of the first 2026 digits after the decimal point?

18. _____

19. A fair six-sided die is rolled repeatedly until each of the numbers 1, 2 and 3 has appeared at least once, and then the rolling stops. What is the expected (average) number of rolls?

19. _____

20. A triangle has three different whole-number side lengths, and its perimeter is 30. How many such triangles are there? (Congruent triangles count as the same triangle.)

20. _____

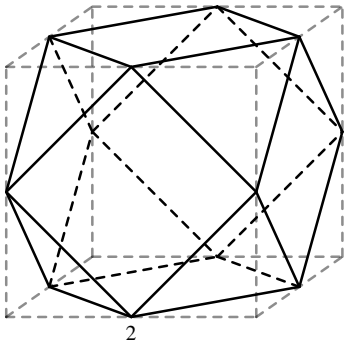
21. The number 10^{10} (ten billion) has many positive divisors, such as 1, 2, 4 and 10^{10} itself. How many of its positive divisors are three-digit numbers?

21. _____

22. Sixteen dots are arranged in a 4 by 4 square array, with neighbouring dots in each row and column 1 unit apart. How many triangles with positive area have all three of their vertices among these 16 dots?

22. _____

23. A solid cube has edge length 2. At each of its 8 corners, a small piece is sliced off by the plane through the midpoints of the three edges that meet at that corner. What is the total surface area of the solid that remains? Express your answer in simplest radical form.



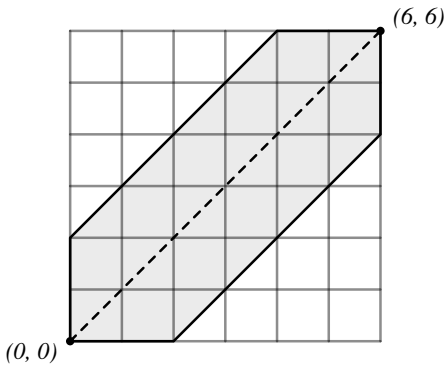
23. _____

24. A triangle has side lengths 5, 6 and 7. A straight cut is made whose two endpoints lie on the sides of the triangle (on two different sides), dividing the triangle into two regions of equal area. What is the shortest possible length of such a cut? Express your answer in simplest radical form.

24. _____

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25. A path goes from $(0, 0)$ to $(6, 6)$ in 12 unit steps, each step going either one unit right or one unit up. The path must stay close to the line $y = x$: at every point along the way, the difference between the path's x-coordinate and its y-coordinate is at most 2 (in either direction). How many such paths are there?



25. _____

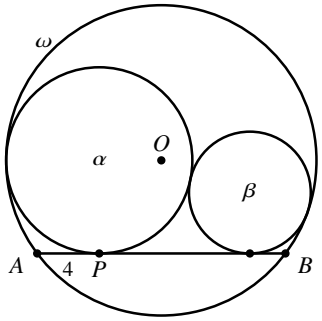
26. Each of a and b is chosen from the integers $1, 2, 3, \ldots, 12$ (they may be equal), and the fraction $\frac{a}{b}$ is formed. How many different numbers can $\frac{a}{b}$ equal?

26. _____

27. The system of equations $x^2 + y = 11$ and $y^2 + x = 11$ has exactly four solutions (x, y) in real numbers. For each solution, compute $x^2 + y^2$. What is the sum of these four values?

27. _____

28. Circle ω has center O and radius 10. Chord $\overline{AB}$ of ω has length 16, and P is the point of $\overline{AB}$ with $AP = 4$. Circle α lies inside ω on the same side of line AB as O, is tangent to line AB at P, and is tangent to ω. Circle β also lies inside ω on the same side of line AB as O; it is tangent to ω, tangent to α from outside, and tangent to line AB at a point between P and B. What is the radius of β? Express your answer in simplest radical form.



28. _____

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29. How many ordered triples (a, b, c) of positive integers satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$? (The numbers need not be different, and, for example, $(3, 7, 42)$ and $(7, 3, 42)$ count as different triples.)

29. _____

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30. The numbers $1, 2, 3, 4, 5, 6, 7$ are to be arranged in a row. The arrangement is called a zigzag if each of the five numbers that is not at an end of the row is either greater than both of its neighbours or less than both of its neighbours. (For example, 3 1 5 2 7 4 6 is a zigzag.) How many zigzags are there?

30. _____

Answer Key — MathCounts State Sprint Practice Contest 2

1. 296

2. 17

3. $\sqrt{3}$

4. 24

5. 69

6. 216

7. 24

8. 5988

9. 40

10. $\frac{7}{20}$

11. 6

12. $\frac{19}{2}$

13. 2

14. $\frac{5\sqrt{13}}{3}$

15. $\frac{14}{3}$

16. 5

17. 10

18. 9121

19. 11

20. 12

21. 14

22. 516

23. $12 + 4\sqrt{3}$

24. $2\sqrt{3}$

25. 486

26. 91

27. 88

28. $\frac{9+3\sqrt{5}}{4}$

29. 46

30. 544

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Solutions — MathCounts State Sprint Practice Contest 2

1. Answer: 296. **The idea.** "At least one vowel" has many cases, but its opposite, "no vowel at all", is easy. Count all strings with no equal neighbours, then subtract the ones built only from B, C, D.

Work it out. With no equal neighbours, the first letter has 5 choices and each later letter has 4 (anything except the letter just before it): $5 \cdot 4 \cdot 4 \cdot 4 = 320$ strings. Using only the consonants B, C, D, the first letter has 3 choices and each later letter has 2: $3 \cdot 2 \cdot 2 \cdot 2 = 24$ strings. So $320 - 24 = 296$ passwords contain a vowel.

Check. For 2-letter strings the same method gives $5 \cdot 4 - 3 \cdot 2 = 14$; listing them directly, the pairs with an A or E and different letters are the 20 ordered pairs of different letters minus the 6 ordered pairs from $\{B, C, D\}$, which is 14.

2. Answer: 17. **The idea.** A climber is completely determined by its set of digits: write the set in increasing order and you get the number. The digit 0 can never appear, since it would have to come first. So we are really counting sets of at least two different digits from 1 to 9 with sum 15, and it is easiest to sort them by size.

Work it out. Two digits: $6 + 9$ and $7 + 8$, so 2 sets. Three digits, listed by the smallest digit: $\{1, 5, 9\}, \{1, 6, 8\}, \{2, 4, 9\}, \{2, 5, 8\}, \{2, 6, 7\}, \{3, 4, 8\}, \{3, 5, 7\}, \{4, 5, 6\}$, so 8 sets. Four digits: $\{1, 2, 3, 9\}, \{1, 2, 4, 8\}, \{1, 2, 5, 7\}, \{1, 3, 4, 7\}, \{1, 3, 5, 6\}, \{2, 3, 4, 6\}$, so 6 sets. Five digits: the smallest possible sum is $1 + 2 + 3 + 4 + 5 = 15$, so only $\{1, 2, 3, 4, 5\}$ works; six or more digits sum to at least 21. The total is $2 + 8 + 6 + 1 = 17$.

Check. The eight three-digit sets are exactly the eight lines (rows, columns and diagonals) of the 3×3 magic square with magic sum 15, a good sign the list is complete. For four digits, the smallest sum is 1+2+3+4 = 10, so a set is 1, 2, 3, 4 with 5 extra spread over the digits, larger digits getting at least as much as smaller ones: the extra amounts $(0, 0, 0, 5), (0, 0, 1, 4), (0, 0, 2, 3), (0, 1, 1, 3), (0, 1, 2, 2), (1, 1, 1, 2$ give exactly the six sets listed.

3. Answer: $\sqrt{3}$. **The idea.** Similar rectangles have the same ratio of longer side to shorter side. Write both ratios and set them equal, taking care to see which side of each small piece is the longer one.

Work it out. Let the original be a by b with $a > b$. The cuts are parallel to the short sides, so they split the long side into three parts, and each piece is b by $\frac{a}{3}$. If $\frac{a}{3}$ were the longer side of a piece, its ratio would be $\frac{a/3}{b} = \frac{1}{3} \cdot \frac{a}{b}$, which cannot equal $\frac{a}{b}$. So b is the longer side of each piece, and similarity gives $\frac{b}{a/3} = \frac{a}{b}$, that is $3b^2 = a^2$ and $\frac{a}{b} = \sqrt{3}$.

Check. A $\sqrt{3}$ by 1 rectangle splits into three 1 by $\frac{\sqrt{3}}{3}$ pieces, whose ratio is $1 \div \frac{\sqrt{3}}{3} = \frac{3}{\sqrt{3}} = \sqrt{3}$, the same as the original.

4. Answer: 24. **The idea.** Work upward through the even numbers, and for each one only test the primes up to half of it, pairing each with its partner. Remember that the two primes must be different, so a split like 11 + 11 does not count.

Work it out. The prime 2 is only useful for 2 + 2, which is not allowed, so both primes are odd. Checking: $16 = 3 + 13 = 5 + 11$ (two ways); $18 = 5 + 13 = 7 + 11$ (two); $20 = 3 + 17 = 7 + 13$

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(two); $22 = 3 + 19 = 5 + 17$, and $11 + 11$ is not allowed (two); $24 = 5 + 19 = 7 + 17 = 11 + 13$ (three). Smaller even numbers have at most two ways ($10 = 3 + 7$ only, since $5 + 5$ is not allowed; $12 = 5 + 7$; $14 = 3 + 11$). So the answer is 24.

Check. For 24 the only odd primes below 12 are $3, 5, 7, 11$, with partners $21, 19, 17, 13$; exactly three of the partners $(19, 17, 13)$ are prime, confirming three ways.

5. Answer: 69**. The idea.** Count column by column: for each integer x, find how many integers y satisfy $y^2 < 25 - x^2$. Symmetry halves the work, and the points exactly on the circle, like $(3, 4)$, must be left out.

Work it out. For $x = 0$: $y^2 < 25$, so $|y| \le 4$, giving 9 values. For $x = \pm 1$: $y^2 < 24$, $|y| \le 4$, 9 values. For $x = \pm 2$: $y^2 < 21$, $|y| \le 4$, 9 values. For $x = \pm 3$: $y^2 < 16$, $|y| \le 3$, 7 values. For $x = \pm 4$: $y^2 < 9$, $|y| \le 2$, 5 values. For $x = \pm 5$ nothing. Total $9 + 2(9 + 9 + 7 + 5) = 69$.

Check. Counting the closed disc instead, the columns would have $11, 9, 9, 9, 7, 1$ points for $x = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ (now $y = \pm 5$, ± 4 and ± 3 appear at $x = 0, \pm 3, \pm 4$): $11 + 2(9 + 9 + 9 + 7 + 1) = 81$. The difference, 12, is exactly the number of lattice points on the circle: $(\pm 5, 0), (0, \pm 5), (\pm 3, \pm 4), (\pm 4, \pm 3)$.

6. Answer: 216**. The idea.** Pair each divisor d with its partner $\frac{N}{d}$: each pair multiplies to N. So if N has k divisors, their product is $N^{k/2}$, and the problem becomes finding N with $N^{k/2} = 6^{24}$.

Work it out. Every divisor of N divides the product 6^{24}, so N itself has no prime factors other than 2 and 3: $N = 2^a 3^b$, with $k = (a + 1)(b + 1)$ divisors. Then $N^{k/2} = 2^{ak/2} 3^{bk/2} = 2^{24} 3^{24}$ forces $a = b$, so $N = 6^a$ with $k = (a + 1)^2$, and we need $\frac{a(a+1)^2}{2} = 24$, that is $a(a + 1)^2 = 48$. Since $a(a + 1)^2$ increases with a and $a = 3$ gives $3 \cdot 16 = 48$, the only solution is $a = 3$, so $N = 6^3 = 216$.

Check. $216 = 2^3 3^3$ has $4 \cdot 4 = 16$ divisors, forming 8 pairs that each multiply to 216, so the product is $216^8 = (6^3)^8 = 6^{24}$. For comparison, $a = 2$ would give $36^{9/2} = 6^9$ and $a = 4$ would give $1296^{25/2} = 6^{50}$.

7. Answer: 24**. The idea.** The overlap is bounded by two pairs of parallel lines, so it is a parallelogram. Its area is a side times the distance to the opposite side, and one side length can be read off from the tilt of 30°.

Work it out. Take the side of the parallelogram that runs along an edge of the 4 cm strip. It stretches across the whole 3 cm strip, crossing it at 30°, so it is the hypotenuse of a right triangle whose leg opposite the 30° angle is 3: its length is $\frac{3}{\sin 30^\circ} = 6$ cm. The opposite side of the parallelogram lies on the other edge of the 4 cm strip, at distance 4. So the area is $6 \cdot 4 = 24$ square centimetres.

Check. Do it the other way round: the side along an edge of the 3 cm strip crosses the 4 cm strip, so it has length $\frac{4}{\sin 30^\circ} = 8$, and the distance to its opposite side is 3; again $8 \cdot 3 = 24$. (If the strips crossed at 90° the overlap would be a 3 by 4 rectangle of area 12; the slant doubles it because $\sin 30^\circ = \frac{1}{2}$.)

8. Answer: 5988**. The idea.** First find how many digits are needed, then make the leading digit as small as possible, and only then worry about divisibility by 4, which depends on the last two digits alone.

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Work it out. Three digits add up to at most 27, so we need at least four digits. With leading digit 3, the other three digits must add to 27, forcing 3999, which is odd. With leading digit 4, the other three add to 26, so they are $9, 9, 8$ in some order: 4899, 4989, 4998; their last two digits 99, 89, 98 are not multiples of 4. With leading digit 5, the other three add to 25. Try the hundreds digit as small as possible: hundreds digit 7 forces 5799 (odd); hundreds digit 8 leaves tens and units adding to 17, giving 5889 or 5898, neither ending in a multiple of 4; hundreds digit 9 leaves 16, giving 5979, 5988, 5997, and 88 is a multiple of 4. So the answer is 5988.

Check. $5+9+8+8 = 30$ and $5988 = 4 \cdot 1497$. Every four-digit number with digit sum 30 below 5988 appears in the list above, and none of them is a multiple of 4.

9. Answer: 40**. The idea.** A weight on the same pan as the object counts as minus, a weight on the other pan counts as plus, so a mass can be weighed exactly when it equals $\pm 1 \pm 4 \pm 12 \pm 36$ with some terms left out. The three big weights are 4 times $1, 3, 9$, and those three reach every whole number from -13 to 13.

Work it out. First, $1, 3, 9$ with signs reach every integer from -13 to 13: using ± 1 and ± 3 gives every integer from -4 to 4, and adding -9, 0 or $+9$ extends this to -13 through 13. So $4, 12, 36$ reach exactly the multiples of 4 from -52 to 52, that is $4k$ for $-13 \le k \le 13$. Adding -1, 0 or $+1$ then gives exactly the numbers $4k-1, 4k, 4k+1$. The positive ones are: $4k$ for $k = 1, \ldots, 13$ (13 masses), $4k+1$ for $k = 0, \ldots, 13$ (14 masses), and $4k-1$ for $k = 1, \ldots, 13$ (13 masses). These are all different, so $13 + 14 + 13 = 40$ masses can be weighed.

Check. The masses that cannot be weighed, up to the total 53, are exactly those that leave remainder 2 when divided by 4: $2, 6, 10, \ldots, 50$, which is 13 numbers, and $53 - 13 = 40$. For example, $23 = 36 - 12 - 1$ (the 12 and 1 gram weights go on the object's pan), while 2 cannot be made, since every combination is within 1 of a multiple of 4.

10. Answer: $\frac{7}{20}$**. The idea.** Instead of going square by square (and double counting), count touching pairs by the direction in which they touch: side by side horizontally, one above the other, or diagonally in one of two directions.

Work it out. Horizontal neighbours: each of the 4 rows has 3 adjacent pairs, 12 in all. Vertical neighbours: likewise 12. Diagonal neighbours: each touching diagonal pair fills one 2×2 block, and there are $3 \cdot 3 = 9$ such blocks, each containing 2 diagonal pairs, so 18. Total touching pairs: $12 + 12 + 18 = 42$. There are $\binom{16}{2} = 120$ pairs in all, so the probability is $\frac{42}{120} = \frac{7}{20}$.

Check. Count neighbours per square: the 4 corner squares have 3 neighbours, the 8 edge squares have 5, and the 4 middle squares have 8. That is $12 + 40 + 32 = 84$, and each touching pair was counted twice, giving 42.

11. Answer: 6**. The idea.** Solving the system directly gives fractions with $1 - k^2$ in the denominator, which are awkward to test. The symmetry of the two equations suggests adding and subtracting them instead: that splits $1 - k^2$ into the two factors $1 + k$ and $1 - k$, each dividing a fixed number.

Work it out. If $k = 1$ or $k = -1$, the lines are parallel and distinct ($x + y = 20$ and $x + y = 4$; $x - y = 20$ and $x - y = -4$), so they do not meet. Otherwise, adding the equations gives $(1 + k)(x + y) = 24$ and subtracting gives $(1 - k)(x - y) = 16$. If x and y are integers, then $1 - k$ divides 16, so $1 - k \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$ and $k \in \{0, 2, 3, -1, 5, -3, 9, -7, 17, -15\}$. Dropping $k = -1$ and requiring $1 + k$ to divide 24 removes $k = 9, 17, -15$ (as 10, 18, -14 do not divide 24), leaving $k = 0, 2, 3, -3, 5, -7$. Conversely, for each of these, $s = x + y = \frac{24}{1+k}$ and

$d = x - y = \frac{16}{1-k}$ are integers, and we still need $x = \frac{s+d}{2}$ and $y = \frac{s-d}{2}$ to be integers, i.e. s and d of the same parity. The pairs (s, d) are $(24, 16)$, $(8, -16)$, $(6, -8)$, $(-12, 4)$, $(4, -4)$, $(-4, 2)$, all even-even, so all 6 values work.

Check. The six intersection points are $(20, 4)$, $(-4, 12)$, $(-1, 7)$, $(-4, -8)$, $(0, 4)$ and $(-1, -3)$. For instance, with $k = -7$: $-1 + (-7)(-3) = 20$ and $(-7)(-1) + (-3) = 4$. With $k = 3$: $-1 + 21 = 20$ and $-3 + 7 = 4$.

12. Answer: $\frac{19}{2}$. The idea. Each left side is one short of a product: $x + y + xy + 1 = (x + 1)(y + 1)$. Adding 1 to each equation turns the system into three products of the numbers $x + 1$, $y + 1$, $z + 1$, and multiplying all three equations gives their full product.

Work it out. Let $a = x + 1$, $b = y + 1$, $c = z + 1$. The equations become $ab = 10$, $bc = 15$ and $ca = 24$. Multiplying them, $(abc)^2 = 10 \cdot 15 \cdot 24 = 3600$, and since a, b, c are all greater than 1, $abc = 60$. Dividing by each equation: $c = \frac{60}{10} = 6$, $a = \frac{60}{15} = 4$, $b = \frac{60}{24} = \frac{5}{2}$. So $x = 3$, $y = \frac{3}{2}$, $z = 5$, and $x + y + z = \frac{19}{2}$.

Check. $3 + \frac{3}{2} + \frac{9}{2} = 9$, $\frac{3}{2} + 5 + \frac{15}{2} = 14$, and $5 + 3 + 15 = 23$. The other choice $abc = -60$ gives $a = -4$, $b = -\frac{5}{2}$, $c = -6$, so x, y, z would be negative, which the problem rules out.

13. Answer: 2. **The idea.** The line crosses the y-axis at $P = (0, 8)$, between A and B. Use $\overline{OP}$, of length 8, as a common base for triangles OPA and OPB: their heights are just $|x_A|$ and $|x_B|$. Vieta's formulas then give $|x_A - x_B|$ without solving for the points.

Work it out. With slope m, the intersections satisfy $x^2 = mx + 8$, i.e. $x^2 - mx - 8 = 0$. The product of the roots is $-8 < 0$, so A and B lie on opposite sides of the y-axis, and the area of OAB is $\frac{1}{2} \cdot 8 \cdot (|x_A| + |x_B|) = 4|x_A - x_B|$. Now $(x_A - x_B)^2 = (x_A + x_B)^2 - 4x_A x_B = m^2 + 32$. Setting $4\sqrt{m^2 + 32} = 24$ gives $m^2 + 32 = 36$, so $m = 2$ (the slope is positive).

Check. For $m = 2$: $x^2 - 2x - 8 = (x - 4)(x + 2)$, so $A = (4, 16)$ and $B = (-2, 4)$. The shoelace formula gives area $\frac{1}{2}|4 \cdot 4 - (-2) \cdot 16| = \frac{1}{2} \cdot 48 = 24$.

14. Answer: $\frac{5\sqrt{13}}{3}$. The idea. First find where the altitude lands by writing AD^2 in two ways. Then look only at the right triangle ABD: the line BP bisects its angle at B, so the Angle Bisector Theorem splits the leg $\overline{AD}$ in the ratio of the other two sides.

Work it out. Let $BD = x$, so $DC = 21 - x$. From the right triangles ABD and ACD: $13^2 - x^2 = AD^2 = 20^2 - (21 - x)^2$, so $169 - x^2 = 400 - 441 + 42x - x^2$, giving $42x = 210$ and $x = 5$. Then $AD = \sqrt{169 - 25} = 12$. In triangle ABD the bisector from B meets the opposite side $\overline{AD}$ at P, so $\frac{DP}{PA} = \frac{BD}{BA} = \frac{5}{13}$ and $DP = \frac{5}{18} \cdot 12 = \frac{10}{3}$. Finally, triangle BDP has a right angle at D, so $BP = \sqrt{5^2 + \left(\frac{10}{3}\right)^2} = \sqrt{\frac{325}{9}} = \frac{5\sqrt{13}}{3}$.

Check. In coordinates, $B = (0, 0)$, $C = (21, 0)$, $A = (5, 12)$: $AC = \sqrt{16^2 + 12^2} = 20$. The bisector at B points along the sum of the unit vectors $\left(\frac{5}{13}, \frac{12}{13}\right)$ and $(1, 0)$, that is along $(18, 12)$, or slope $\frac{2}{3}$. At $x = 5$ it is at height $\frac{10}{3}$, matching DP.

15. Answer: $\frac{14}{3}$. The idea. Work backwards from the last roll. At each stage, Mia should keep a roll exactly when it beats the expected score she would get by rolling on, and the value of the stage she is at follows from that rule.

Work it out. With one roll left she must keep it: expected score $\frac{7}{2}$. With two rolls left, she keeps the first roll when it beats $\frac{7}{2}$, that is when it is 4, 5 or 6, and otherwise rolls once more: the expected score is $\frac{4+5+6}{6} + \frac{3}{6} \cdot \frac{7}{2} = \frac{5}{2} + \frac{7}{4} = \frac{17}{4}$. With three rolls left, she keeps the first roll when it beats $\frac{17}{4}$, that is when it is 5 or 6, and otherwise goes on to the two-roll game: the expected score is $\frac{5+6}{6} + \frac{4}{6} \cdot \frac{17}{4} = \frac{11}{6} + \frac{17}{6} = \frac{28}{6} = \frac{14}{3}$.

Check. Stopping with a 4 on the first roll would be a mistake: it would score 4, less than the $\frac{17}{4}$ the remaining two rolls are worth. Keeping only a 6 on the first roll is also worse: $\frac{6}{6} + \frac{5}{6} \cdot \frac{17}{4} = 1 + \frac{85}{24} = \frac{109}{24} < \frac{112}{24} = \frac{14}{3}$.

16. Answer: 5**. The idea.** Count coin flips in total. Each coin must end up turned over an odd number of times, so the total number of flips is a sum of eleven odd numbers, which is odd. But every move contributes exactly 3 flips. The size bound alone suggests 4 moves; parity is what rules it out.

Work it out. With m moves there are $3m$ flips in all. Since each of the 11 coins is flipped at least once, $3m \geq 11$, so $m \geq 4$. Each coin is flipped an odd number of times, and a sum of eleven odd numbers is odd, so $3m$ is odd, which means m is odd. Hence $m \neq 4$ and $m \geq 5$. Five moves suffice. Number the coins 1 to 11 and turn over $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{6,7,8\}$, $\{9,10,11\}$. Coins 1 and 2 are turned 3 times each and every other coin once, all odd, so every coin ends on tails. The answer is 5.

Check. The five moves make 15 flips: $3+3+9\cdot 1 = 15$. In four moves (12 flips) you would need eleven odd counts adding to 12, impossible since eleven odd numbers add to an odd number. The trap is to stop at $\lceil 11/3 \rceil = 4$.

17. Answer: 10**. The idea.** Treat the x-axis and the line $y = x$ as mirrors. Reflecting P in each line turns the two sides PQ and RP into sides that start at the images, so the whole perimeter becomes one broken path between two fixed points, and a broken path is never shorter than the straight segment.

Work it out. Reflect P in the x-axis to get $P_1 = (7,-1)$, and in the line $y = x$ to get $P_2 = (1,7)$. For any Q on the x-axis, $PQ = P_1Q$, and for any R on $y = x$, $RP = RP_2$. So $PQ + QR + RP = P_1Q + QR + RP_2 \geq P_1P_2 = \sqrt{6^2+8^2} = 10$. Equality needs Q and R on segment P_1P_2. That segment crosses the x-axis at $Q = \left(\frac{25}{4}, 0\right)$ and the line $y = x$ at $R = \left(\frac{25}{7}, \frac{25}{7}\right)$, both in the allowed places, so the least perimeter is 10.

Check. Directly: $PQ = \sqrt{\left(\frac{3}{4}\right)^2 + 1^2} = \frac{5}{4}$, $QR = \sqrt{\left(\frac{75}{28}\right)^2 + \left(\frac{100}{28}\right)^2} = \frac{125}{28}$ and $RP = \sqrt{\left(\frac{24}{7}\right)^2 + \left(\frac{18}{7}\right)^2} = \frac{30}{7}$. Their sum is $\frac{35+125+120}{28} = \frac{280}{28} = 10$. A common slip is to reflect only once, which bounds just two of the three sides.

18. Answer: 9121**. The idea.** The decimal repeats, so the sum is (number of full blocks) × (block sum) plus the sum of a leftover partial block.

Work it out. Long division gives $\frac{1}{13} = 0.\overline{076923}$, a repeating block of 6 digits with digit sum $0+7+6+9+2+3 = 27$. Since $2026 = 6\cdot 337+4$, the first 2026 digits are 337 full blocks followed by the first 4 digits $0, 7, 6, 9$ of another block. The sum is $337 \cdot 27 + (0+7+6+9) = 9099 + 22 = 9121$.

Check. The period really is 6: $13 \cdot 76923 = 999999$, so $\frac{1}{13} = \frac{076923}{999999}$. A nice sanity check on the block sum: $076 + 923 = 999$, so paired digits 0+9, 7+2, 6+3 each give 9, and $3 \cdot 9 = 27$.

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19. Answer: 11**. The idea.** Split the rolling into three stages: waiting for the first of 1, 2, 3 to appear, then for a second one of them, then for the last. The expected total is the sum of the expected stage lengths, and each stage is a simple wait for an event of fixed probability: if each roll succeeds with probability p, the expected wait is $\frac{1}{p}$ rolls.

Work it out. Stage 1: any of the 3 targets counts, so $p = \frac{3}{6}$ and the expected wait is 2. Stage 2: one target is already seen, and 2 of the faces are new targets, so $p = \frac{2}{6}$ and the wait is 3. Stage 3: only 1 face helps, so $p = \frac{1}{6}$ and the wait is 6. The expected number of rolls is $2 + 3 + 6 = 11$.

Check. Why is the wait $\frac{1}{p}$? If W is the expected wait, the first roll either succeeds or leaves you where you started: $W = 1 + (1 - p)W$, so $W = \frac{1}{p}$. The last stage dominates, as expected: once two targets have shown up you need one specific face. A common slip is to use $\frac{6}{3}$ for every stage, giving 6, or to wait for all six faces.

20. Answer: 12**. The idea.** Name the sides $a < b < c$. The triangle inequality only needs checking for the longest side: $a + b > c$. Combined with $a + b + c = 30$ this pins c into a narrow window, and then count (a, b) for each c.

Work it out. $a + b > c$ and $a + b = 30 - c$ give $c < 15$. Since c is strictly the largest, $3c > 30$, so $c \geq 11$. For each c, we need $a < b < c$ with $a + b = 30 - c$: $c = 11$: $a + b = 19$ with $b \leq 10$: only $(9, 10)$. That is 1. $c = 12$: $a + b = 18$ with $10 \leq b \leq 11$: $(8, 10)$, $(7, 11)$. That is 2. $c = 13$: $a + b = 17$ with $9 \leq b \leq 12$: 4. $c = 14$: $a + b = 16$ with $9 \leq b \leq 13$: 5. Total $1 + 2 + 4 + 5 = 12$.

Check. The lower limit for b each time is "just over half of $a + b$" (so that $a < b$), and the upper limit is $c - 1$. For example $c = 14$: $b \in \{9, \ldots, 13\}$ gives $a \in \{7, \ldots, 3\}$, all positive and less than b. And $c = 15$ would give a flat triangle ($a + b = 15 = c$), which is not allowed.

21. Answer: 14**. The idea.** $10^{10} = 2^{10} \cdot 5^{10}$, so every divisor is $2^a 5^b$ with $0 \leq a, b \leq 10$. Fix the (few possible) powers of 5 and ask which powers of 2 push the product into the range 100 to 999.

Work it out. $b = 0$: $2^a \in [100, 999]$: $128, 256, 512$. That is 3. $b = 1$: $5 \cdot 2^a$ with $2^a \in [20, 199]$: $2^a = 32, 64, 128$ (giving $160, 320, 640$). That is 3. $b = 2$: $25 \cdot 2^a$ with $2^a \in [4, 39]$: $4, 8, 16, 32$ (giving $100, 200, 400, 800$). That is 4. $b = 3$: $125 \cdot 2^a$ with $2^a \leq 7$: $1, 2, 4$ (giving $125, 250, 500$). That is 3. $b = 4$: $625 \cdot 2^a$ with $2^a \leq 1$: just 625. That is 1. $b \geq 5$: $5^5 = 3125$ is already too big. Total: $3 + 3 + 4 + 3 + 1 = 14$.

Check. Every exponent used is at most 9, so the cap of 10 never bites. The divisors in order: $100, 125, 128, 160, 200, 250, 256, 320, 400, 500, 512, 625, 640, 800$: fourteen of them.

22. Answer: 516**. The idea.** Any three dots form a triangle unless they lie on one line. So count all triples and subtract the collinear ones, sorting the lines by direction.

Work it out. There are $\binom{16}{3} = 560$ triples. Collinear triples: rows: 4 rows, each with $\binom{4}{3} = 4$ triples, giving 16; columns: likewise 16; the two long diagonals (4 dots each): $2 \cdot 4 = 8$; the diagonals with exactly 3 dots: there are two in each direction, so 4 more; any other direction (slope ± 2 or $\pm\frac{1}{2}$ or steeper) needs a span of at least 5 dots in one direction to fit three points, which a 4 by 4 array does not have: 0. Total collinear: $16 + 16 + 8 + 4 = 44$, so there are $560 - 44 = 516$ triangles.

Check. Try the same method on a 3 by 3 array: $\binom{9}{3} - (3 + 3 + 2) = 76$, the well-known count. Here, the easy-to-miss cases are the four short diagonals; forgetting them gives 520.

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23. Answer: $12 + 4\sqrt{3}$. **The idea.** The remaining solid has two kinds of faces: what is left of each original face, and a new face created by each cut. Find the shape of each kind.

Work it out. Each face of the cube is a 2 by 2 square. Cutting off its four corners along the lines joining midpoints of adjacent edges leaves the square whose vertices are the four edge midpoints: its side is the distance between midpoints of adjacent edges, $\sqrt{1^2 + 1^2} = \sqrt{2}$, so its area is 2. Six faces give 12. Each cut exposes a triangle whose vertices are three edge midpoints around one corner; each side is again $\sqrt{2}$, so it is equilateral with area $\frac{\sqrt{3}}{4} \cdot (\sqrt{2})^2 = \frac{\sqrt{3}}{2}$. Eight cuts give $4\sqrt{3}$. The total is $12 + 4\sqrt{3}$.

Check. The pieces cut off do not overlap: two cuts at the ends of one edge meet only at that edge's midpoint. Area bookkeeping on one face: $4 - 4 \cdot \frac{1}{2} = 2$, since each face loses four right triangles with legs 1 and 1. (The solid is a cuboctahedron: 6 squares, 8 triangles.)

24. Answer: $2\sqrt{3}$. **The idea.** Any such cut slices off a small triangle at one corner. At a corner with angle A between sides b and c, a cut with ends at distances x and y from the corner cuts off area $\frac{1}{2}xy \sin A$, and halving the triangle means $xy = \frac{bc}{2}$: the product is fixed. By the law of cosines the cut has length squared $x^2 + y^2 - 2xy \cos A \geq 2xy - 2xy \cos A$, with equality when $x = y$. So for each corner the best cut is isosceles, and we only have to compare three corners.

Work it out. With $xy = \frac{bc}{2}$, the minimum squared length at that corner is $bc(1 - \cos A) = bc - \frac{b^2+c^2-a^2}{2} = \frac{a^2-(b-c)^2}{2}$, where a is the side opposite the corner. Corner opposite 5 (between 6 and 7): $\frac{25-1}{2} = 12$. Corner opposite 6 (between 5 and 7): $\frac{36-4}{2} = 16$. Corner opposite 7 (between 5 and 6): $\frac{49-1}{2} = 24$. The smallest is 12, at the corner opposite the side 5. There $x = y = \sqrt{\frac{6 \cdot 7}{2}} = \sqrt{21} \approx 4.58$, which fits on both the side 6 and the side 7, so this cut really exists. Its length is $\sqrt{12} = 2\sqrt{3}$.

Check. Direct computation at that corner: $\cos A = \frac{36+49-25}{2 \cdot 42} = \frac{5}{7}$, so the cut has length squared $21 + 21 - 2 \cdot 21 \cdot \frac{5}{7} = 42 - 30 = 12$. Compare the shortest median, to the side 7: its squared length is $\frac{2 \cdot 25 + 2 \cdot 36 - 49}{4} = \frac{73}{4} > 12$, so a slanted cut beats every median. Note the winning corner is the smallest angle, and the fitting check matters: at a corner with very unequal sides the isosceles cut might not fit.

25. Answer: 486. **The idea.** Track $d = x - y$. Each step changes d by ± 1, and d must stay in $\{-2, -1, 0, 1, 2\}$. Look at the path two steps at a time: after an even number of steps d is -2, 0 or 2, and we can count how the path moves between these states.

Work it out. From $d = 0$, two steps can go RU or UR (back to 0), RR (to 2) or UU (to -2): 2 ways to return, 1 way to each side. From $d = 2$, the next step R would reach 3, which is forbidden; so the pair must be UR (back to 2) or UU (to 0): 1 way each. Likewise from -2. Let a_n be the number of allowed paths that are at $d = 0$ after $2n$ steps, and b_n the number at $d = 2$ (the same as at $d = -2$, by symmetry). Then $a_{n+1} = 2a_n + 2b_n$ and $b_{n+1} = a_n + b_n$. So $a_{n+1} = 2b_{n+1}$ for every $n \geq 0$, and then $b_{n+1} = a_n + b_n = 2b_n + b_n = 3b_n$ once $n \geq 1$. Starting from $a_0 = 1$, $b_0 = 0$: $b_1 = 1$, $a_1 = 2$, and then $b_n = 3^{n-1}$, $a_n = 2 \cdot 3^{n-1}$. Reaching $(6, 6)$ means $d = 0$ after 12 steps: $a_6 = 2 \cdot 3^5 = 486$.

Check. Small cases by hand: to $(1, 1)$ there are 2 paths ($a_1 = 2$); to $(2, 2)$ all $\binom{4}{2} = 6$ paths are allowed (you cannot get 3 away in only 2 steps each way), and $a_2 = 2 \cdot 3 = 6$. To $(3, 3)$: a path breaks the rule only if d reaches ± 3, which needs three R's before any U (or the reverse), so only RRRUUU and UUURRR fail, leaving $20 - 2 = 18 = 2 \cdot 3^2$.

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26. Answer: 91. **The idea.** Every value is equal to exactly one fraction in lowest terms, and if $\frac{a}{b}$ has $a, b \le 12$, so does its reduced form. So count the pairs (a, b) with $1 \le a, b \le 12$ and $\gcd(a, b) = 1$. Organise by the larger of the two numbers, using the count of numbers below n that are relatively prime to n.

Work it out. The value 1 comes only from $\frac{1}{1}$ in lowest terms. For values less than 1, the reduced form $\frac{a}{b}$ has $a < b$ and $\gcd(a, b) = 1$; for a given b there are $\varphi(b)$ choices of a, where $\varphi(b)$ counts the numbers from 1 to b relatively prime to b: $\varphi(2), \ldots, \varphi(12) = 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4$, with sum 45. The values greater than 1 are exactly the reciprocals of these, another 45. Total: $45 + 45 + 1 = 91$.

Check. Out of the 144 ordered pairs, 91 are relatively prime pairs; the other 53 are repeats of values already counted. Quick spot check with 1 to 3: the values are $1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, 2, 3, \frac{3}{2}$, that is $7 = 2(1 + 2) + 1$.

27. Answer: 88. **The idea.** Subtracting the two equations makes them factor, which splits the system into two easy cases: $y = x$ or $x + y = 1$. In each case the solutions come from a quadratic whose roots are not nice, but only symmetric sums of the roots are needed, and Vieta's formulas give those directly.

Work it out. Subtracting, $x^2 - y^2 + y - x = 0$, i.e. $(x - y)(x + y - 1) = 0$. Case $y = x$: then $x^2 + x - 11 = 0$, with two real roots x_1, x_2 (discriminant 45), $x_1 + x_2 = -1$ and $x_1 x_2 = -11$. Each solution (x_i, x_i) gives $2x_i^2$, and $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = 1 + 22 = 23$. This case contributes $2 \cdot 23 = 46$. Case $y = 1 - x$: then $x^2 + 1 - x = 11$, i.e. $x^2 - x - 10 = 0$, with two real roots x_3, x_4 (discriminant 41), $x_3 + x_4 = 1$ and $x_3 x_4 = -10$. Here $y = 1 - x$ is the other root of the same quadratic, so the two solutions are (x_3, x_4) and (x_4, x_3), and each gives $x_3^2 + x_4^2 = 1 + 20 = 21$. This case contributes 42. The two cases do not overlap (that would need $x = y = \frac{1}{2}$, but $\frac{1}{4} + \frac{1}{2} \ne 11$), so there are exactly four solutions and the sum is $46 + 42 = 88$.

Check. Numerically, $x^2 + x - 11 = 0$ has roots about 2.854 and -3.854, giving $x^2 + y^2 \approx 16.29$ and 29.71 (sum 46); $x^2 - x - 10 = 0$ has roots about 3.702 and -2.702, and $3.702^2 + 2.702^2 \approx 21$ for each of the two ordered pairs. For instance $(3.702, -2.702)$: $3.702^2 - 2.702 \approx 11.00$ and $2.702^2 + 3.702 \approx 11.00$. The trap is to forget the case $x + y = 1$, which gives 46.

28. Answer: $\frac{9+3\sqrt{5}}{4}$. **The idea.** Use coordinates with the chord horizontal. A circle sitting on line AB with radius r has its center exactly r above the line, so each tangency is a "distance between centers" equation. The two equations for β both contain its unknown horizontal position; eliminating it leaves a radical equation in r, and the side condition (between P and B) picks out one of its two roots.

Work it out. Place $O = (0, 0)$. The chord is at distance $\sqrt{10^2 - 8^2} = 6$ from O, so it lies on $y = -6$, with $A = (-8, -6)$, $B = (8, -6)$ and $P = (-4, -6)$. Circle α has center $(-4, -6 + a)$, where a is its radius, and internal tangency means that center is $10 - a$ from O: $16 + (a - 6)^2 = (10 - a)^2$, so $16 + 36 - 12a = 100 - 20a$, $a = 6$, and the center of α is $(-4, 0)$. Circle β has radius r and center $(x, r - 6)$ with $-4 < x < 8$. Tangent to ω from inside: $x^2 + (r - 6)^2 = (10 - r)^2$, so $x^2 = 64 - 8r$. Tangent to α from outside: $(x + 4)^2 + (r - 6)^2 = (6 + r)^2$, so $(x + 4)^2 = 24r$. Subtracting, $8x + 16 = 32r - 64$, so $x = 4r - 10$. Substituting into $x^2 = 64 - 8r$: $16r^2 - 80r + 100 = 64 - 8r$, i.e. $16r^2 - 72r + 36 = 0$, i.e. $4r^2 - 18r + 9 = 0$. So $r = \frac{18 \pm \sqrt{324 - 144}}{8} = \frac{9 \pm 3\sqrt{5}}{4}$. The condition $x > -4$ means $4r - 10 > -4$, i.e. $r > \frac{3}{2}$, which rules out $\frac{9-3\sqrt{5}}{4} \approx 0.57$. (That smaller circle is the one tucked between α and A.) So the radius of β is $\frac{9+3\sqrt{5}}{4}$.

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Check. $r \approx 3.927$ gives $x = 4r-10 \approx 5.708$, between -4 and 8 as required, so β touches the chord itself between P and B. Distances: from O to $(5.708, -2.073)$ is $\sqrt{32.58 + 4.30} \approx 6.073 = 10-r$, and from $(-4, 0)$ it is $\sqrt{94.25 + 4.30} \approx 9.927 = 6 + r$. The key lesson is the last step: the algebra produces two circles, and only the side condition says which one is meant.

29. Answer: 46. **The idea.** First find the triples with $a \le b \le c$. The smallest number carries the largest fraction, which traps a in a short range; then for each a, the two-variable equation factors (Simon's Favorite Factoring Trick). Finally count the orderings of each sorted triple.

Work it out. With $a \le b \le c$: $\frac{1}{a} < \frac{1}{2}$ forces $a \ge 3$, and $\frac{1}{2} \le \frac{3}{a}$ forces $a \le 6$. $a = 3$: $\frac{1}{b} + \frac{1}{c} = \frac{1}{6}$, i.e. $bc - 6b - 6c = 0$, i.e. $(b-6)(c-6) = 36$. Both factors are positive (since $\frac{1}{b} < \frac{1}{6}$ means $b > 6$), and with $b \le c$ the factor pairs $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$, $6 \cdot 6$ give $(b, c) = (7, 42), (8, 24), (9, 18), (10, 15), (12, 12)$: five triples. $a = 4$: $\frac{1}{b} + \frac{1}{c} = \frac{1}{4}$, so $(b-4)(c-4) = 16$: $(5, 20), (6, 12), (8, 8)$: three triples. $a = 5$: $\frac{1}{b} + \frac{1}{c} = \frac{3}{10}$, so $3bc = 10b + 10c$ and $(3b-10)(3c-10) = 100$. With $b \ge 5$ the first factor is at least 5 and at most 10 (it is the smaller factor), and it must be 2 more than a multiple of 3: only $5 \cdot 20$ works, giving $(b, c) = (5, 10)$: one triple. $a = 6$: $b, c \ge 6$ and $\frac{1}{b} + \frac{1}{c} = \frac{1}{3}$ force $b = c = 6$: one triple. Now count orders. All different: $(3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (4, 5, 20), (4, 6, 12)$: 6 triples $\times 6$ orders $= 36$. Two equal: $(3, 12, 12), (4, 8, 8), (5, 5, 10)$: $3 \times 3 = 9$. All equal: $(6, 6, 6)$: 1. Total $36 + 9 + 1 = 46$.

Check. Spot check each sorted triple, e.g. $\frac{1}{5} + \frac{1}{5} + \frac{1}{10} = \frac{2+2+1}{10} = \frac{1}{2}$ and $\frac{1}{3} + \frac{1}{10} + \frac{1}{15} = \frac{10+3+2}{30} = \frac{1}{2}$. For $a = 5$ the rejected factor pair $10 \cdot 10$ would need $3b = 20$, not an integer.

30. Answer: 544. **The idea.** Look at where the largest number goes. In a zigzag it must be a "peak" (it is bigger than its neighbours), and it splits the row into a left part and a right part, each of which is a smaller zigzag with a forced direction next to the peak. That gives a recursion.

Work it out. By flipping left to right and by the symmetry $x \mapsto m + 1 - x$, exactly half of the zigzags of $m \ge 2$ numbers have any one prescribed direction at one end, and for $m = 0$ or 1 there is one "zigzag" that fits. Let E_m be this count: $E_0 = E_1 = 1$, and $E_m = Z_m/2$ for $m \ge 2$, where Z_m is the number of zigzags of m numbers. To build a zigzag of n numbers, place the largest one with k numbers to its left and $n - 1 - k$ to its right. Choose which numbers go left ($\binom{n-1}{k}$ ways). The left block must be a zigzag whose last number is a valley (smaller than the number before it), and the right block a zigzag whose first number is a valley: E_k and E_{n-1-k} ways. Any such choice gives a zigzag, and different choices give different zigzags. So $Z_n = \sum_{k=0}^{n-1} \binom{n-1}{k} E_k E_{n-1-k}$. Now build up: $Z_2 = 1+1 = 2$, so $E_2 = 1$. $Z_3 = 1+2+1 = 4$, so $E_3 = 2$. $Z_4 = 2+3+3+2 = 10$, so $E_4 = 5$. $Z_5 = 5+4\cdot2+6\cdot1+4\cdot2+5 = 32$, so $E_5 = 16$. $Z_6 = 16+5\cdot5+10\cdot2+10\cdot2+5\cdot5+16 = 122$, so $E_6 = 61$. $Z_7 = 61+6\cdot16+15\cdot5+20\cdot2\cdot2+15\cdot5+6\cdot16+61 = 61+96+75+80+75+96+61 = 544$.

Check. Small cases by listing: for $1, 2, 3$ the zigzags are $132, 231, 213, 312$, four of them, matching $Z_3 = 4$. For $1, 2, 3, 4$ there are 10 ($1324, 1423, 2314, 2413, 3412$ and their mirror images $4231, \ldots$ under $x \mapsto 5 - x$), matching $Z_4 = 10$. The example 3152746 has the 7 in position 5, with left block 3152 (ending in a valley 2) and right block 46 (starting with a valley 4).

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