MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A straight line passes through the points $(0, 4)$ and $(6, 0)$. It crosses the line $y = x$ at exactly one point. What is the x-coordinate of that point? Express your answer as a common fraction. 1. _____

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2. Two opposite vertices of a square are $(1, 2)$ and $(7, 10)$. What is the area of the square, in square units? 2. _____

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3. A bag holds 3 red, 4 blue and 5 green marbles. Without looking, you take marbles out of the bag one at a time and do not put them back. What is the fewest number of marbles you must take out to be certain that you have at least 2 green marbles? 3. _____

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4. Lena writes the positive integers in order but skips every multiple of 3: $1, 2, 4, 5, 7, 8, 10, 11, \ldots$. What is the 100th number she writes? 4. _____

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5. A class is collecting money for a gift. If every student pays \$4, they are \$10 short of the price. If every student pays \$5 instead, they have \$15 more than they need. What is the price of the gift, in dollars? 5. _____

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6. A spinner has three equal sections numbered 1, 2 and 3. It is spun three times. What is the probability that the three numbers spun add up to 6? Express your answer as a common fraction. 6. _____

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7. Cube A has a space diagonal (the segment from one corner, through the inside of the cube, to the opposite corner) of length 6. Cube B has a face diagonal (a diagonal of one of its square faces) of length 6. What is the ratio of the surface area of cube A to the surface area of cube B? Express your answer as a common fraction. 7. _____

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8. Right now, the ages of a mother and her two children add up to 64 years. In 6 years, the mother's age will equal the sum of her two children's ages at that time. How old is the mother now? 8. _____

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9. A parking garage charges \$2.00 for the first hour or any part of it, and \$1.50 for each additional hour or any part of an additional hour. Omar paid \$9.50 to park there. What is the greatest number of minutes he could have parked? 9. _____

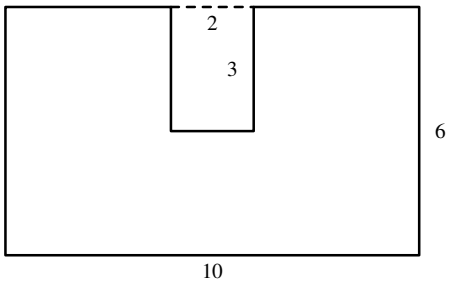
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

10. A positive divisor of 360 is chosen. It must be a multiple of 4, but it must not be a multiple of 8. How many different divisors of 360 could be chosen?

10. _____

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11. A rectangle measures 10 units by 6 units. A rectangular notch 2 units wide and 3 units deep is cut out of the middle of one of the 10-unit sides, so the notch's 2-unit side lies along that side of the rectangle. What is the perimeter of the remaining shape, in units?



11. _____

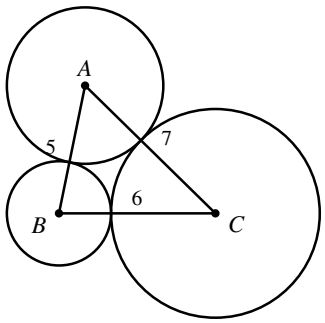
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12. The Fibonacci numbers begin $1, 1, 2, 3, 5, 8, 13, \ldots$; the first two are 1, and each one after that is the sum of the two before it. What is the units digit of the 100th Fibonacci number?

12. _____

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13. Triangle ABC has $AB = 5$, $BC = 6$ and $CA = 7$. Three circles are drawn, centered at A, B and C, so that each circle is tangent to the other two and no circle lies inside another. What is the radius of the smallest of the three circles?



13. _____

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14. In an arithmetic sequence, the 3rd term is 17 and the 10th term is 52. How many of the first 100 terms of the sequence are multiples of 7?

14. _____

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15. Four friends each choose one day of the week at random, independently, with all seven days equally likely for each friend. What is the probability that exactly three different days are chosen (so two of the friends choose the same day and the other two choose two other days, different from each other)? Express your answer as a common fraction.

15. _____

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16. Partway through its season, a soccer team had won exactly 40% of the games it had played. It then won 8 of its next 10 games, after which it had won exactly 50% of all the games it had played. How many games had the team played before those 10 games?

16. _____

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17. Two fair six-sided dice, one red and one blue, are rolled. The number on the red die becomes the tens digit and the number on the blue die the units digit of a two-digit number; for example, red 4 and blue 1 give 41. What is the probability that this two-digit number is prime? Express your answer as a common fraction.

17. _____

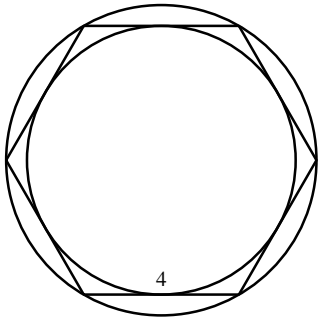
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18. In some three-digit numbers the middle digit is the average of the first and last digits, for example 135, 777 and 840. How many three-digit numbers have this property?

18. _____

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19. A regular hexagon has side length 4. One circle passes through all six of its vertices, and a smaller circle with the same center touches all six of its sides. What is the area of the ring-shaped region between the two circles? Express your answer in terms of π.



19. _____

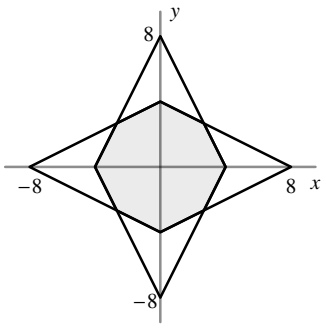
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20. A faulty car odometer never shows the digits 3 or 7. In every place it skips straight from 2 to 4 and from 6 to 8, so after 2 it shows 4, after 6 it shows 8, after 29 it shows 40, and so on; otherwise it counts normally. It started at 0 and now reads 1250. How many miles has the car actually travelled?

20. _____

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21. In the coordinate plane, consider all the points (x, y) that satisfy both $|x| + 2|y| \leq 8$ and $2|x| + |y| \leq 8$. These points fill a region. What is the area of that region, in square units? Express your answer as a common fraction.

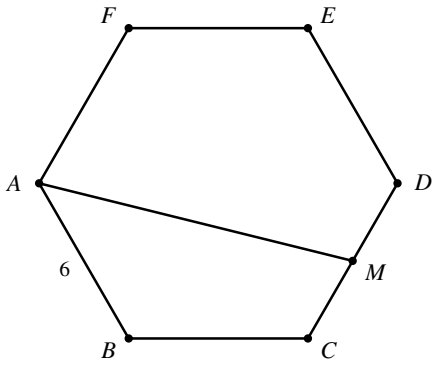


21. _____

22. In a sequence of numbers, the first term is 3, and each term from the third term on equals the sum of ALL the terms before it. The tenth term is 1536. What is the sum of the first ten terms?

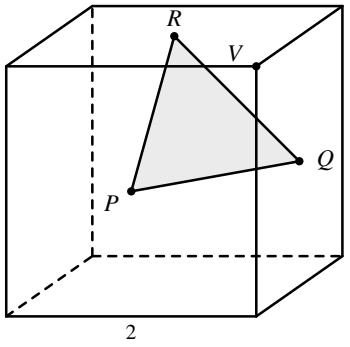
22. _____

23. Regular hexagon $ABCDEF$ has side length 6, with its vertices labelled in order around it. Point M is the midpoint of side $\overline{CD}$. What is the length of $\overline{AM}$? Express your answer in simplest radical form.



23. _____

24. A cube has edge length 2. Let V be one corner of the cube, and let P, Q and R be the centers of the three faces of the cube that meet at V. What is the distance from V to the plane that passes through P, Q and R? Express your answer in simplest radical form.



24. _____

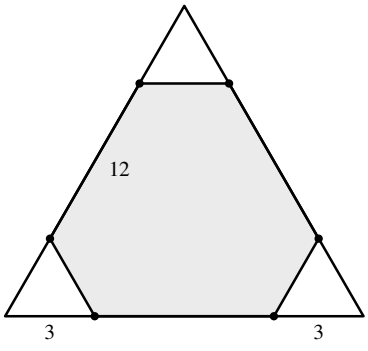
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25. Three identical red beads and three identical blue beads are placed at six equally spaced points around a circle, one bead at each point. Two arrangements are considered the same if one can be rotated into the other; arrangements that are mirror images of each other but cannot be rotated into each other count as different. How many different arrangements are there?

25. _____

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26. An equilateral triangle has side length 12. On each side, mark the two points that are 3 units from the two ends of that side, giving six points in all. These six points are the corners of a hexagon. What is the area of the hexagon? Express your answer in simplest radical form.



26. _____

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27. For how many integers n from 1 to 1000, inclusive, is the least common multiple of n and 24 equal to the least common multiple of n and 36?

27. _____

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28. A real number x is greater than 1 and satisfies $x^2 + \frac{1}{x^2} = 14$. What is the value of $x^3 - \frac{1}{x^3}$? Express your answer in simplest radical form.

28. _____

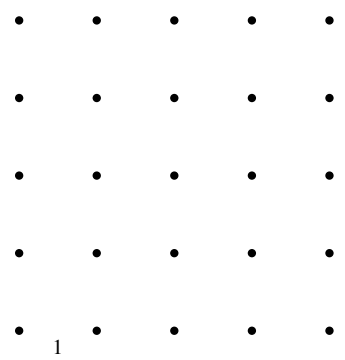
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29. Consider all the positive divisors of 540 that are even numbers. What is the sum of their reciprocals? (For example, 2 is one such divisor and contributes $\frac{1}{2}$.) Express your answer as a common fraction.

29. _____

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30. A 5×5 square array of 25 dots is drawn, with neighbouring dots 1 unit apart in each row and column. How many squares, of any size and in any position (tilted or not), have all four of their corners at dots of the array?



30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 8

1. $\frac{12}{5}$

2. 50

3. 9

4. 149

5. 110

6. $\frac{7}{27}$

7. $\frac{2}{3}$

8. 35

9. 360

10. 6

11. 38

12. 5

13. 2

14. 15

15. $\frac{180}{343}$

16. 30

17. $\frac{2}{9}$

18. 45

19. 4π

20. 672

21. $\frac{128}{3}$

22. 3072

23. $3\sqrt{13}$

24. $\frac{2\sqrt{3}}{3}$

25. 4

26. $\frac{117\sqrt{3}}{4}$

27. 13

28. $30\sqrt{3}$

29. $\frac{4}{3}$

30. 50

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Solutions — MathCounts Chapter Sprint Practice Contest 8

1. **Answer:** $\frac{12}{5}$. **The idea.** A line that cuts the axes at $(6, 0)$ and $(0, 4)$ can be written straight away in intercept form, $\frac{x}{6} + \frac{y}{4} = 1$. On the line $y = x$ the two coordinates are equal, so put $y = x$ and solve one easy equation.

   Work it out. Check the intercept form: $(6, 0)$ gives $1 + 0 = 1$ and $(0, 4)$ gives $0 + 1 = 1$. Setting $y = x$: $\frac{x}{6} + \frac{x}{4} = 1$, that is, $\frac{2x+3x}{12} = 1$, so $5x = 12$ and $x = \frac{12}{5}$.

   Check. In slope form the line is $y = 4 - \frac{2}{3}x$. At $x = \frac{12}{5}$ this gives $y = 4 - \frac{8}{5} = \frac{12}{5}$, equal to x, as it should be.

2. **Answer:** 50. **The idea.** There is no need to find the other two vertices. The two given points are the ends of a diagonal, and a square can be cut along a diagonal d into two right isosceles triangles, so its area is $\frac{d^2}{2}$.

   Work it out. The diagonal runs 6 across and 8 up, so $d^2 = 6^2 + 8^2 = 100$ and $d = 10$. The area is $\frac{100}{2} = 50$.

   Check. The centre is $(4, 6)$, and the half-diagonal $(3, 4)$ turned a quarter turn is $(-4, 3)$, so the other vertices are $(0, 9)$ and $(8, 3)$. The side from $(1, 2)$ to $(8, 3)$ has squared length $7^2 + 1^2 = 50$, which is the area.

3. **Answer:** 9. **The idea.** "Certain" means it must work even with the worst possible luck. The worst luck is to pull out every marble that is not green before the greens start coming.

   Work it out. With 8 marbles you could have all 3 red, all 4 blue and just 1 green, so 8 is not enough. With 9 marbles, at most $3 + 4 = 7$ of them can be red or blue, so at least $9 - 7 = 2$ are green. The answer is 9.

   Check. A tempting wrong answer is $3 + 4 + 1 = 8$ (forgetting that the second green is needed) or $2 + 4 = 6$ (planning only for the larger non-green colour). The worst case must include every non-green marble, 7 of them, plus 2 greens.

4. **Answer:** 149. **The idea.** Group the integers into blocks of three: $\{1, 2, 3\}, \{4, 5, 6\}, \ldots$. Each block gives exactly two numbers to Lena's list (the first two of the block), so the list moves in steps of two per block.

   Work it out. Since $100 = 2 \cdot 50$, the 100th number is the second number Lena keeps from the 50th block. The 50th block is $\{148, 149, 150\}$, and she keeps 148 and 149. So the 100th number is 149.

   Check. Up to 149 there are 149 integers, of which $\lfloor 149/3 \rfloor = 49$ are multiples of 3, leaving $149 - 49 = 100$ kept numbers, and 149 itself is kept.

5. **Answer:** 110. **The idea.** Going from \$4 to \$5 each, every student adds one dollar. Those extra dollars turn a \$10 shortfall into a \$15 surplus, a swing of $10 + 15 = 25$ dollars, so the number of extra dollars, and hence the number of students, is 25.

   Work it out. There are 25 students. At \$4 each they collect \$100, which is \$10 short, so the gift costs $100 + 10 = 110$ dollars.

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Check. At \$5 each the 25 students collect \$125, which is $125 - 110 = 15$ dollars more than needed. In algebra: $4n + 10 = 5n - 15$ gives $n = 25$.

6. Answer: $\frac{7}{27}$. **The idea.** There are $3^3 = 27$ equally likely ordered results. List the collections of three numbers from $\{1, 2, 3\}$ that add to 6, then count how many orders each one can come in.

Work it out. If a 3 appears, the other two add to 3, so they are 1 and 2: the collection $\{1, 2, 3\}$, which can come in $3! = 6$ orders. (Two 3s would leave 0, impossible.) If no 3 appears, all three numbers are 1 or 2 and add to 6, which forces 2, 2, 2: one order. That is $6 + 1 = 7$ outcomes, and the probability is $\frac{7}{27}$.

Check. The average spin is 2, so 6 is the most likely total. Counting totals $3, 4, 5, 6, 7, 8, 9$ gives $1, 3, 6, 7, 6, 3, 1$ outcomes, which add to 27; the middle one is 7.

7. Answer: $\frac{2}{3}$. **The idea.** For a cube of edge e, a face diagonal has length $e\sqrt{2}$ and a space diagonal has length $e\sqrt{3}$ (the space diagonal is the hypotenuse of a right triangle with legs e and $e\sqrt{2}$). Surface area is $6e^2$, so the ratio of surface areas is just the ratio of the values of e^2.

Work it out. For cube A: $3e_A^2 = 36$, so $e_A^2 = 12$. For cube B: $2e_B^2 = 36$, so $e_B^2 = 18$. The ratio of surface areas is $\frac{6 \cdot 12}{6 \cdot 18} = \frac{12}{18} = \frac{2}{3}$.

Check. The surface areas are 72 and 108, and $\frac{72}{108} = \frac{2}{3}$. It makes sense that cube A is the smaller one: the same length 6 is its longest diagonal but only a face diagonal of cube B.

8. Answer: 35. **The idea.** Each year the mother gets 1 year older, but the two children together get 2 years older. So every year the children's total gains 1 year on the mother. In 6 years they must have caught up exactly, so right now the mother is 6 years ahead of the children's total.

Work it out. Let c be the sum of the children's ages now. The mother is $c + 6$, so $(c + 6) + c = 64$, giving $2c = 58$, $c = 29$, and the mother is $29 + 6 = 35$.

Check. Now: $35 + 29 = 64$. In 6 years the mother is 41 and the children's total is $29 + 2 \cdot 6 = 41$. (Equation form: $m + 6 = c + 12$ and $m + c = 64$.) The tempting slip is to add only 6 to the children's total, which would make the two ages equal now.

9. Answer: 360. **The idea.** Every hour that has been started is charged in full. So first find how many hours Omar was charged for, then take the longest stay that is charged exactly that many hours.

Work it out. After the first hour's \$2.00, the remaining \$7.50 pays for $\frac{7.50}{1.50} = 5$ additional hours, so Omar was charged for 6 hours. Any stay longer than 5 hours and no longer than 6 hours costs exactly this, so the longest possible stay is 6 hours, or 360 minutes.

Check. A stay of 361 minutes starts a seventh hour and costs $2.00 + 6 \cdot 1.50 = \$11.00$, while 300 minutes costs only $2.00 + 4 \cdot 1.50 = \$8.00$.

10. Answer: 6. **The idea.** Write $360 = 2^3 \cdot 3^2 \cdot 5$. Every divisor is $2^a \cdot 3^b \cdot 5^c$ with $0 \le a \le 3$, $0 \le b \le 2$, $0 \le c \le 1$. Being a multiple of 4 but not of 8 means the power of 2 is exactly 2^2.

Work it out. Fix $a = 2$. Then b has 3 choices and c has 2 choices, so there are $3 \cdot 2 = 6$ such divisors.

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Check. They are $4 \cdot 1, 4 \cdot 3, 4 \cdot 5, 4 \cdot 9, 4 \cdot 15, 4 \cdot 45$, that is, $4, 12, 20, 36, 60, 180$. Each divides 360, each is a multiple of 4, and none is a multiple of 8 (each is 4 times an odd number).

11. Answer: 38**. The idea.** Walk around the new outline. Along the notched side, the 2 units of edge that were removed are replaced by the notch's bottom, which also has length 2, so the horizontal edges add up to the same total as before. What is new are the notch's two walls, each 3 units long.

Work it out. The rectangle's perimeter is $2(10 + 6) = 32$. The notch keeps every horizontal and vertical total the same except that it adds its two vertical walls, $3 + 3 = 6$. The new perimeter is $32 + 6 = 38$.

Check. List the edges going around: along the notched side 4, down the wall 3, across the notch 2, up the wall 3, then 4 more; then 6, 10 and 6. Total $4 + 3 + 2 + 3 + 4 + 6 + 10 + 6 = 38$. (Cutting the same piece from a corner would not change the perimeter at all, which is why a corner cut is a trap and a middle cut is not.)

12. Answer: 5**. The idea.** The units digit is decided by the remainders on division by 2 and by 5. Each of those remainders repeats in a short cycle, because each remainder depends only on the two remainders before it.

Work it out. Mod 2 the sequence goes odd, odd, even, odd, odd, even, . . .: a Fibonacci number is even exactly when its position is a multiple of 3. Since 100 is not a multiple of 3, the 100th Fibonacci number is odd.

Mod 5 the remainders are $1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0$, and then $1, 1$ again, so the pattern repeats every 20 terms. A 0 appears at every position that is a multiple of 5. Since 100 is a multiple of 5, the 100th Fibonacci number is a multiple of 5.

An odd multiple of 5 ends in 5.

Check. The small cases fit: $F_5 = 5$, $F_{10} = 55$, $F_{20} = 6765$ and $F_{25} = 75025$ end in 5, while $F_{15} = 610$ and $F_{30} = 832040$, whose positions are multiples of 3 as well, end in 0.

13. Answer: 2**. The idea.** When two circles touch from the outside, the distance between their centers is the sum of their radii. So each side of the triangle is the sum of two of the three radii, which gives three equations; adding them all is the quick way to solve.

Work it out. Let the radii be a, b and c for the circles at A, B and C. Then $a + b = 5$, $b + c = 6$ and $c + a = 7$. Adding: $2(a + b + c) = 18$, so $a + b + c = 9$. Subtracting each equation from this: $c = 9 - 5 = 4$, $a = 9 - 6 = 3$ and $b = 9 - 7 = 2$. The smallest radius is 2, for the circle centered at B.

Check. $3 + 2 = 5 = AB$, $2 + 4 = 6 = BC$ and $4 + 3 = 7 = CA$, and all three radii are positive. The circle at B, where the two shortest sides meet, is the smallest, as it should be: it is the vertex opposite the longest side.

14. Answer: 15**. The idea.** First pin the sequence down: going from the 3rd term to the 10th term takes 7 steps of the common difference. Then look at the terms' remainders on division by 7, which repeat in a cycle of length 7.

Work it out. The 7 steps add $52 - 17 = 35$, so the common difference is 5, and the 1st term is $17 - 2 \cdot 5 = 7$. The nth term is $7 + 5(n - 1)$. This is a multiple of 7 exactly when $5(n - 1)$ is, and

since 5 and 7 share no factor, exactly when $n-1$ is a multiple of 7. So $n = 1, 8, 15, \ldots, 99$, which is $\frac{99-1}{7} + 1 = 15$ terms.

Check. The terms for $n = 1, 8, 15$ are $7, 42, 77$, all multiples of 7, and they go up by 35 each time; the terms in between (such as $12, 17, 22, 27, 32, 37$) are not. The last one, for $n = 99$, is $7 + 5 \cdot 98 = 497 = 7 \cdot 71$.

15. Answer: $\frac{180}{343}$**. The idea.** Count the ordered choices, out of $7^4 = 2401$. Build a good outcome in steps: first decide which two friends share a day, then hand out days to the three "groups" (the pair and the two single friends), all different.

Work it out. The pair of friends who match can be chosen in $\binom{4}{2} = 6$ ways. Then the pair's shared day can be any of 7, the next friend's day any of the other 6, and the last friend's any of the remaining 5. That gives $6 \cdot 7 \cdot 6 \cdot 5 = 1260$ outcomes, and the probability is $\frac{1260}{2401} = \frac{180}{343}$ (divide top and bottom by 7).

Check. Sort all 2401 outcomes by how many different days appear. All four different: $7 \cdot 6 \cdot 5 \cdot 4 = 840$. One day for everyone: 7. Exactly two days: choose the two days (21 ways) and give each friend one of them, not all the same ($2^4 - 2 = 14$), for 294. Then $840 + 7 + 294 + 1260 = 2401$, as it must.

16. Answer: 30**. The idea.** Call the earlier number of games n. The number of wins can be counted two ways after the 10 new games: as the old wins plus 8, and as half of all $n + 10$ games. Setting these equal gives one equation for n.

Work it out. Before the run the team had $0.4n$ wins. Afterwards it has $0.4n + 8$ wins out of $n + 10$ games, and this is half: $0.4n + 8 = 0.5(n + 10) = 0.5n + 5$. So $0.1n = 3$ and $n = 30$.

Check. 40% of 30 is 12 wins. After the run: $12 + 8 = 20$ wins in $30 + 10 = 40$ games, exactly 50%. (A quick feel for it: the 10 games went 80% instead of 50%, a surplus of 3 wins, and that surplus has to make up the 10-point shortfall on the first n games, so $0.1n = 3$.)

17. Answer: $\frac{2}{9}$**. The idea.** All 36 outcomes are equally likely and give 36 different numbers. A prime greater than 10 is odd and does not end in 5, so the blue die must show 1 or 3. That leaves only 12 numbers to test.

Work it out. Units digit 1: $11, 21, 31, 41, 51, 61$; of these $21 = 3 \cdot 7$ and $51 = 3 \cdot 17$ are not prime, leaving 4 primes. Units digit 3: $13, 23, 33, 43, 53, 63$; of these $33 = 3 \cdot 11$ and $63 = 7 \cdot 9$ are not prime, leaving 4 primes. So 8 of the 36 outcomes work, and the probability is $\frac{8}{36} = \frac{2}{9}$.

Check. The eight primes are $11, 13, 23, 31, 41, 43, 53, 61$. The numbers ruled out at the start are all even (blue $2, 4, 6$) or multiples of 5 (blue 5), so none of them could have been prime.

18. Answer: 45**. The idea.** Once the first and last digits are chosen, the middle digit is forced to be their average, and it is automatically a single digit between them. So the only condition is that the average be a whole number, that is, that the first and last digits add to an even number.

Work it out. Let the first digit be a (1 to 9) and the last be c (0 to 9); we need $a + c$ even, so a and c have the same parity. Both odd: 5 choices for a and 5 for c, giving 25. Both even: $a \in \{2, 4, 6, 8\}$ (4 choices, since $a \neq 0$) and $c \in \{0, 2, 4, 6, 8\}$ (5 choices), giving 20. Total $25 + 20 = 45$.

Check. Count by the middle digit b instead: we need $a + c = 2b$. For $b = 0$: only 000, which is not allowed, so 0. For $b = 1, \ldots, 4$ the pairs (a, c) with $a \geq 1$ number $2b$: $2 + 4 + 6 + 8 = 20$. For

$b = 5, \ldots, 9$ they number $2(9-b)+1$: $9+7+5+3+1 = 25$. Total 45.

19. Answer: 4π. **The idea.** The hexagon is made of six equilateral triangles of side 4 meeting at the center. The big circle's radius is the distance from the center to a vertex, and the small circle's radius is the distance from the center to the midpoint of a side, which is the height of one of those triangles.

Work it out. The big radius is 4. The small radius is the height of an equilateral triangle of side 4: $\sqrt{4^2-2^2} = \sqrt{12} = 2\sqrt{3}$. The ring's area is $\pi \cdot 4^2 - \pi \cdot (2\sqrt{3})^2 = 16\pi - 12\pi = 4\pi$.

Check. The center, a vertex and the midpoint of a side next to it form a right triangle with legs r and 2 and hypotenuse R, so $R^2 - r^2 = 2^2 = 4$ and the ring's area is 4π directly: it only depends on half the side length.

20. Answer: 672. **The idea.** Each place of the odometer uses only eight symbols, 0, 1, 2, 4, 5, 6, 8, 9, so the odometer is really counting in base 8 with unusual names for the digits.

Work it out. Translate each symbol to its true base-8 value: 0, 1, 2 stay, 4 means 3, 5 means 4, 6 means 5, 8 means 6, 9 means 7. The reading 1250 becomes the base-8 number 1240_8, which is $1 \cdot 512 + 2 \cdot 64 + 4 \cdot 8 + 0 = 512 + 128 + 32 = 672$ miles.

Check. Count the readings from 0 to 999 directly: each of the three places has 8 choices, so $8^3 = 512$ readings, meaning the display reaches 1000 after exactly 512 miles. From 1000 to 1250 it has the same pattern as 000 to 250, which is $2 \cdot 64 + 4 \cdot 8 = 160$ more miles (there the tens symbol 5 stands for the base-8 digit 4). Total 672.

21. Answer: $\frac{128}{3}$. **The idea.** Replacing x by $-x$ or y by $-y$ changes nothing, so the region is symmetric across both axes: find the part in the first quadrant and multiply by 4. In the first quadrant the absolute values disappear and each condition is just a half-plane below a line.

Work it out. For $x, y \geq 0$ the conditions are $x + 2y \leq 8$ and $2x + y \leq 8$. The first line meets the axes at $(8,0)$ and $(0,4)$, the second at $(4,0)$ and $(0,8)$, so near each axis the stricter line wins: the region reaches $(4,0)$ on the x-axis and $(0,4)$ on the y-axis. The lines cross where $x+2y = 2x+y$, i.e. $x = y$, and then $3x = 8$: the point $\left(\frac{8}{3}, \frac{8}{3}\right)$. So the first-quadrant piece is the quadrilateral with corners $(0,0)$, $(4,0)$, $\left(\frac{8}{3}, \frac{8}{3}\right)$, $(0,4)$. Split it along the diagonal from $(0,0)$ to $\left(\frac{8}{3}, \frac{8}{3}\right)$ into two triangles, each with base 4 on an axis and height $\frac{8}{3}$: area $2 \cdot \frac{1}{2} \cdot 4 \cdot \frac{8}{3} = \frac{32}{3}$. The whole region has area $4 \cdot \frac{32}{3} = \frac{128}{3}$.

Check. Each diamond alone has area half the product of its diagonals: $\frac{16 \cdot 8}{2} = 64$. The overlap must be less than 64, and it contains the square $|x| + |y| \leq 4$ of area 32; indeed $32 < \frac{128}{3} \approx 42.7 < 64$. Also, the region is an octagon with vertices $(\pm 4, 0)$, $(0, \pm 4)$, $\left(\pm\frac{8}{3}, \pm\frac{8}{3}\right)$; the shoelace formula on the first-quadrant piece gives $\frac{1}{2}\left(4 \cdot \frac{8}{3} + \frac{8}{3} \cdot 4\right) = \frac{32}{3}$ again.

22. Answer: 3072. **The idea.** Compare two neighbouring terms from the third term on. The next term is the sum of everything so far, which is (the sum of everything before the current term) plus the current term, that is, the current term plus itself. So from the third term on, each term is double the one before, and the sum of the first ten terms is simply the eleventh term.

Work it out. Let the second term be x. The third term is $3 + x$, and from there the terms double: $3+x$, $2(3+x)$, $4(3+x)$, ..., so the tenth term is $2^7(3+x) = 128(3+x)$. Then $128(3+x) = 1536$

gives $3 + x = 12$, so $x = 9$. The sum of the first ten terms is, by the rule, the eleventh term, which is $2 \cdot 1536 = 3072$.

Check. The terms are $3, 9, 12, 24, 48, 96, 192, 384, 768, 1536$. The first two add to $3 + 9 = 12$, and the other eight are $12 \cdot 1, 12 \cdot 2, \ldots, 12 \cdot 128$, which add to $12 \cdot 255$. The total is $12 + 12 \cdot 255 = 12 \cdot 256 = 3072$.

23. Answer: $3\sqrt{13}$. **The idea.** Look for a right angle. The vertices of a regular hexagon lie on a circle, and $\overline{AD}$ joins opposite vertices, so it is a diameter. An angle drawn in a semicircle is a right angle, so $\angle ACD = 90°$, and AM is the hypotenuse of right triangle ACM.

Work it out. The hexagon splits into six equilateral triangles of side 6, so its circle has radius 6 and $AD = 12$. In right triangle ACD, $CD = 6$ and $AD = 12$, so $AC = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}$. With $CM = 3$, $AM = \sqrt{AC^2 + CM^2} = \sqrt{108 + 9} = \sqrt{117} = 3\sqrt{13}$.

Check. Coordinates: centre at the origin, $A = (6, 0)$, $C = (-3, 3\sqrt{3})$, $D = (-6, 0)$. Then $M = \left(-\frac{9}{2}, \frac{3\sqrt{3}}{2}\right)$ and $AM^2 = \left(\frac{21}{2}\right)^2 + \frac{27}{4} = \frac{441+27}{4} = 117$, so $AM = 3\sqrt{13} \approx 10.8$, a bit less than the diameter 12, as it should be.

24. Answer: $\frac{2\sqrt{3}}{3}$. **The idea.** Everything is symmetric about the space diagonal of the cube that starts at V: turning the cube a third of a turn about that diagonal permutes P, Q, R. So the point of the plane nearest to V lies on that diagonal, and it must be the centre (centroid) of triangle PQR.

Work it out. Put V at the origin with the cube filling $0 \le x, y, z \le 2$. The three faces through V are $x = 0$, $y = 0$, $z = 0$, with centres $P = (0, 1, 1)$, $Q = (1, 0, 1)$, $R = (1, 1, 0)$. Their centroid is $G = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, which lies on the diagonal $x = y = z$. So the distance is $VG = \sqrt{3 \cdot \frac{4}{9}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check. Use volume instead. The points P, Q, R all satisfy $x + y + z = 2$, so that is the plane. It cuts off from the corner V a pyramid whose three edges along the axes have length 2 (it meets the axes at $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$), so its volume is $\frac{1}{3} \cdot \frac{1}{2} \cdot 2 \cdot 2 \cdot 2 = \frac{4}{3}$. Its slanted face is an equilateral triangle of side $2\sqrt{2}$, with area $\frac{\sqrt{3}}{4} \cdot 8 = 2\sqrt{3}$. The height from V to that face is $\frac{3 \cdot \frac{4}{3}}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, the same.

25. Answer: 4. **The idea.** First count the arrangements with the six points held fixed, then group them into families that are rotations of each other. A family has 6 members unless the pattern looks the same after a smaller turn, so the only work is to find those special patterns.

Work it out. With the points fixed, choose which 3 of the 6 points get red beads: $\binom{6}{3} = 20$ arrangements. Rotating an arrangement by $1, 2, \ldots, 5$ steps gives 6 versions. They are all different unless the pattern repeats after fewer than 6 steps. A pattern that repeats every 2 steps is $RBRBRB$, whose family has only 2 members ($RBRBRB$ and $BRBRBR$). A pattern that repeats every 3 steps would contain the same three-bead block twice, so it would have an even number of red beads, which is impossible with 3. So the other $20 - 2 = 18$ arrangements fall into families of 6: that is 3 families. In all there are $3 + 1 = 4$ arrangements.

Check. List them by how the reds sit around the ring: all three together ($RRRBBB$); all three apart ($RBRBRB$); and two together with one alone, which comes in two versions, $RRBRBB$ and $RRBBRB$. Going clockwise from the pair, the lone red is either 1 or 2 blue beads later; these two are mirror images, but no rotation turns one into the other, so they both count. That gives 4.

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26. Answer: $\frac{117\sqrt{3}}{4}$**. The idea.** The hexagon is the big triangle with its three corners cut off. At each corner the cut-off piece has two sides of length 3 meeting at a 60° angle, so it is itself an equilateral triangle, of side 3.

Work it out. An equilateral triangle of side s has area $\frac{\sqrt{3}}{4}s^2$. The big triangle has area $\frac{\sqrt{3}}{4} \cdot 144 = 36\sqrt{3}$ and each corner piece has area $\frac{\sqrt{3}}{4} \cdot 9 = \frac{9\sqrt{3}}{4}$. The hexagon has area $36\sqrt{3} - 3 \cdot \frac{9\sqrt{3}}{4} = \frac{144\sqrt{3}-27\sqrt{3}}{4} = \frac{117\sqrt{3}}{4}$.

Check. Count in small triangles of side 3: the big triangle (side $12 = 4 \cdot 3$) is made of $4^2 = 16$ of them, and the hexagon is what remains after removing 3, so it is $\frac{13}{16}$ of $36\sqrt{3}$, which is $\frac{117\sqrt{3}}{4}$. The shoelace formula on the six points gives the same value.

27. Answer: 13**. The idea.** A least common multiple is built one prime at a time: for each prime it takes the larger of the two exponents. Since $24 = 2^3 \cdot 3$ and $36 = 2^2 \cdot 3^2$, the two lcms can only differ in their powers of 2 and 3, so compare those separately.

Work it out. Write $n = 2^x \cdot 3^y \cdot m$ with m not divisible by 2 or 3. Both lcms contain exactly m from the other primes. Power of 2: $\max(x, 3)$ against $\max(x, 2)$. These agree exactly when $x \geq 3$ (if $x \leq 2$ they are 3 and 2). Power of 3: $\max(y, 1)$ against $\max(y, 2)$, which agree exactly when $y \geq 2$ (if $y \leq 1$ they are 1 and 2). So the condition is exactly that n is a multiple of $2^3 \cdot 3^2 = 72$. From 1 to 1000 there are $\left\lfloor \frac{1000}{72} \right\rfloor = 13$ of them ($72 \cdot 13 = 936$ and $72 \cdot 14 = 1008$).

Check. For $n = 72$: $\mathrm{lcm}(72, 24) = 72 = \mathrm{lcm}(72, 36)$. For $n = 36$ (only 2^2): $\mathrm{lcm}(36, 24) = 72$ but $\mathrm{lcm}(36, 36) = 36$. For $n = 24$ (only 3^1): $\mathrm{lcm}(24, 24) = 24$ but $\mathrm{lcm}(24, 36) = 72$. So both conditions are really needed.

28. Answer: $30\sqrt{3}$**. The idea.** Do not solve for x. Squaring $x - \frac{1}{x}$ produces $x^2 + \frac{1}{x^2}$ (the middle term is a constant, because $x \cdot \frac{1}{x} = 1$), and $x^3 - \frac{1}{x^3}$ factors as a difference of cubes with $x - \frac{1}{x}$ as one factor.

Work it out. $\left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2} = 14 - 2 = 12$. Since $x > 1$, we have $\frac{1}{x} < 1 < x$, so $x - \frac{1}{x}$ is positive and equals $\sqrt{12} = 2\sqrt{3}$. By the difference of cubes, $x^3 - \frac{1}{x^3} = \left(x - \frac{1}{x}\right)\left(x^2 + x \cdot \frac{1}{x} + \frac{1}{x^2}\right) = 2\sqrt{3}\,(14 + 1) = 30\sqrt{3}$.

Check. Find x after all: $\left(x + \frac{1}{x}\right)^2 = 14 + 2 = 16$, so $x + \frac{1}{x} = 4$; together with $x - \frac{1}{x} = 2\sqrt{3}$ this gives $x = 2 + \sqrt{3}$ and $\frac{1}{x} = 2 - \sqrt{3}$ (indeed their product is $4 - 3 = 1$). Then $x^3 = (2 + \sqrt{3})^3 = 8 + 12\sqrt{3} + 18 + 3\sqrt{3} = 26 + 15\sqrt{3}$ and $\frac{1}{x^3} = 26 - 15\sqrt{3}$; the difference is $30\sqrt{3}$.

29. Answer: $\frac{4}{3}$**. The idea.** Adding reciprocals of divisors is easy if you pair each divisor d of n with its partner n/d: then $\sum \frac{1}{d} = \sum \frac{n/d}{n} = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of the divisors. First turn "even divisors of 540" into "all divisors of something".

Work it out. An even divisor of 540 is $2m$ where m divides 270, and every divisor m of 270 gives one. So the sum is $\sum_{m|270} \frac{1}{2m} = \frac{1}{2} \cdot \frac{\sigma(270)}{270}$. Since $270 = 2 \cdot 3^3 \cdot 5$, $\sigma(270) = (1 + 2)(1 + 3 + 9 + 27)(1 + 5) = 3 \cdot 40 \cdot 6 = 720$. The sum is $\frac{1}{2} \cdot \frac{720}{270} = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}$.

Check. Do it the other way: all divisors of $540 = 2^2 \cdot 3^3 \cdot 5$ give $\frac{\sigma(540)}{540} = \frac{7 \cdot 40 \cdot 6}{540} = \frac{28}{9}$; the odd divisors are the divisors of 135, giving $\frac{\sigma(135)}{135} = \frac{40 \cdot 6}{135} = \frac{16}{9}$. The even ones give $\frac{28}{9} - \frac{16}{9} = \frac{12}{9} = \frac{4}{3}$.

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30. Answer: 50**. The idea.** Every such square sits snugly inside an upright "frame": the smallest upright square around it, whose four sides each contain one corner of the tilted square. So count frames, and count the squares inside each frame.

Work it out. Take a square whose side goes p units across and q units up from one corner to the next, with $p \geq 1$ and $q \geq 0$. Its frame is an upright square of side $k = p + q$, and each corner of the square sits on a side of the frame, q units from a corner of the frame. For a frame of side k there are k choices, $q = 0, 1, \ldots, k-1$ ($q = 0$ is the frame itself). An upright $k \times k$ frame fits in the array in $(5-k)^2$ positions. The total is $\sum_{k=1}^{4} k(5-k)^2 = 1 \cdot 16 + 2 \cdot 9 + 3 \cdot 4 + 4 \cdot 1 = 50$.

Check. The upright squares alone number $16+9+4+1 = 30$, and the tilted ones number $1 \cdot 9 = 9$ in 2×2 frames, $2 \cdot 4 = 8$ in 3×3 frames and $3 \cdot 1 = 3$ in the 4×4 frame, 20 in all; $30 + 20 = 50$. A computer check over all pairs of dots finds the same 50 squares.

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