# MathCastle

STATE COMPETITION

# Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

**Name** _____

**School** _____

**Chapter** _____

**INSTRUCTIONS**

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

| | |
|---|---|
| Total Correct | Scorer's Initials |

**DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.**

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**1.** Ann is older than Bo. Ann's age now is exactly twice the age Bo was at the moment when Ann was as old as Bo is now. The sum of their present ages is 42 years. How old is Ann now, in years?

1. _____

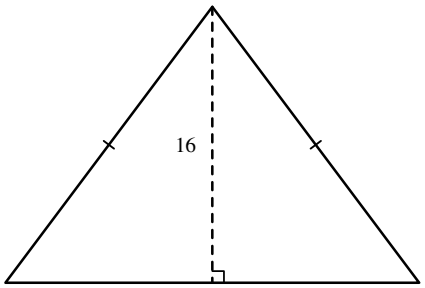
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**2.** Three different prime numbers have a sum of exactly 50. Among all such sets of three primes, what is the largest possible value of their product?

2. _____

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**3.** An isosceles triangle has two sides of equal length, called its legs, and a third side called its base. Its perimeter is 64 cm, and the altitude drawn from the vertex between the legs to the base is 16 cm long. What is the area of the triangle, in square centimetres?




3. _____

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**4.** A door code is a string of four digits, each from 0 to 9; a code may begin with 0. A code is called a *step code* if each digit after the first differs from the digit just before it by exactly 1. For example, 3232, 0123 and 5654 are step codes, but 3235 and 4444 are not. How many step codes are there?

4. _____

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**5.** What is the smallest positive integer $q$ for which there is an integer $p$ with $\frac{5}{8} < \frac{p}{q} < \frac{7}{11}$?

5. _____

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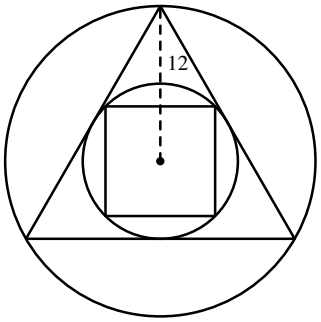
**6.** For some prime numbers $p$, the number $5p+1$ is a perfect square (the square of a whole number). What is the sum of all prime numbers $p$ with this property?

6. _____

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**7.** An equilateral triangle has all three of its vertices on a circle of radius 12. A second circle is drawn inside the triangle, touching all three of its sides. Then a square is drawn with all four of its vertices on this second circle. What is the area of the region that lies inside the second circle but outside the square? Express your answer in terms of $\pi$.




7. _____

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**8.** A two-digit positive integer (from 10 to 99 inclusive) is chosen at random, each being equally likely. What is the probability that its tens digit and its units digit differ by 6 or more? Express your answer as a common fraction.

8. _____

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**9.** Nonzero real numbers $a$ and $b$ satisfy $a+\dfrac{1}{b}=5$ and $b+\dfrac{1}{a}=10$. What is the value of $a^2b^2+\dfrac{1}{a^2b^2}$?

9. _____

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**10.** A jar holds only red and blue marbles, in the ratio 5 : 7. Kai removes the same number of red marbles as blue marbles, and afterwards the ratio of red to blue is 1 : 2. He then puts 18 new red marbles into the jar, and now the jar holds equal numbers of red and blue marbles. How many marbles were in the jar at the very start?

10. _____

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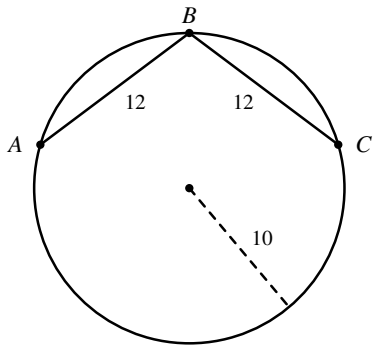
**11.** Two positive integers have a sum of 78 and a least common multiple of 216. What is the positive difference of the two integers?

11. _____

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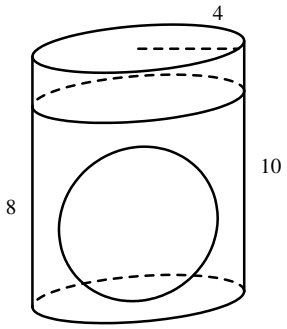
**12.** Points $A$, $B$ and $C$ lie on a circle of radius 10, with $A$ and $C$ different points, and $AB = BC = 12$. What is the length of $AC$? Express your answer as a common fraction.




12. _____

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**13.** A closed-bottom cylindrical tank has an inside radius of 4 cm and an inside height of 10 cm, and it is filled with water to a depth of 8 cm. A solid metal ball of radius 3 cm is lowered slowly into the tank until it rests on the bottom. How many cubic centimetres of water spill over the top of the tank? Express your answer in terms of $\pi$.




13. _____

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**14.** A regular decagon (a polygon with 10 equal sides and 10 equal angles) has 35 diagonals. How many of these 35 diagonals are parallel to none of the decagon's 10 sides?

14. _____

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**15.** The numbers $2^n + 3^n$ for $n = 1, 2, 3, 4, \ldots$ begin $5, 13, 35, 97, \ldots$. For how many integers $n$ with $1 \leq n \leq 200$ is $2^n + 3^n$ divisible by 77?

15. _____

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**16.** Two different integers are chosen at random from $1, 2, 3, \ldots, 30$, with every pair of different integers equally likely. What is the probability that their product is a multiple of 6? Express your answer as a common fraction.

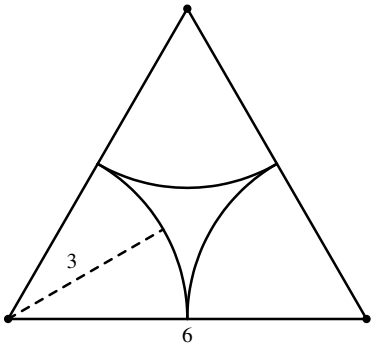
16. _____

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**17.** For a positive integer $n$, let $S(n)$ be the sum of the digits of $n$; for example, $S(2026) = 2+0+2+6 = 10$. What is the sum of all positive integers $n$ for which $n + S(n) = 2027$?

17. _____

**18.** A lawn has the shape of an equilateral triangle with side length 6 metres. A sprinkler at each of the three corners waters every point of the lawn that is within 3 metres of it. What is the area, in square metres, of the part of the lawn that gets no water? Express your answer in simplest radical form, in terms of $\pi$.




18. _____

**19.** A list of positive integers has mean 20. Its largest number appears only once, and so does its smallest number. If the largest number is removed, the mean of the numbers left is 18. If instead the smallest number is removed (and the largest kept), the mean of the numbers left is 22. The largest number is 36 more than the smallest. What is the largest number?

19. _____

**20.** Among all fractions $\frac{p}{q}$ with $p$ and $q$ positive integers and $q \leq 20$, which one is closest to $\frac{3}{7}$ without being equal to $\frac{3}{7}$? Express your answer as a common fraction.

20. _____

**21.** Five different squares of a $4 \times 4$ grid are chosen at random, with every set of five squares equally likely. What is the probability that every row and every column of the grid contains at least one of the chosen squares? Express your answer as a common fraction.

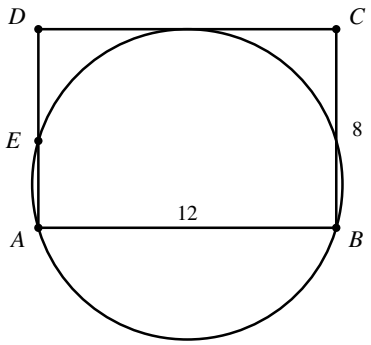
21. _____

**22.** How many points $(x, y, z)$ in space, with $x$, $y$ and $z$ all integers, satisfy $|x| + |y| + |z| \leq 4$?

22. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**23.** Rectangle $ABCD$ has $AB = 12$ and $BC = 8$, with its vertices labelled in order around it. A circle passes through $A$ and $B$ and is tangent to the line $CD$. Besides $A$, this circle meets side $\overline{AD}$ at exactly one other point, $E$. What is the length $AE$? Express your answer as a common fraction.




23. _____

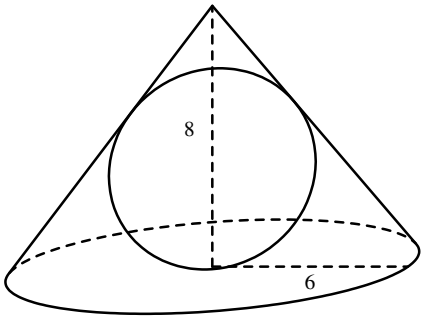
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**24.** How many positive integers $n \leq 800$ have exactly three 1's when written in base 2? (For example, 11 is 1011 in base 2, so it has exactly three 1's.)

24. _____

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**25.** A closed right circular cone has base radius 6 and height 8. A solid ball is placed inside it so that the ball touches the cone's flat base and also touches the cone's curved surface all the way around (along a full circle). What fraction of the cone's volume does the ball take up? Express your answer as a common fraction.




25. _____

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**26.** A fair six-sided die is rolled over and over, and the rolling stops as soon as the last four rolls show four different numbers. For example, the rolls $3, 3, 5, 3, 1, 6$ stop after the sixth roll: $5, 3, 1, 6$ are all different, but no four earlier consecutive rolls were. What is the expected (average) number of rolls? Express your answer as a common fraction.

26. _____

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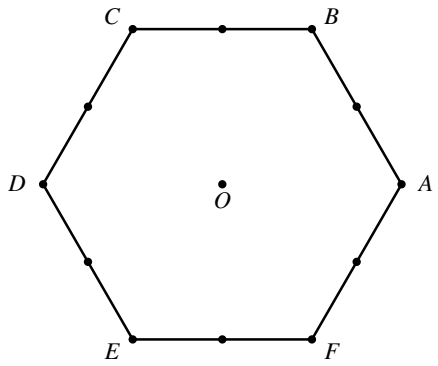
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**27.** Tetrahedron $ABCD$ has $AB = CD = 5$, $AC = BD = 6$ and $AD = BC = 7$ (each pair of opposite edges has equal length). What is the volume of the tetrahedron? Express your answer in simplest radical form.

27. _____

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**28.** Regular hexagon $ABCDEF$ has centre $O$. Thirteen points are marked: the centre $O$, the six vertices, and the six midpoints of the sides. A point $X$ is chosen uniformly at random inside the hexagon. What is the probability that, of the thirteen marked points, the one closest to $X$ is a vertex of the hexagon? Express your answer as a common fraction.




28. _____

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**29.** How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, have the property that no element of the subset is exactly twice another element of the subset, and no element is exactly three times another element of the subset?

29. _____

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**30.** Each of the 8 vertices of a cube is coloured with one of 4 colours so that the two ends of every edge of the cube get different colours. Not every colour has to be used, and the cube is fixed in place, so colourings that differ by a rotation of the cube count as different. How many such colourings are there?

30. _____

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# Answer Key — MathCounts State Sprint Practice Contest 1

**1.** 24

**2.** 1102

**3.** 192

**4.** 64

**5.** 19

**6.** 10

**7.** $36\pi - 72$

**8.** $\frac{8}{45}$

**9.** 2302

**10.** 108

**11.** 30

**12.** $\frac{96}{5}$

**13.** $4\pi$

**14.** 20

**15.** 7

**16.** $\frac{37}{87}$

**17.** 4016

**18.** $9\sqrt{3} - \frac{9\pi}{2}$

**19.** 38

**20.** $\frac{8}{19}$

**21.** $\frac{9}{91}$

**22.** 129

**23.** $\frac{7}{2}$

**24.** 118

**25.** $\frac{3}{8}$

**26.** $\frac{38}{5}$

**27.** $2\sqrt{95}$

**28.** $\frac{1}{4}$

**29.** 588

**30.** 2652

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# Solutions — MathCounts State Sprint Practice Contest 1

1. **Answer:** 24. **The idea.** Age problems untangle once you notice the one thing that never changes: the difference between two people's ages. Let Ann be $A$ and Bo be $B$ now, and let $d = A - B$ be the gap.

   **Work it out.** Ann was as old as Bo is now exactly $d$ years ago. At that moment Bo was $B - d = B - (A - B) = 2B - A$. The statement says $A = 2(2B - A)$, so $3A = 4B$, and Ann and Bo are in the ratio $4 : 3$. With $A + B = 42$, this gives $A = \frac{4}{7} \cdot 42 = 24$ and $B = 18$.

   **Check.** The gap is 6 years. Six years ago Ann was 18, Bo's age now, and Bo was 12. Ann's present age, 24, is indeed twice 12. A common slip is to use Bo's present age in place of "the age Bo was", which gives $A = 2B$ and the wrong answer 28.

2. **Answer:** 1102. **The idea.** Parity first. Every prime except 2 is odd, and the sum of three odd numbers is odd, so it cannot be 50. Hence one of the primes must be 2, and the other two are different odd primes adding to 48.

   **Work it out.** The pairs of different odd primes with sum 48 are $5 + 43$, $7 + 41$, $11 + 37$, $17 + 31$ and $19 + 29$ (the pair $24 + 24$ is neither prime nor allowed, and $13 + 35$, $23 + 25$ fail since 35 and 25 are composite). For a fixed sum, a product is larger the closer the two numbers are, so the best pair is 19 and 29. The largest product is $2 \cdot 19 \cdot 29 = 2 \cdot 551 = 1102$.

   **Check.** The next closest pair gives $2 \cdot 17 \cdot 31 = 1054 < 1102$. Two primes cannot both be 2 (they must be different), so there is no other way to fix the parity.

3. **Answer:** 192. **The idea.** The altitude from the apex of an isosceles triangle hits the base at its midpoint, splitting the triangle into two right triangles. Name the half-base $x$; the perimeter then fixes each leg as $32 - x$, and the Pythagorean theorem gives an equation whose squared terms cancel.

   **Work it out.** Half the perimeter is $32 = x + \text{leg}$, so a leg is $32 - x$. In one right half, $(32 - x)^2 = x^2 + 16^2$, that is $1024 - 64x + x^2 = x^2 + 256$. The $x^2$ terms cancel: $64x = 768$, so $x = 12$. The base is 24, each leg is 20, and the area is $\frac{1}{2} \cdot 24 \cdot 16 = 192$.

   **Check.** Each half is a 12-16-20 right triangle (4 times 3-4-5), and the perimeter is 20+20+24 = 64.

4. **Answer:** 64. **The idea.** Think of a step code as a walk on the number line: start at the first digit and take three steps, each $+1$ or $-1$. If there were no edges, every start would allow $2^3 = 8$ walks, for $10 \cdot 8 = 80$ in all. The only thing that can go wrong is stepping off the digits, below 0 or above 9, so count those bad walks and subtract.

   **Work it out.** Walks that go below 0 must start low. Starting at 0: any walk whose first step is down (4 walks) plus the walk $0, 1, 0, -1$ (1 more), so 5 bad. Starting at 1: the first two steps must both be down, $1, 0, -1$, then anything (2 bad). Starting at 2: all three steps down (1 bad). That is $5 + 2 + 1 = 8$ walks falling below 0. By the symmetry $d \mapsto 9 - d$, exactly 8 walks rise above 9, and no walk of three steps can do both. So there are $80 - 16 = 64$ step codes.

   **Check.** Count by first digit directly. Starting at 0 the codes are $0101, 0121, 0123$ (3); at 1 there are $8 - 2 = 6$; at 2, $8 - 1 = 7$; starts 3 through 6 never reach an edge, 8 each; and $7, 8, 9$ mirror $2, 1, 0$. Total $3 + 6 + 7 + 4 \cdot 8 + 7 + 6 + 3 = 64$. Forgetting the edges gives the tempting wrong answer 80.

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**5. Answer:** 19**. The idea.** Clear the denominators of the two inequalities and look at the positive integers they produce. From $\frac{5}{8} < \frac{p}{q}$ we get $8p - 5q > 0$, and from $\frac{p}{q} < \frac{7}{11}$ we get $7q - 11p > 0$. Both are whole numbers, so each is at least 1. The trick is to find a combination of these two numbers that equals $q$ itself.

**Work it out.** Let $x = 8p-5q \geq 1$ and $y = 7q-11p \geq 1$. Then $11x+8y = 88p-55q+56q-88p = q$. So $q = 11x + 8y \geq 11 + 8 = 19$. Equality needs $x = y = 1$: $8p - 5q = 1$ and $7q - 11p = 1$ with $q = 19$ give $8p = 96$, so $p = 12$. Indeed $\frac{5}{8} = 0.625 < \frac{12}{19} \approx 0.6316 < \frac{7}{11} \approx 0.6364$. The smallest denominator is 19.

**Check.** Notice $\frac{12}{19}$ is the fraction you get by adding the numerators and the denominators, $\frac{5+7}{8+11}$; that works here because $7 \cdot 8 - 5 \cdot 11 = 1$. A direct scan agrees: the interval has width $\frac{7}{11} - \frac{5}{8} = \frac{1}{88}$, and for each $q \leq 18$ one checks that the multiples of $\frac{1}{q}$ just above 0.625 already pass 0.6364 (for example $q = 11$: $\frac{7}{11}$ is the endpoint itself, and $q = 16$: $\frac{10}{16} = 0.625$ and $\frac{11}{16} = 0.6875$).

**6. Answer:** 10**. The idea.** If $5p + 1 = k^2$, then $5p = k^2 - 1 = (k - 1)(k + 1)$: the number $5p$ splits as a product of two positive integers that differ by exactly 2. But $5p$ has very few divisors, so there are very few ways this can happen.

**Work it out.** Here $k \geq 2$, so $k - 1$ and $k + 1$ are positive divisors of $5p$ with product $5p$. If $p = 5$, then $5p + 1 = 26$ is not a square. Otherwise the positive divisors of $5p$ are only $1, 5, p, 5p$, so the pair $(k - 1,\ k + 1)$, smaller number first, is $(1, 5p)$, $(5, p)$ or $(p, 5)$.

The pair $(1, 5p)$ needs $5p = 3$, impossible. The pair $(5, p)$ needs $p = 7$, which is prime: $5 \cdot 7 + 1 = 36 = 6^2$. The pair $(p, 5)$ needs $p = 3$, which is prime: $5 \cdot 3 + 1 = 16 = 4^2$. So the primes are 3 and 7, and their sum is 10.

**Check.** Small primes directly: $p = 2, 3, 5, 7, 11, 13$ give $11, 16, 26, 36, 56, 66$, and only 16 and 36 are squares. The easy slip is to try only $k - 1 = 5$, find $p = 7$, and miss $p = 3$, where the prime is the smaller factor.

**7. Answer:** $36\pi - 72$**. The idea.** In an equilateral triangle the centre of the circumscribed circle and the centre of the inscribed circle are the same point, and it sits two thirds of the way down each median. So the inradius is exactly half the circumradius.

**Work it out.** Each median runs from a vertex (distance 12 from the centre) through the centre to the midpoint of the opposite side. The centre splits the median in the ratio $2 : 1$, so the distance from the centre to each side is $\frac{12}{2} = 6$. The second circle has radius 6 and area $36\pi$.

A square inscribed in a circle of radius 6 has diagonals equal to the diameter, 12. The area of a square is half the product of its diagonals: $\frac{1}{2} \cdot 12 \cdot 12 = 72$. The region inside the circle but outside the square has area $36\pi - 72$.

**Check.** The square's side is $\frac{12}{\sqrt{2}} = 6\sqrt{2}$, and $(6\sqrt{2})^2 = 72$. As a sanity check, $36\pi - 72 \approx 41.1$, a bit more than half the square's area, which fits a picture of four circular caps around the square.

**8. Answer:** $\frac{8}{45}$**. The idea.** Organise by the tens digit, which runs from 1 to 9, and for each one count the units digits 0 to 9 that are at least 6 away from it.

**Work it out.** Tens digit 1: units 7, 8, 9 (3). Tens digit 2: 8, 9 (2). Tens digit 3: 9 (1). Tens digits 4 and 5: none, since the far digits would have to be below 0 or above 9. Tens digit 6: 0 (1). Tens digit 7: 0, 1 (2). Tens digit 8: 0, 1, 2 (3). Tens digit 9: 0, 1, 2, 3 (4). The total is 3+2+1+1+2+3+4 = 16 out of 90 numbers, so the probability is $\frac{16}{90} = \frac{8}{45}$.

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**Check.** Count by unordered digit pairs instead. Pairs of digits $\{p, q\}$ from 0 to 9 with $q - p \geq 6$: difference 6 gives 4 pairs, 7 gives 3, 8 gives 2, 9 gives 1, so 10 pairs. Each pair makes two numbers, except that a pair containing 0 can only be written with 0 second; there are 4 such pairs ($\{0,6\}, \{0,7\}, \{0,8\}, \{0,9\}$). Total $2 \cdot 10 - 4 = 16$.

**9. Answer:** 2302**. The idea.** Solving for $a$ and $b$ separately leads to messy square roots. But the target only involves the product $ab$, and multiplying the two equations produces exactly that product together with its reciprocal.

**Work it out.** Multiply: $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + a \cdot \frac{1}{a} + \frac{1}{b} \cdot b + \frac{1}{ab} = ab + 2 + \frac{1}{ab}$. The left side is $5 \cdot 10 = 50$, so $ab + \frac{1}{ab} = 48$. Now square: $\left(ab + \frac{1}{ab}\right)^2 = a^2b^2 + 2 + \frac{1}{a^2b^2}$, so $a^2b^2 + \frac{1}{a^2b^2} = 48^2 - 2 = 2304 - 2 = 2302$.

**Check.** Such numbers really exist. Dividing the equations gives $\frac{(ab+1)/b}{(ab+1)/a} = \frac{a}{b} = \frac{1}{2}$, so $b = 2a$, and the first equation becomes $a + \frac{1}{2a} = 5$, that is $2a^2 - 10a + 1 = 0$, with real roots $a = \frac{5 \pm \sqrt{23}}{2}$. Then $ab = 2a^2 = 10a - 1 = 24 \pm 5\sqrt{23}$ and $\frac{1}{ab} = 24 \mp 5\sqrt{23}$ (their product is $576 - 575 = 1$), so $ab + \frac{1}{ab} = 48$ as found.

**10. Answer:** 108**. The idea.** Removing equal numbers of each colour does not change the difference between the counts. Track that difference through each step.

**Work it out.** Start with $5k$ red and $7k$ blue, a difference of $2k$. After removing equal numbers, the difference is still $2k$, and red : blue = 1 : 2 means blue exceeds red by exactly the red count. So there are now $2k$ red and $4k$ blue marbles, and Kai removed $3k$ of each. Adding 18 red makes the counts equal: $2k + 18 = 4k$, so $k = 9$. The jar started with $5k + 7k = 12k = 108$ marbles.

**Check.** Start: 45 red, 63 blue (5 : 7). Remove 27 of each: 18 red, 36 blue (1 : 2). Add 18 red: 36 and 36.

**11. Answer:** 30**. The idea.** Pull out the greatest common divisor. Write the numbers as $a = dx$ and $b = dy$, where $d$ is their greatest common divisor and $x$, $y$ share no common factor. Then $a + b = d(x + y)$ and the least common multiple is $dxy$. The key fact is that $x + y$ and $xy$ share no prime factor either: a prime dividing $xy$ divides $x$ or $y$, and if it also divided $x + y$ it would divide the other one too. So the greatest common divisor of the sum and the least common multiple is exactly $d$.

**Work it out.** $78 = 2 \cdot 3 \cdot 13$ and $216 = 2^3 \cdot 3^3$, so their greatest common divisor is 6, and $d = 6$. Then $x + y = \frac{78}{6} = 13$ and $xy = \frac{216}{6} = 36$. So $x$ and $y$ are the roots of $z^2 - 13z + 36 = (z - 4)(z - 9)$, namely 4 and 9. The numbers are $6 \cdot 4 = 24$ and $6 \cdot 9 = 54$, and their difference is $54 - 24 = 30$.

**Check.** $24 + 54 = 78$. With $24 = 2^3 \cdot 3$ and $54 = 2 \cdot 3^3$, the least common multiple takes the higher power of each prime: $2^3 \cdot 3^3 = 216$. The two numbers are forced, so the answer is unique.

**12. Answer:** $\frac{96}{5}$**. The idea.** Let $O$ be the centre. Both $A$ and $C$ are 10 from $O$ and 12 from $B$, so they are mirror images of each other across line $OB$. Then $AC$ is twice the distance from $A$ to line $OB$, and that distance is a height of triangle $OAB$, which we can find through its area.

**Work it out.** Triangle $OAB$ has sides $OA = OB = 10$ and $AB = 12$. The height from $O$ to $AB$ lands at the midpoint of $AB$ and has length $\sqrt{10^2 - 6^2} = 8$, so the area of $OAB$ is $\frac{1}{2} \cdot 12 \cdot 8 = 48$.

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Using $OB = 10$ as the base instead, the height from $A$ to line $OB$ is $h$ with $\frac{1}{2} \cdot 10 \cdot h = 48$, so $h = \frac{48}{5}$. Hence $AC = 2h = \frac{96}{5}$.

**Check.** Coordinates: put $O = (0, 0)$ and $B = (0, 10)$. A point $(x, y)$ with $x^2 + y^2 = 100$ and $x^2 + (y - 10)^2 = 144$ gives, on subtracting, $-20y + 100 = 44$, so $y = \frac{14}{5}$ and $x^2 = 100 - \frac{196}{25} = \frac{2304}{25}$, $x = \pm\frac{48}{5}$. The two points are $\frac{96}{5}$ apart. Note $\frac{96}{5} = 19.2 < 20$, the diameter, as it must be.

**13. Answer:** $4\pi$. **The idea.** Water spills only when there is no longer room for it. Compare the total volume that must fit below the rim (water plus the submerged part of the ball) with the tank's capacity. Along the way, check how much of the ball ends up under water.

**Work it out.** The ball's diameter is 6, less than the tank's diameter 8, so it sits on the bottom and reaches a height of 6 cm. The water already stands at 8 cm, higher than the top of the ball, and water only rises when the ball goes in, so the ball ends up completely submerged. Its volume is $\frac{4}{3}\pi \cdot 3^3 = 36\pi$.

The tank holds $\pi \cdot 4^2 \cdot 10 = 160\pi$; the water is $\pi \cdot 4^2 \cdot 8 = 128\pi$, leaving $32\pi$ of empty space. Water plus ball is $128\pi + 36\pi = 164\pi$, which is $4\pi$ more than the tank holds. So $4\pi$ cubic centimetres spill.

**Check.** Without the rim the water would rise by $\frac{36\pi}{16\pi} = 2.25$ cm to 10.25 cm, which is 0.25 cm above the rim; a layer 0.25 cm deep over the full $16\pi$ cross section is $4\pi$. The trap answer $36\pi$ forgets the empty space above the water.

**14. Answer:** 20. **The idea.** Label the vertices $0, 1, \ldots, 9$ around the decagon. The segment joining vertices $i$ and $j$ is perpendicular to the line through the centre that bisects the arc between them, and that line depends only on $i + j$ (modulo 10). So two chords are parallel exactly when their label sums agree modulo 10.

**Work it out.** The sides join $i$ and $i + 1$, whose sums $2i + 1$ are odd and run through all five odd values modulo 10. So a diagonal is parallel to some side exactly when $i + j$ is odd, that is, when its endpoints are an odd number of steps apart around the decagon. Diagonals skip 2, 3, 4 or 5 steps. Those 3 steps apart: 10 of them. Those 5 steps apart (the long diagonals through the centre): 5 of them. That is 15 diagonals parallel to a side, so $35 - 15 = 20$ are parallel to none.

**Check.** The 20 are the diagonals 2 steps apart (10) and 4 steps apart (10); their label sums are even. For instance the diagonal $\{0, 2\}$ is parallel to the diagonal $\{9, 3\}$ (both have sum 2 mod 10), but no side has an even sum. A long diagonal such as $\{0, 5\}$ has odd sum 5 and is parallel to the sides $\{2, 3\}$ and $\{7, 8\}$, which is easy to see in a picture of the decagon.

**15. Answer:** 7. **The idea.** Since $77 = 7 \cdot 11$, the number must be divisible by 7 and by 11, and each prime can be handled on its own. Modulo a prime, $2^n + 3^n \equiv 0$ says $3^n \equiv -2^n$. Multiply both sides by a power of the number that "undoes" 2, and the condition becomes a single power equal to $-1$, whose pattern repeats in a short cycle.

**Work it out.** Modulo 7: since $4 \cdot 2 = 8 \equiv 1$, multiplying $3^n \equiv -2^n$ by $4^n$ gives $12^n \equiv -1$, that is $5^n \equiv -1 \equiv 6 \pmod 7$. The powers of 5 modulo 7 run $5, 4, 6, 2, 3, 1$ and then repeat with period 6, so $5^n \equiv 6$ exactly when $n \equiv 3 \pmod 6$.

Modulo 11: since $6 \cdot 2 = 12 \equiv 1$, multiplying by $6^n$ gives $18^n \equiv -1$, that is $7^n \equiv 10 \pmod{11}$. The powers of 7 modulo 11 run $7, 5, 2, 3, 10, 4, 6, 9, 8, 1$ and repeat with period 10, so $7^n \equiv 10$ exactly when $n \equiv 5 \pmod{10}$.

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We need both. $n \equiv 3 \pmod 6$ means $n$ is an odd multiple of 3, and $n \equiv 5 \pmod{10}$ means $n$ is an odd multiple of 5. Together: $n$ is an odd multiple of 15. In $1 \le n \le 200$ these are $15, 45, 75, 105, 135, 165, 195$, which is 7 values.

**Check.** $n = 3$: $8 + 27 = 35$, divisible by 7 but not 11. $n = 5$: $32 + 243 = 275 = 25 \cdot 11$, divisible by 11 but not 7. $n = 15$: $2^{15} + 3^{15} = 32{,}768 + 14{,}348{,}907 = 14{,}381{,}675 = 77 \cdot 186{,}775$. A student who stops at $n \equiv 3 \pmod 6$ for 7 alone, or at odd $n$ only, badly overcounts.

**16. Answer:** $\frac{37}{87}$**. The idea.** A product is a multiple of 6 when the two numbers together supply a factor 2 and a factor 3. Sort the thirty numbers by which of these factors they have; the good pairs then split into two easy kinds.

**Work it out.** Among 1 to 30: the multiples of 6 are 5 numbers; the even numbers that are not multiples of 3 are $15 - 5 = 10$; the odd multiples of 3 are $10 - 5 = 5$; the remaining 10 numbers have neither factor. There are $\binom{30}{2} = 435$ pairs. A pair containing a multiple of 6 always works: $435 - \binom{25}{2} = 435 - 300 = 135$ pairs. A pair with no multiple of 6 works only if one number is even but not a multiple of 3 and the other is an odd multiple of 3: $10 \cdot 5 = 50$ pairs. So $135 + 50 = 185$ pairs work, and the probability is $\frac{185}{435} = \frac{37}{87}$.

**Check.** Count the failures instead; a failing pair has no multiple of 6. It fails if both numbers lack a factor 3 (both from the $10 + 10 = 20$ such numbers: $\binom{20}{2} = 190$) or both are odd (both from the $5 + 10 = 15$ odd numbers: $\binom{15}{2} = 105$); both happen when both come from the 10 numbers with neither factor ($\binom{10}{2} = 45$). Failures: $190 + 105 - 45 = 250$, and $435 - 250 = 185$.

**17. Answer:** 4016**. The idea.** Two facts cut the search down to a handful of numbers. First, $S(n)$ is small, so $n$ must be close to 2027. Second, a number and its digit sum leave the same remainder on division by 9, so $n + S(n)$ behaves like $2n$ modulo 9, which pins down $n$ modulo 9.

**Work it out.** Any solution has $n < 2027$. Among the integers below 2027, the largest digit sum is $S(1999) = 28$, so $n = 2027 - S(n) \ge 2027 - 28 = 1999$. Next, $n \equiv S(n) \pmod 9$, so $2n \equiv 2027 \equiv 2 \pmod 9$ (since $2 + 0 + 2 + 7 = 11$ leaves remainder 2), and because 2 is invertible modulo 9, $n \equiv 1 \pmod 9$. The numbers from 1999 to 2026 that leave remainder 1 on division by 9 are $1999, 2008, 2017, 2026$. Test them: $1999 + 28 = 2027$ works; $2008 + 10 = 2018$ fails; $2017 + 10 = 2027$ works; $2026 + 10 = 2036$ fails. The solutions are 1999 and 2017, and their sum is $1999 + 2017 = 4016$.

**Check.** A direct look at the other numbers from 2000 to 2026: for $n = 20ab$ we need $2000 + 10a + b + 2 + a + b = 2027$, that is $11a + 2b = 25$, whose only digit solution is $a = 1, b = 7$, giving 2017. For $n = 19ab$ with $n \ge 1999$ only 1999 is possible, and it works. So exactly 1999 and 2017, total 4016.

**18. Answer:** $9\sqrt{3} - \frac{9\pi}{2}$**. The idea.** Each watered region is a circle of radius 3, exactly half the side. Two neighbouring circles touch at the midpoint of the side between their centres and do not overlap, and each circle stays well away from the opposite side, so inside the lawn each sprinkler waters exactly a $60^\circ$ sector.

**Work it out.** Two corners are 6 apart and each circle has radius 3, so the circles meet only at the midpoint of that side: no overlap. The distance from a corner to the opposite side is the height $3\sqrt{3} \approx 5.2 > 3$, so the circle around a corner does not reach the opposite side; inside the triangle it is exactly the sector cut out by the $60^\circ$ angle at that corner, with area $\frac{60}{360} \cdot 9\pi = \frac{3\pi}{2}$. The lawn has area $\frac{\sqrt{3}}{4} \cdot 36 = 9\sqrt{3}$. The dry part has area $9\sqrt{3} - 3 \cdot \frac{3\pi}{2} = 9\sqrt{3} - \frac{9\pi}{2}$.

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**Check.** The three sectors together are 180° of a circle of radius 3, i.e. half a circle, $\frac{9\pi}{2} \approx 14.14$, which is less than the lawn's $9\sqrt{3} \approx 15.59$; the dry curvy triangle in the middle has area about 1.45, a sensible small positive number.

**19. Answer:** 38**. The idea.** A mean is a total divided by a count, so work with totals. If the list has $n$ numbers, each removal statement gives the removed number in terms of $n$, and the known gap between them then pins down $n$.

**Work it out.** Let the list have $n$ numbers, so its total is $20n$. Removing the largest number $L$ leaves $n-1$ numbers with total $18(n-1)$, so $L = 20n - 18(n-1) = 2n + 18$. Removing the smallest number $s$ leaves total $22(n-1)$, so $s = 20n - 22(n-1) = 22 - 2n$. Then $L - s = 4n - 4 = 36$, so $n = 10$, and $L = 2 \cdot 10 + 18 = 38$ (and $s = 2$).

**Check.** Such a list exists: 2, 38 and eight 20's have total 200 and mean 20. Without the 38: $\frac{162}{9} = 18$. Without the 2: $\frac{198}{9} = 22$. And $38 - 2 = 36$.

**20. Answer:** $\frac{8}{19}$**. The idea.** The distance is $\left|\frac{p}{q} - \frac{3}{7}\right| = \frac{|7p-3q|}{7q}$. The top is a positive integer, so it is at least 1; the best candidates make it exactly 1 with $q$ as large as possible.

**Work it out.** Any fraction with $|7p - 3q| \geq 2$ is at least $\frac{2}{7q} \geq \frac{2}{140} = \frac{1}{70}$ away. With $|7p - 3q| = 1$ the distance is $\frac{1}{7q}$, smaller for larger $q$. We need $3q \equiv \pm 1 \pmod 7$, that is $q \equiv 5$ or $q \equiv 2 \pmod 7$. The largest such $q \leq 20$ is $q = 19$ (since $3 \cdot 19 = 57 = 7 \cdot 8 + 1$), giving $p = 8$ and distance $\frac{1}{133}$. Every other fraction is farther: the next candidate $q = 16$ ($7 \cdot 7 - 48 = 1$) gives $\frac{7}{16}$ at distance $\frac{1}{112}$, and fractions with $|7p - 3q| \geq 2$ are at least $\frac{1}{70}$ away. So the answer is $\frac{8}{19}$.

**Check.** $\frac{3}{7} - \frac{8}{19} = \frac{57-56}{133} = \frac{1}{133}$, and $\frac{1}{133} < \frac{1}{112} < \frac{1}{70}$. Note $q = 20$ does not help: $3 \cdot 20 = 60 = 56 + 4$, so the nearest fraction with denominator 20 is off by at least $\frac{3}{140}$.

**21. Answer:** $\frac{9}{91}$**. The idea.** Five squares spread over four rows with none empty must fill the rows in the pattern 2, 1, 1, 1, and the same holds for the columns. So build a good set row by row: first the one row that holds two squares, then the other three rows, which must between them reach the two columns still missing.

**Work it out.** There are $\binom{16}{5} = 4368$ sets. For a good set, choose the row with two squares (4 ways) and its two columns ($\binom{4}{2} = 6$ ways). Each of the other three rows gets exactly one square, and together they must cover the two columns not yet used. Of the $4^3 = 64$ ways to place those three squares, $3^3 = 27$ miss the first of the missing columns and 27 miss the second, while $2^3 = 8$ miss both, so by inclusion–exclusion $64 - 27 - 27 + 8 = 18$ ways cover both. That gives $4 \cdot 6 \cdot 18 = 432$ good sets, and the probability is $\frac{432}{4368} = \frac{9}{91}$.

**Check.** Count again by where the doubled row $R$ and the doubled column $C$ cross. If the square at $(R, C)$ is chosen, removing it leaves one square in each row and each column, a rook arrangement ($4! = 24$ of them), and the extra square is any of the other 12: 288 sets. If it is not chosen, pick $R$ and $C$ (16 ways), the two other squares in row $R$ (3 ways) and in column $C$ (3 ways); the one row and one column still empty must then meet in the fifth square: $16 \cdot 9 = 144$ sets. Total $288 + 144 = 432$.

**22. Answer:** 129**. The idea.** Count the points with $|x|+|y|+|z|$ equal to exactly $k$, for $k = 0, 1, 2, 3, 4$. For a fixed $k$, sort the points by how many coordinates are zero: the nonzero coordinates can be chosen positive first and then given signs.

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**Work it out.** For $k \geq 1$: points with exactly one nonzero coordinate: choose it (3 ways) and its sign (2): 6 points. Exactly two nonzero: choose which two (3 ways), split $k$ into two positive parts ($k-1$ ways), and choose signs (4): $12(k-1)$. All three nonzero: split $k$ into three positive parts ($\binom{k-1}{2}$ ways) and choose signs (8): $8\binom{k-1}{2}$. So the number with sum exactly $k$ is $6 + 12(k-1) + 4(k-1)(k-2) = 4k^2 + 2$. For $k = 1, 2, 3, 4$ this gives $6, 18, 38, 66$. Adding the origin: $1 + 6 + 18 + 38 + 66 = 129$.

**Check.** Slice by $z$ instead. The plane layer at height $z$ is the square $|x| + |y| \leq 4 - |z|$, which holds $2m^2 + 2m + 1$ lattice points when $m = 4 - |z|$: $41, 25, 13, 5, 1$ for $m = 4, 3, 2, 1, 0$. Total $41 + 2(25 + 13 + 5 + 1) = 41 + 88 = 129$.

**23. Answer: $\frac{7}{2}$. The idea.** By symmetry the centre lies on the perpendicular bisector of $AB$, and tangency to $CD$ means the radius equals the centre's distance to line $CD$. That gives one equation for the centre. Then the chord on line $AD$ is found from the centre's height.

**Work it out.** Put $A = (0, 0)$, $B = (12, 0)$, $C = (12, 8)$, $D = (0, 8)$. The centre is $(6, k)$ and the radius is $8 - k$ (the distance to the line $y = 8$). Since $A$ is on the circle, $6^2 + k^2 = (8-k)^2$, so $36 = 64 - 16k$ and $k = \frac{7}{4}$, radius $\frac{25}{4}$. The line $AD$ is $x = 0$; the centre is at height $\frac{7}{4}$, and the chord of a circle is bisected by the perpendicular from the centre, so the two points of the circle on $x = 0$ are symmetric about $y = \frac{7}{4}$. One of them is $A$ at $y = 0$, so the other is at $y = \frac{7}{2}$, which lies on side $AD$ since $\frac{7}{2} < 8$. So $AE = \frac{7}{2}$.

**Check.** $E = (0, \frac{7}{2})$ is at distance $\sqrt{36 + (\frac{7}{4})^2} = \sqrt{\frac{576+49}{16}} = \frac{25}{4}$ from the centre. Also by power of the point $D$: $DE \cdot DA$ should equal the square of the tangent length from $D$ to the tangency point $(6, 8)$: $\frac{9}{2} \cdot 8 = 36 = 6^2$.

**24. Answer:** 118**. The idea.** Every positive integer below $1024 = 2^{10}$ uses at most 10 binary places, so numbers below 1024 with three 1's correspond exactly to choices of 3 of the 10 places. Then remove the few that are larger than 800.

**Work it out.** Numbers below 1024 with exactly three 1's: $\binom{10}{3} = 120$. In base 2, $800 = 512 + 256 + 32 = 1100100000_2$. A number with three 1's exceeds 800 only if it uses both 512 and 256 (without them the number is at most $512 + 128 + 64 = 704$). With $512 + 256 = 768$, the third 1 adds a power of two, and the total exceeds 800 only when that power is more than 32: 64 or 128, giving 832 and 896. ($768 + 32 = 800$ itself is allowed.) So the count is $120 - 2 = 118$.

**Check.** Count upward by the largest 1: if it is at place value $2^j$ ($j = 2, \ldots, 8$), the other two 1's come from $j$ lower places, giving $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{8}{2} = \binom{9}{3} = 84$ numbers below 512; with the top 1 at 512, the other two can be any pair from the 9 lower places (36 ways) except the two pairs $\{256, 64\}$ and $\{256, 128\}$: 34. Total $84 + 34 = 118$.

**25. Answer: $\frac{3}{8}$. The idea.** Slice through the cone's axis. The cross-section is an isosceles triangle, and the ball's cross-section is a circle tangent to all three sides of it: the ball's radius is the inradius of that triangle, which is area divided by semiperimeter.

**Work it out.** The axial cross-section is a triangle with base 12 and height 8, so its equal sides are $\sqrt{6^2 + 8^2} = 10$. Its area is $\frac{1}{2} \cdot 12 \cdot 8 = 48$ and its semiperimeter is $\frac{10+10+12}{2} = 16$, so the inradius is $\frac{48}{16} = 3$. The ball has volume $\frac{4}{3}\pi \cdot 27 = 36\pi$ and the cone has volume $\frac{1}{3}\pi \cdot 36 \cdot 8 = 96\pi$. The fraction is $\frac{36\pi}{96\pi} = \frac{3}{8}$.

**Check.** The ball's centre is at height 3 on the axis. The apex is at height 8, so the centre is 5 from

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the apex; the slant side makes a right triangle with the axis with sides $6, 8, 10$, so the distance from a point on the axis 5 below the apex to the slant side is $5 \cdot \frac{6}{10} = 3$, equal to the radius, as it should be.

**26. Answer: $\frac{38}{5}$. The idea.** All that matters is the length $k$ of the longest block of all-different numbers ending with the latest roll. If the next roll is new to that block, the block grows to $k+1$; if it equals the number $j$ rolls back (for $j = 1, \ldots, k$), the new block is just the last $j$ rolls, of length $j$. Write $e_k$ for the expected number of further rolls from a block of length $k$ and solve the small system.

**Work it out.** From $k = 1$: a repeat (probability $\frac{1}{6}$) keeps $k = 1$, anything else makes $k = 2$, so $e_1 = 1 + \frac{1}{6}e_1 + \frac{5}{6}e_2$, i.e. $e_1 = e_2 + \frac{6}{5}$. From $k = 2$: $e_2 = 1 + \frac{1}{6}e_1 + \frac{1}{6}e_2 + \frac{4}{6}e_3$. Substituting $e_1 = e_2 + \frac{6}{5}$ gives $\frac{2}{3}e_2 = \frac{6}{5} + \frac{2}{3}e_3$, so $e_2 = e_3 + \frac{9}{5}$. From $k = 3$ a new number (probability $\frac{3}{6}$) ends the game: $e_3 = 1 + \frac{1}{6}(e_1 + e_2 + e_3)$. With $e_2 = e_3 + \frac{9}{5}$ and $e_1 = e_3 + 3$ this reads $e_3 = 1 + \frac{1}{6}\left(3e_3 + \frac{24}{5}\right) = \frac{9}{5} + \frac{1}{2}e_3$, so $e_3 = \frac{18}{5}$, $e_2 = \frac{27}{5}$ and $e_1 = \frac{33}{5}$. The first roll always makes a block of length 1, so the expected total is $1 + \frac{33}{5} = \frac{38}{5}$.

**Check.** Put the values back into the $k = 3$ equation: $1 + \frac{1}{6} \cdot \frac{33+27+18}{5} = 1 + \frac{13}{5} = \frac{18}{5} = e_3$. The same method for three different numbers in a row gives $f_1 = f_2 + \frac{6}{5}$ and $f_2 = 1 + \frac{1}{6}(f_1 + f_2)$, so $f_2 = \frac{9}{5}$, $f_1 = 3$ and a total of 4; the four-number game should take longer, and $\frac{38}{5} = 7.6 > 4$. It is also at least the 4 rolls any game needs.

**27. Answer: $2\sqrt{95}$. The idea.** A tetrahedron whose opposite edges are equal fits inside a rectangular box with its six edges as diagonals of the six faces of the box: take four corners of the box, no two joined by a box edge. Then the tetrahedron is the box with four congruent corner pieces cut off.

**Work it out.** Let the box be $a \times b \times c$. The three pairs of opposite face diagonals must have lengths 5, 6 and 7: $a^2 + b^2 = 25$, $b^2 + c^2 = 36$, $a^2 + c^2 = 49$. Adding, $a^2 + b^2 + c^2 = 55$, so $c^2 = 30$, $a^2 = 19$, $b^2 = 6$, all positive, so the box exists. Each cut-off corner is a right-corner pyramid with volume $\frac{1}{6}abc$, so the tetrahedron has volume $abc - 4 \cdot \frac{abc}{6} = \frac{abc}{3}$. Here $abc = \sqrt{19 \cdot 6 \cdot 30} = \sqrt{3420} = 6\sqrt{95}$, so the volume is $2\sqrt{95}$.

**Check.** By the Cayley–Menger determinant for the six given edges, $288V^2 = 109440$, so $V^2 = 380 = 4 \cdot 95$ and $V = 2\sqrt{95} \approx 19.5$, which is sensibly less than the volume of a regular tetrahedron with edge 7 (about 40.4).

**28. Answer: $\frac{1}{4}$. The idea.** The points closer to a marked point $V$ than to another marked point $Q$ form a half-plane bounded by the perpendicular bisector of $VQ$. So the region belonging to vertex $A$ is cut out by a few bisectors, and only the nearby marked points matter: the midpoints of the two sides at $A$, and the centre $O$.

**Work it out.** Let the side be 4. The midpoints $M$ and $N$ of sides $AB$ and $AF$ are 2 from $A$, so the bisectors of $AM$ and $AN$ are the lines perpendicular to $AB$ and $AF$ at distance 1 from $A$. These two lines are symmetric about line $AO$, and they meet on $AO$ at distance $\frac{1}{\cos 60^\circ} = 2$ from $A$, since each side makes a $60^\circ$ angle with $AO$. The bisector of $AO$ is perpendicular to $AO$ at distance $\frac{4}{2} = 2$ from $A$ (it is the line $BF$), so it passes through the same meeting point and cuts nothing extra off. So $A$'s region is the kite with vertices $A$, the two feet at distance 1 along $AB$ and $AF$, and the point 2 from $A$ on $AO$. It is two right triangles with legs 1 and $\sqrt{3}$ (the other leg is $1 \cdot \tan 60^\circ$), so its area is $\sqrt{3}$. No other marked point cuts into this kite: every point $X$ of the kite is within 2 of

$A$, while every other marked point $Q$ is at least 4 from $A$, so $XQ \geq AQ - AX \geq 4 - 2 \geq AX$. Six vertices give $6\sqrt{3}$, while the hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 16 = 24\sqrt{3}$. The probability is $\frac{6\sqrt{3}}{24\sqrt{3}} = \frac{1}{4}$.

**Check.** The centre's region: the bisectors toward the midpoints are at distance $\sqrt{3}$ (half the apothem $2\sqrt{3}$), forming a hexagon of half the size, area $\frac{1}{4}$ of the whole; its corners are at distance 2 from $O$ toward the vertices, exactly on the bisectors toward the vertices, so the vertices cut nothing off. That leaves $1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$ for the midpoints, a clean split $1 : 1 : 2$ consistent with the three kinds of region fitting together.

**29. Answer:** 588**. The idea.** Doubling or tripling a number never changes its part that is coprime to 6. So split 1 to 12 into families $m \cdot 2^a 3^b$ with the same $m$ (coprime to 6); conflicts only happen inside a family, and the families can be handled separately and multiplied.

**Work it out.** The families are: $m = 1$: $\{1, 2, 3, 4, 6, 8, 9, 12\}$; $m = 5$: $\{5, 10\}$; $m = 7$: $\{7\}$; $m = 11$: $\{11\}$. The family $\{5, 10\}$ allows 3 choices (not both), and $\{7\}$, $\{11\}$ allow 2 each. The big family is a staircase grid, with $2^a 3^b$ in row $b$, column $a$: row 0 is $1, 2, 4, 8$; row 1 is $3, 6, 12$; row 2 is 9. Conflicts are grid neighbours: 1-2, 2-4, 4-8, 3-6, 6-12 (doubles) and 1-3, 2-6, 4-12, 3-9 (triples). Sort by the choice in row 1 (no two neighbours among $3, 6, 12$: $\emptyset$, $\{3\}$, $\{6\}$, $\{12\}$, $\{3, 12\}$). Row 0 must avoid the numbers directly below (a third of) the chosen ones, and 9 is allowed unless 3 is chosen. A path of $k$ free numbers with no two neighbours has $2, 3, 5, 8$ choices for $k = 1, 2, 3, 4$.

$\emptyset$: row 0 free, 8 ways, times 2 for 9: 16. $\{3\}$: row 0 avoids 1, path $2, 4, 8$: 5; 9 banned: 5. $\{6\}$: avoid 2: $\{1\}$ and path $4, 8$: $2 \cdot 3 = 6$, times 2: 12. $\{12\}$: avoid 4: path $1, 2$ and $\{8\}$: $3 \cdot 2 = 6$, times 2: 12. $\{3, 12\}$: avoid 1 and 4: $\{2\}$, $\{8\}$: 4; 9 banned: 4. The big family allows $16+5+12+12+4 = 49$ choices. Total: $49 \cdot 3 \cdot 2 \cdot 2 = 588$.

**Check.** Ignoring the triple rule, the big family would split into the chains $1, 2, 4, 8$ (8 choices), $3, 6, 12$ (5) and 9 (2), i.e. 80 choices; the triple rule should cut this down, and $49 < 80$. As a small test, for $\{1, \ldots, 4\}$ the method gives family $\{1, 2, 3, 4\}$ with conflicts 1-2, 2-4, 1-3; listing by hand: $\emptyset$, four singletons, $\{1, 4\}$, $\{2, 3\}$, $\{3, 4\}$, and no triple: 8, which the row method also gives ($\emptyset$ in row 1: path $1, 2, 4$ gives 5; $\{3\}$: path $2, 4$ gives 3).

**30. Answer:** 2652**. The idea.** Split the vertices into two groups of four, $P$ and $Q$, so that every edge joins a $P$ vertex to a $Q$ vertex (alternate corners, like the two interlocking tetrahedra in a cube). Each $Q$ vertex is joined to exactly three $P$ vertices: all of them except the one diagonally opposite. So colour $P$ first; then each $Q$ vertex independently has $4 - m$ choices, where $m$ is the number of different colours on its three $P$ neighbours.

**Work it out.** Sort the colourings of $P$ by how the four vertices share colours.

All one colour: 4 colourings of $P$; every $Q$ vertex sees 1 colour, $3^4 = 81$ ways: 324.

Three alike and one different: $4 \cdot 3 \cdot 4 = 48$ colourings; the $Q$ vertex opposite the odd one sees 1 colour (3 choices) and the other three see 2 (2 each): 24 ways, total 1152.

Two and two: the partner of one fixed vertex can be chosen in 3 ways, and the two pairs get two different colours in order, $4 \cdot 3 = 12$ ways: 36 colourings; every $Q$ vertex sees both colours, $2^4 = 16$ ways: 576.

Two alike, two others different: $\binom{4}{2} = 6$ places for the pair, 4 colours for it and $3 \cdot 2$ for the two singles: 144 colourings. A $Q$ vertex opposite a pair vertex sees 3 colours (1 choice); one opposite a single sees 2 (2 choices): 4 ways, total 576.

All four different: 24 colourings; each $Q$ vertex sees 3 colours: 1 way, total 24.

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Sum: $324 + 1152 + 576 + 576 + 24 = 2652$.

**Check.** The colourings of $P$ counted add up to $4 + 48 + 36 + 144 + 24 = 256 = 4^4$, as they must. The same method with 3 colours gives $3 \cdot 16 + 24 \cdot 2 + 18 \cdot 1 + 36 \cdot 0 = 114$, and with 2 colours gives 2 (the checkerboard colourings), both of which a direct count confirms.

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