MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Kai lives far from school. On days when he walks to school and rides the bus home, his two trips take 90 minutes in all. On days when he rides the bus both ways, his two trips take 30 minutes in all. The bus trip takes the same time in each direction, and so does the walk. How many minutes in all would his two trips take on a day when he walks both ways?

1. _____

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2. A perfect square is the square of an integer, such as $49 = 7^2$. How many perfect squares are greater than 1000 and less than 2026?

2. _____

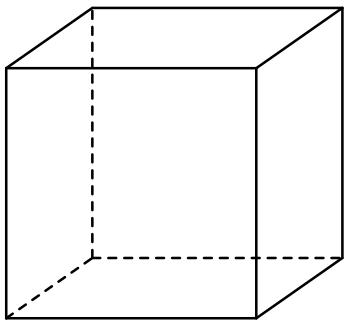
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3. One angle of an isosceles triangle measures $40°$. Depending on where that angle is, the largest angle of the triangle can have more than one possible measure. What is the sum, in degrees, of all the different possible measures of the largest angle?

3. _____

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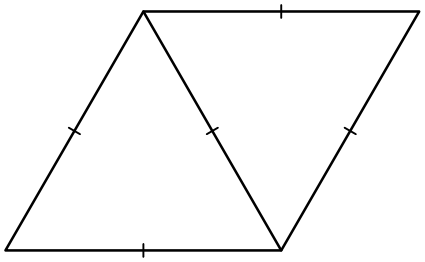
4. Two edges of a cube are called skew if they do not share a vertex and are not parallel. (Two different edges that point in the same direction, even on opposite sides of the cube, count as parallel.) How many unordered pairs of edges of a cube are skew?



4. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

5. A rhombus has perimeter 24, and one of its two diagonals is exactly as long as a side of the rhombus. What is the area of the rhombus? Express your answer in simplest radical form.



5. _____

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6. Ten friends share one pizza, taking turns. The first friend eats $\frac{1}{3}$ of the pizza. The second eats $\frac{1}{4}$ of what is left. The third eats $\frac{1}{5}$ of what is then left, and so on: each friend eats a fraction of the remaining pizza whose denominator is one more than the previous friend's. The ninth friend eats $\frac{1}{11}$ of what remains, and the tenth friend eats everything that is left. What fraction of the whole pizza does the tenth friend eat? Express your answer as a common fraction.

6. _____

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7. Mia picks three consecutive positive integers, squares each of them, and adds the three squares. Her total is 770. What is the sum of the three integers she picked?

7. _____

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8. A composite number is a positive integer greater than 1 that is not prime. What is the smallest positive integer n such that all five of the numbers n, $n+1$, $n+2$, $n+3$ and $n+4$ are composite?

8. _____

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9. The odd numbers $1, 3, 5, 7, \ldots$ are written in order in rows: row 1 holds one number, row 2 holds the next two numbers, row 3 the next three, and so on, so that row n holds n numbers. Row 1 is 1, row 2 is $3, 5$, row 3 is $7, 9, 11$, and row 4 is $13, 15, 17, 19$. What is the sum of the numbers in row 20?

9. _____

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10. How many positive integers less than 1000 are written using only the digits 1, 2 and 3 (each digit may be used any number of times, or not at all) and are divisible by 3?

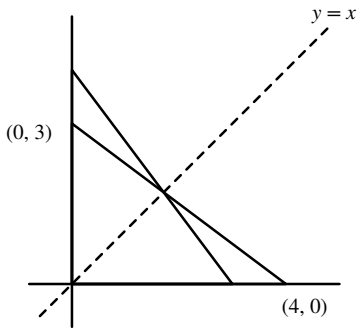
10. _____

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11. Tess lists the positive multiples of 11 in order, $11, 22, 33, \ldots$, and adds up the digits of each one. For example, 11 has digit sum $1 + 1 = 2$. What is the first multiple on her list whose digits add up to exactly 11? 11. _____

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12. In the coordinate plane, the triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 3)$ is reflected across the line $y = x$. What is the area of the region covered by the original triangle, its reflection, or both? Express your answer as a common fraction.



12. _____

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13. What is the smallest integer greater than 1000 that can be written as $2^a \cdot 3^b$, where a and b are both positive integers? 13. _____

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14. In a 5-kilometre race, Ana runs at a steady pace of 4 minutes per kilometre. Bea starts 2 minutes after Ana, from the same starting line, and runs at a steady pace of $3\frac{1}{2}$ minutes per kilometre. How many kilometres from the start is Bea when she catches up with Ana? 14. _____

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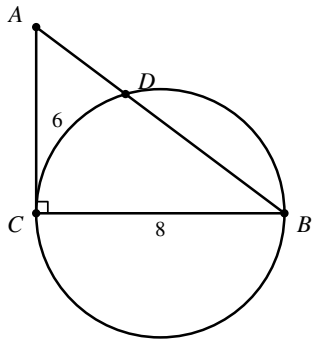
15. The numbers $1, 2, 3, 4, 5$ are written in a row in a random order, with all 120 orders equally likely. What is the probability that 1 appears somewhere before 2, 2 appears somewhere before 3, AND 4 appears somewhere before 5? Express your answer as a common fraction. 15. _____

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16. A train makes the same trip every day at a steady speed, and the trip takes 2 hours. If the train went 20 km/h faster, the trip would take 30 minutes less. How many minutes would the trip take if the train instead went 20 km/h slower than its usual speed? 16. _____

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17. Right triangle ABC has its right angle at C, with $CA = 6$ and $CB = 8$. The circle that has segment $\overline{CB}$ as a diameter passes through B and crosses the hypotenuse $\overline{AB}$ at one other point, D. What is the length AD? Express your answer as a common fraction.



17. _____

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18. Lena writes down every whole number from 1 to 400, inclusive, and then adds up all the individual digits she wrote. (For example, the number 237 contributes $2 + 3 + 7 = 12$.) What total does she get?

18. _____

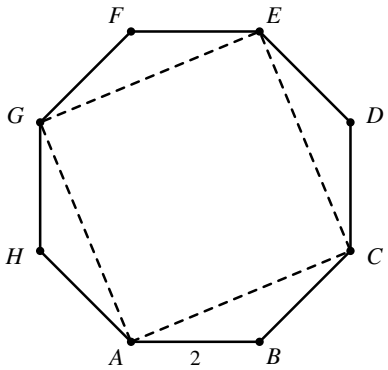
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19. Real numbers a, b, c and d satisfy $4^a = 5$, $5^b = 6$, $6^c = 7$ and $7^d = 8$. What is the value of the product $abcd$? Express your answer as a common fraction.

19. _____

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20. A regular octagon $ABCDEFGH$ has side length 2. Joining every other vertex, A, C, E and G, forms a square inside the octagon. What is the area of square $ACEG$? Express your answer in simplest radical form.



20. _____

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21. Let $f(x) = \dfrac{x+3}{x-1}$. Start with the number 7 and apply f over and over: 7, then $f(7)$, then $f(f(7))$, and so on. What number do you have after applying f exactly 2025 times? Express your answer as a common fraction.

21. _____

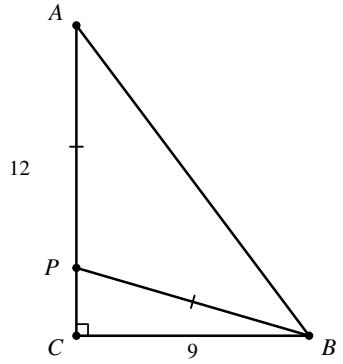
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22. A 10×10 multiplication table has rows and columns numbered 1 to 10, and the entry in row r and column c is the product rc. The main diagonal holds the 10 entries with $r = c$. What is the sum of the 45 entries that lie below the main diagonal (those with $r > c$)?

22. _____

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23. Right triangle ABC has its right angle at C, with $BC = 9$ and $AC = 12$. Point P lies on leg $\overline{AC}$ and is the same distance from A as it is from B. What is the length CP? Express your answer as a common fraction.



23. _____

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24. In some three-digit numbers exactly two of the three digits are equal, for example 110, 424 and 899 (but not 777 or 123). How many even three-digit numbers have exactly two equal digits?

24. _____

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25. Some positive integers can be written as the sum of two different powers of 2 and also as the sum of two different powers of 3; for example, $10 = 8 + 2 = 9 + 1$. (The powers of 2 are $1, 2, 4, 8, \ldots$ and the powers of 3 are $1, 3, 9, 27, \ldots$.) What is the sum of all such integers less than 1000?

25. _____

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26. Seven people, A, B, C, D, E, F and G, sit around a round table with seven seats. Two seatings count as the same if one is a rotation of the other (but a mirror image counts as different). A and B insist on sitting in adjacent seats, and C and D refuse to sit in adjacent seats. How many different seatings are possible?

26. _____

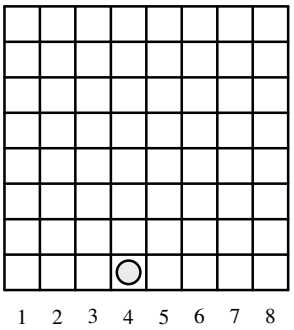
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27. Triangle ABC has $AB = 25$, $AC = 26$ and $BC = 17$. Three points are marked on side $\overline{BC}$: the foot H of the altitude from A, the point D where the bisector of $\angle BAC$ meets $\overline{BC}$, and the midpoint M of $\overline{BC}$. What is the value of the ratio $\dfrac{HD}{DM}$?

27. _____

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28. The columns of an 8×8 board are numbered 1 to 8 from left to right. A checker starts on the bottom row in column 4. Each move takes it one square diagonally upward: up one row and either one column left or one column right. It may never leave the board. After 7 moves it is on the top row. How many different sequences of 7 moves can it make?



28. _____

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29. A positive integer N has at least six positive divisors. Its three smallest positive divisors add up to 9, and its three largest positive divisors add up to 1035. What is N?

29. _____

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30. For some positive integers n, the number $n^2 + 20n - 1924$ is a perfect square (for example, $n = 35$ gives $1 = 1^2$). Find the sum of all positive integers n for which $n^2 + 20n - 1924$ is a perfect square.

30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 7

1. 150

2. 14

3. 170

4. 24

5. $18\sqrt{3}$

6. $\frac{2}{11}$

7. 48

8. 24

9. 8000

10. 13

11. 209

12. $\frac{48}{7}$

13. 1152

14. 4

15. $\frac{1}{12}$

16. 180

17. $\frac{18}{5}$

18. 4204

19. $\frac{3}{2}$

20. $8 + 4\sqrt{2}$

21. $\frac{5}{3}$

22. 1320

23. $\frac{21}{8}$

24. 118

25. 58

26. 144

27. 8

28. 103

29. 675

30. 824

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Solutions — MathCounts Chapter Sprint Practice Contest 7

1. **Answer:** 150**. The idea.** The bus-both-ways day tells you what one bus trip costs. Remove it from the mixed day to find one walk, then double.

   Work it out. Two bus trips take 30 minutes, so one bus trip takes 15 minutes. On the mixed day, the walk takes $90 - 15 = 75$ minutes. Walking both ways takes $2 \cdot 75 = 150$ minutes.

   Check. Compare the days: switching one bus trip to a walk adds $90-30 = 60$ minutes. Switching the second bus trip as well adds another 60, so walking both ways takes $90 + 60 = 150$ minutes. The tempting slip is $90 \cdot 2 = 180$ or $90 + 30 = 120$, which ignore that the mixed day already contains one walk.

2. **Answer:** 14**. The idea.** Do not list the squares; find the smallest and largest numbers whose squares land in the range, and count the integers between them.

   Work it out. $31^2 = 961 < 1000$ but $32^2 = 1024 > 1000$, so the smallest square in range is 32^2. At the top, $45^2 = 2025 < 2026$ but $46^2 = 2116$, so the largest is 45^2. The squares are $32^2, 33^2, \ldots, 45^2$, and there are $45 - 32 + 1 = 14$ of them.

   Check. Watch the top end: $2025 = 45^2$ is just below 2026, so it counts. At the bottom, 961 is just below 1000, so it does not. Forgetting the +1 when counting from 32 to 45 gives the wrong answer 13.

3. **Answer:** 170**. The idea.** In an isosceles triangle, a given angle is either the angle between the two equal sides (the apex) or one of the two equal base angles. Try both.

   Work it out. If 40° is the apex, the base angles are each $\frac{180-40}{2} = 70^\circ$, and the largest angle is 70°. If 40° is a base angle, the other base angle is also 40° and the apex is $180 - 80 = 100^\circ$, which is the largest. The possible largest angles are 70° and 100°, with sum 170.

   Check. The triangles are $40, 70, 70$ and $40, 40, 100$; both add to 180° and both have two equal angles. There is no third case: the 40° angle is either one of the equal pair or the odd one out.

4. **Answer:** 24**. The idea.** Every pair of the 12 edges is exactly one of three kinds: meeting at a vertex, parallel, or skew. Count the first two kinds, which are easy, and subtract.

   Work it out. There are $\binom{12}{2} = 66$ pairs of edges.

   Meeting pairs: each of the 8 vertices has 3 edges, giving $\binom{3}{2} = 3$ pairs meeting there, and two different edges share at most one vertex, so there are $8 \cdot 3 = 24$ meeting pairs.

   Parallel pairs: the edges point in 3 directions, with 4 edges in each direction, giving $3 \cdot \binom{4}{2} = 18$ parallel pairs.

   Skew pairs: $66 - 24 - 18 = 24$.

   Check. Count directly from one edge, say a vertical one. It meets 4 edges (two at each end) and is parallel to 3, so it is skew to $11 - 4 - 3 = 4$ edges. Each of the 12 edges is skew to 4 others, and each pair is counted twice: $\frac{12 \cdot 4}{2} = 24$.

5. **Answer:** $18\sqrt{3}$**. The idea.** A diagonal cuts a rhombus into two triangles whose other two sides are sides of the rhombus. If that diagonal equals a side, both triangles are equilateral.

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Work it out. Each side is $24 \div 4 = 6$. The diagonal of length 6 splits the rhombus into two triangles with sides 6, 6, 6. An equilateral triangle of side 6 has height $3\sqrt{3}$ and area $\frac{1}{2} \cdot 6 \cdot 3\sqrt{3} = 9\sqrt{3}$. The rhombus has area $2 \cdot 9\sqrt{3} = 18\sqrt{3}$.

Check. The other diagonal is twice the height of an equilateral triangle: $6\sqrt{3}$. The area of a rhombus is half the product of its diagonals: $\frac{1}{2} \cdot 6 \cdot 6\sqrt{3} = 18\sqrt{3}$. (The equal diagonal must be the shorter one: the angles are 60° and 120°, and the longer diagonal $6\sqrt{3}$ is longer than a side.)

6. Answer: $\frac{2}{11}$**. The idea.** Track what is LEFT, not what is eaten. Eating $\frac{1}{k}$ of what remains leaves $\frac{k-1}{k}$ of it, so the leftover is a product of fractions, and that product telescopes.

Work it out. After the first friend, $\frac{2}{3}$ is left. After the second, $\frac{2}{3} \cdot \frac{3}{4}$. After the ninth friend, the amount left is

$$\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \frac{10}{11}.$$

Each numerator cancels the previous denominator, leaving $\frac{2}{11}$. The tenth friend eats all of it: $\frac{2}{11}$.

Check. Follow the first few steps: after two friends $\frac{2}{3} \cdot \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$ is left, and after three $\frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$. The pattern is $\frac{2}{k+2}$ left after k friends, and $k = 9$ gives $\frac{2}{11}$. Surprisingly, the last friend gets more than the second, third, or any other friend except the first.

7. Answer: 48**. The idea.** Name the MIDDLE number n, so the three are $n-1$, n, $n+1$. The cross terms then cancel and the equation becomes simple.

Work it out. $(n-1)^2 + n^2 + (n+1)^2 = (n^2 - 2n + 1) + n^2 + (n^2 + 2n + 1) = 3n^2 + 2$. Setting $3n^2 + 2 = 770$ gives $3n^2 = 768$, $n^2 = 256$ and $n = 16$ (positive). The numbers are 15, 16, 17, and their sum is 48.

Check. $225 + 256 + 289 = 770$. Also, $770 \div 3 \approx 257$ is the average square, so the middle number is about $\sqrt{256} = 16$, as found.

8. Answer: 24**. The idea.** Five composites in a row sit strictly between two consecutive primes, so look for the first pair of consecutive primes that differ by at least 6.

Work it out. The primes begin $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$ with gaps $1, 2, 2, 4, 2, 4, 2, 4, 6, \ldots$. Every gap before 23 is at most 4, so no five consecutive numbers below 23 avoid the primes (and 1 is not composite either). The gap from 23 to 29 leaves $24, 25, 26, 27, 28$, all composite. So $n = 24$.

Check. $24 = 4 \cdot 6$, $25 = 5^2$, $26 = 2 \cdot 13$, $27 = 3^3$, $28 = 4 \cdot 7$. Any smaller start $n \le 23$ puts one of 19 or 23 (or an earlier prime) among the five numbers.

9. Answer: 8000**. The idea.** Add up the first few rows: 1, 3+5 = 8, 7+9+11 = 27, 13+15+17+19 = 64. These are the cubes $1^3, 2^3, 3^3, 4^3$. To see why, note that row n is centered on n^2 (the middle of row 3 is 9 and the middle of row 4 is 16), and n numbers spread evenly around n^2 add up to $n \cdot n^2$.

Work it out. Rows 1 to 19 use $1 + 2 + \cdots + 19 = 190$ odd numbers. So row 20 starts with the 191st odd number, $2 \cdot 191 - 1 = 381$, and ends with the 210th, 419. Its 20 numbers average $(381 + 419)/2 = 400 = 20^2$, so their sum is $20 \cdot 400 = 8000 = 20^3$.

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Check. Another route: rows 1 to 20 hold the first 210 odd numbers, which add up to 210^2, and rows 1 to 19 hold the first 190, which add up to 190^2. The difference is $210^2 - 190^2 = 400 \cdot 20 = 8000$.

10. Answer: 13**. The idea.** A number is divisible by 3 exactly when its digit sum is. The digits 1, 2, 3 leave the three different remainders 1, 2, 0 when divided by 3, so once all the other digits are chosen, exactly one choice of the last digit makes the digit sum a multiple of 3.

Work it out. One-digit numbers: only 3 works, so 1. Two-digit numbers: the first digit can be any of 3 choices and then the last digit is forced, so 3. Three-digit numbers: the first two digits have $3 \cdot 3 = 9$ choices and then the last digit is forced, so 9. In all, $1 + 3 + 9 = 13$.

Check. The two-digit ones are 12, 21 and 33. Of the 27 three-digit numbers made from these digits, exactly a third, 9, are multiples of 3, for example 111, 123, 132, 213 and 222.

11. Answer: 209**. The idea.** Multiples of 11 have a very regular digit pattern, and for a long stretch that pattern forces the digit sum to be even. Find where the pattern breaks.

Work it out. The two-digit multiples $11, 22, \ldots, 99$ have digit sums $2, 4, \ldots, 18$, all even. Next, $110, 121, 132, \ldots, 198$ are $11k$ for $k = 10$ to 18: their digits are $1, m, m-1$ with $m = k - 9$ (for example 143 and 198), so the digit sums are $2m = 2, 4, \ldots, 18$, again all even, so none equals 11. The next multiple is $209 = 11 \cdot 19$, with digit sum $2 + 0 + 9 = 11$.

Check. Why even? A multiple of 11 has its alternating digit sum divisible by 11; for these small numbers it is exactly 0, meaning (first digit + last digit) equals the middle digit, so the total digit sum is twice the middle digit. For 209 the alternating sum is $2 - 0 + 9 = 11$, not 0, which is what lets the digit sum be odd.

12. Answer: $\frac{48}{7}$**. The idea.** Area of the union = area of one + area of the other − area of the overlap. The two triangles are mirror images, so the overlap is symmetric about $y = x$ and its corner on that line is where the two slanted sides cross.

Work it out. Reflecting swaps coordinates, so the image has vertices $(0,0)$, $(0,4)$ and $(3,0)$. Each triangle has area $\frac{1}{2} \cdot 4 \cdot 3 = 6$.

The slanted sides are $3x + 4y = 12$ and $4x + 3y = 12$. They cross on $y = x$: $7x = 12$, so at $\left(\frac{12}{7}, \frac{12}{7}\right)$.

The overlap is the quadrilateral with vertices $(0,0)$, $(3,0)$, $\left(\frac{12}{7}, \frac{12}{7}\right)$, $(0,3)$. The diagonal from $(0,0)$ to the crossing point splits it into two triangles, each with base 3 along an axis and height $\frac{12}{7}$, so each has area $\frac{1}{2} \cdot 3 \cdot \frac{12}{7} = \frac{18}{7}$, and the overlap has area $\frac{36}{7}$.

The union has area $6 + 6 - \frac{36}{7} = \frac{48}{7}$.

Check. Add the pieces outside the overlap instead: the original triangle sticks out as the triangle $(3,0)$, $(4,0)$, $\left(\frac{12}{7}, \frac{12}{7}\right)$ with base 1 and height $\frac{12}{7}$, area $\frac{6}{7}$, and the image sticks out by the same amount. So the union is $\frac{36}{7} + 2 \cdot \frac{6}{7} = \frac{48}{7}$.

13. Answer: 1152**. The idea.** There are only a few possible powers of 3. For each one, find the smallest power of 2 that pushes the product past 1000, then compare.

Work it out. ($1024 = 2^{10}$ does not count, since b must be positive.)

$b = 1$: $3 \cdot 2^9 = 1536$.

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$b = 2$: $9 \cdot 2^7 = 1152$ (since $9 \cdot 64 = 576$ is too small).

$b = 3$: $27 \cdot 2^6 = 1728$ ($27 \cdot 32 = 864$ is too small).

$b = 4$: $81 \cdot 2^4 = 1296$.

$b = 5$: $243 \cdot 2^3 = 1944$.

$b = 6$: $729 \cdot 2 = 1458$.

For $b \geq 7$, $3^b \cdot 2 \geq 4374$. The smallest is 1152.

Check. $1152 = 2^7 \cdot 3^2 = 128 \cdot 9$. Could anything between 1000 and 1152 work? Such a number would be $2^a 3^b$ with the list above covering every b, so no.

14. Answer: 4**. The idea.** The paces are given in minutes per kilometre, so compare the two runners kilometre by kilometre rather than converting to speeds.

Work it out. On each kilometre, Bea takes $\frac{1}{2}$ minute less than Ana. She starts 2 minutes behind in time, so she has made up the whole gap after $2 \div \frac{1}{2} = 4$ kilometres. That is before the 5-kilometre finish, so the catch happens at 4 kilometres.

Check. Ana reaches the 4-kilometre mark after $4 \cdot 4 = 16$ minutes. Bea reaches it $2 + 4 \cdot 3\frac{1}{2} = 2 + 14 = 16$ minutes after Ana started. They are there at the same moment.

15. Answer: $\frac{1}{12}$**. The idea.** Only the relative order inside $\{1, 2, 3\}$ and inside $\{4, 5\}$ matters, and those two relative orders are independent of each other in a random shuffle.

Work it out. The three numbers $1, 2, 3$ appear in one of $3! = 6$ equally likely relative orders, and exactly one of them is $1, 2, 3$: probability $\frac{1}{6}$. The pair $4, 5$ is in the right order with probability $\frac{1}{2}$, whatever happens to $1, 2, 3$. So the probability is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

Check. Count directly: choose which 3 of the 5 places hold $1, 2, 3$ ($\binom{5}{3} = 10$ ways); then $1, 2, 3$ fill them in increasing order and $4, 5$ fill the other two in increasing order. That is 10 good orders out of 120, and $\frac{10}{120} = \frac{1}{12}$.

16. Answer: 180**. The idea.** The length of the trip does not change, so write it two ways (usual speed times usual time, faster speed times shorter time) and set them equal. That gives the usual speed, and then the distance, and then any other trip time.

Work it out. Let the usual speed be v km/h. The distance is $2v$ at the usual speed and $1.5(v + 20)$ at the faster speed, since 30 minutes less than 2 hours is 1.5 hours. So $2v = 1.5v + 30$, which gives $0.5v = 30$ and $v = 60$. The trip is $2 \cdot 60 = 120$ km. At $60 - 20 = 40$ km/h it takes $\frac{120}{40} = 3$ hours, which is 180 minutes.

Check. At 80 km/h the 120 km take 1.5 hours, exactly 30 minutes less than 2 hours. Notice the answer is not just "30 minutes more": slowing down by 20 km/h costs a full hour, because time depends on $\frac{1}{\text{speed}}$, not on the speed itself.

17. Answer: $\frac{18}{5}$**. The idea.** Any point of a circle sees a diameter at a right angle. Since D is on the circle with diameter $\overline{CB}$, $\angle CDB = 90°$, so $\overline{CD}$ is the altitude from the right angle to the hypotenuse. Then similar triangles finish the job.

Work it out. The hypotenuse is $AB = \sqrt{6^2 + 8^2} = 10$. Triangles ADC and ACB share the angle at A and both have a right angle (at D and at C), so they are similar, with $\frac{AD}{AC} = \frac{AC}{AB}$. Hence $AD = \frac{AC^2}{AB} = \frac{36}{10} = \frac{18}{5}$.

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Check. With coordinates $C = (0, 0)$, $A = (0, 6)$, $B = (8, 0)$, the circle is $(x-4)^2 + y^2 = 16$ and the hypotenuse is $y = 6 - \frac{3}{4}x$. Substituting gives $\frac{25}{16}x^2 - 17x + 36 = 0$, whose roots are $x = 8$ (the point B) and $x = \frac{72}{25}$. Then $D = \left(\frac{72}{25}, \frac{96}{25}\right)$ and $AD = \sqrt{\left(\frac{72}{25}\right)^2 + \left(\frac{54}{25}\right)^2} = \frac{90}{25} = \frac{18}{5}$.

18. Answer: 4204**. The idea.** Pad the numbers 0 to 399 with leading zeros to get 000, 001, ..., 399. Padding adds only zeros, so the digit sum is unchanged, and now each place has a regular pattern. Then add the single number 400 separately.

Work it out. In the 400 strings 000 to 399, the units digit runs through 0 to 9 exactly 40 times, contributing $40 \cdot 45 = 1800$. The tens digit also takes each value 0 to 9 exactly 40 times: another 1800. The hundreds digit is $0, 1, 2, 3$, each 100 times: $100 \cdot (0 + 1 + 2 + 3) = 600$. That is $1800 + 1800 + 600 = 4200$ for 1 to 399 (the number 0 adds nothing). Adding 400, whose digits sum to 4, gives 4204.

Check. The block 0 to 99 alone gives $2 \cdot 10 \cdot 45 = 900$. The blocks 100–199, 200–299, 300–399 each repeat those 900 plus 100 times the hundreds digit: 1000, 1100, 1200. Total $900 + 1000 + 1100 + 1200 + 4 = 4204$.

19. Answer: $\frac{3}{2}$**. The idea.** Don't find the exponents one by one. Feed each equation into the next: raising both sides of an equation to a power lets you replace a base with a power of the previous base.

Work it out. Start with $8 = 7^d$. Since $7 = 6^c$, this gives $8 = (6^c)^d = 6^{cd}$. Since $6 = 5^b$, it becomes $8 = 5^{bcd}$, and since $5 = 4^a$, it becomes $8 = 4^{abcd}$. Now $8 = 2^3$ and $4^{abcd} = 2^{2abcd}$, so $2abcd = 3$ and $abcd = \dfrac{3}{2}$.

Check. In logarithm form, $a = \frac{\log 5}{\log 4}$, $b = \frac{\log 6}{\log 5}$, $c = \frac{\log 7}{\log 6}$, $d = \frac{\log 8}{\log 7}$. In the product everything cancels except $\frac{\log 8}{\log 4} = \frac{3\log 2}{2\log 2} = \frac{3}{2}$. As a rough check, $a \approx 1.16$, $b \approx 1.11$, $c \approx 1.09$ and $d \approx 1.07$, whose product is about 1.5.

20. Answer: $8 + 4\sqrt{2}$**. The idea.** The square is the octagon with four identical triangles cut off: $\triangle ABC$, $\triangle CDE$, $\triangle EFG$ and $\triangle GHA$. So find the octagon's area (by boxing it in a square) and subtract four of these triangles.

Work it out. Extend four alternate sides to box the octagon in a square. Each corner that is cut off is a right isosceles triangle with hypotenuse 2 and legs $\sqrt{2}$, so the box has side $2 + 2\sqrt{2}$. The octagon's area is $(2+2\sqrt{2})^2 - 4 \cdot \frac{1}{2}(\sqrt{2})^2 = (12 + 8\sqrt{2}) - 4 = 8 + 8\sqrt{2}$. To find the area of $\triangle ABC$, extend side $\overline{AB}$ past B and drop the perpendicular from C to that line, meeting it at P. The line BP makes a 45° angle with side $\overline{BC}$ (each exterior angle of a regular octagon is $\frac{360°}{8} = 45°$), so $\triangle BPC$ is a right isosceles triangle with hypotenuse $BC = 2$, and $BP = CP = \sqrt{2}$. Taking $AB = 2$ as the base, the height of $\triangle ABC$ is $CP = \sqrt{2}$, so its area is $\frac{1}{2} \cdot 2 \cdot \sqrt{2} = \sqrt{2}$. The square has area $8 + 8\sqrt{2} - 4\sqrt{2} = 8 + 4\sqrt{2}$.

Check. Compute the side AC directly, using the same right triangle APC. Its legs are $AP = AB + BP = 2 + \sqrt{2}$ and $CP = \sqrt{2}$, so $AC^2 = (2+\sqrt{2})^2 + (\sqrt{2})^2 = (6 + 4\sqrt{2}) + 2 = 8 + 4\sqrt{2}$. The square's area is $AC^2 = 8 + 4\sqrt{2}$.

21. Answer: $\frac{5}{3}$**. The idea.** Compute a couple of steps and look for a cycle. Here the cycle has length 2, because f undoes itself: $f(f(x)) = x$. Then only the parity of 2025 matters.

Work it out. $f(7) = \frac{10}{6} = \frac{5}{3}$, and $f\left(\frac{5}{3}\right) = \frac{\frac{5}{3}+3}{\frac{5}{3}-1} = \frac{14/3}{2/3} = 7$. So the values go $7, \frac{5}{3}, 7, \frac{5}{3}, \ldots$: after an even number of steps you are back at 7, and after an odd number you are at $\frac{5}{3}$. Since 2025 is odd, the result is $\frac{5}{3}$.

Check. In general, $f(f(x)) = \frac{\frac{x+3}{x-1}+3}{\frac{x+3}{x-1}-1} = \frac{(x+3)+3(x-1)}{(x+3)-(x-1)} = \frac{4x}{4} = x$ for every $x \neq 1$, so the two-step cycle is no accident. The tempting slip is to answer 7, which is where you are after an even number of steps such as 2026.

22. Answer: 1320**. The idea.** Do not add 45 products. The whole table is easy to add because it factors, the diagonal is just the squares, and the table is symmetric, so the part below the diagonal is half of what remains.

Work it out. The sum of all 100 entries is $\sum_r \sum_c rc = (1+2+\cdots+10)(1+2+\cdots+10) = 55^2 = 3025$. The diagonal entries are $1^2+2^2+\cdots+10^2 = 385$. The other 90 entries add to $3025 - 385 = 2640$, and since entry (r, c) equals entry (c, r), the parts above and below the diagonal are equal. The part below adds to $\frac{2640}{2} = 1320$.

Check. Add by columns: below the diagonal in column c are the entries $c \cdot (c+1), \ldots, c \cdot 10$, whose sum is $c\left(55 - \frac{c(c+1)}{2}\right)$. For $c = 1, \ldots, 9$ these are 54, 104, 147, 176, 200, 204, 196, 144, 99 (and 0 for $c = 10$), which add to 1320.

23. Answer: $\frac{21}{8}$**. The idea.** "The same distance from A as from B" means $PB = PA$. Both lengths can be written in terms of CP: PA is the rest of the leg, and PB is the hypotenuse of the small right triangle PCB. Setting them equal gives one equation.

Work it out. Let $CP = x$. Then $PA = 12 - x$, and since $\angle C = 90°$, $PB = \sqrt{x^2 + 9^2}$. From $PB = PA$: $x^2 + 81 = (12 - x)^2 = 144 - 24x + x^2$. The x^2 terms cancel, leaving $24x = 63$, so $x = \frac{63}{24} = \frac{21}{8}$.

Check. Then $PA = 12 - \frac{21}{8} = \frac{75}{8}$ and $PB = \sqrt{\left(\frac{21}{8}\right)^2 + 9^2} = \frac{\sqrt{441+5184}}{8} = \frac{\sqrt{5625}}{8} = \frac{75}{8}$, equal as required. Another way to see it: P is on the perpendicular bisector of $\overline{AB}$ (hypotenuse 15, midpoint M with $AM = \frac{15}{2}$). Right triangles AMP and ACB share the angle at A, so $\frac{AP}{AB} = \frac{AM}{AC}$, giving $AP = \frac{15 \cdot 15/2}{12} = \frac{75}{8}$ and $CP = 12 - \frac{75}{8} = \frac{21}{8}$.

24. Answer: 118**. The idea.** Counting the numbers directly needs care with the digit 0, which cannot come first but can come last. It is cleaner to count all even three-digit numbers and remove the ones whose digits are all different or all the same.

Work it out. Write the number as $\overline{abc}$ with $a \neq 0$ and c even. There are $9 \cdot 10 \cdot 5 = 450$ even three-digit numbers. All digits different: if $c = 0$, then a has 9 choices and b has 8, giving 72; if c is $2, 4, 6$ or 8, then a has 8 choices (not 0, not c) and b has 8, giving $4 \cdot 64 = 256$. All three digits equal and even: $222, 444, 666, 888$, which is 4. So $450 - 72 - 256 - 4 = 118$.

Check. Count directly by which two places match. $a = b \neq c$: an odd a (5 choices) allows 5 even c, and an even a (4 choices) allows 4, for 41. $a = c \neq b$: a is $2, 4, 6$ or 8 and b has 9 choices, for 36. $b = c \neq a$: $c = 0$ gives 9 choices of a, and each of the other 4 even c gives 8, for 41. Total $41 + 36 + 41 = 118$.

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25. Answer: 58**. The idea.** A number is a sum of two different powers of 2 exactly when its base-2 form has exactly two digits equal to 1. There are only a few sums of two different powers of 3 below 1000, so list those and test each one.

Work it out. The powers of 3 below 1000 are $1, 3, 9, 27, 81, 243, 729$, giving $\binom{7}{2} = 21$ sums, all below 1000: $4, 10, 28, 82, 244, 730$ (with 1); $12, 30, 84, 246, 732$ (with 3); $36, 90, 252, 738$ (with 9); $108, 270, 756$; $324, 810$; 972. Now test each for being $2^a + 2^b$ with $a \neq b$: $10 = 8+2$, $12 = 8+4$ and $36 = 32 + 4$ work. The rest fail, for example $4 = 2^2$ has only one 1 in base 2, $28 = 16 + 8 + 4$, $82 = 64 + 16 + 2$, $84 = 64 + 16 + 4$, $90 = 64 + 16 + 8 + 2$, $108 = 64 + 32 + 8 + 4$, and every sum from 244 up has at least three 1s in base 2. The sum is $10 + 12 + 36 = 58$.

Check. In base 2: $10 = 1010_2$, $12 = 1100_2$ and $36 = 100100_2$ each have exactly two 1s, while for instance $30 = 11110_2$, $244 = 11110100_2$ and $972 = 1111001100_2$ have more. A computer search finds no other examples even far beyond 1000.

26. Answer: 144**. The idea.** Glue A and B into a single block so their condition takes care of itself. Then count the seatings with C and D adjacent and subtract them, rather than counting the "apart" seatings directly.

Work it out. With A and B glued there are 6 units around the table. n units in a circle, up to rotation, can be arranged in $(n-1)!$ ways, so there are $5! = 120$ arrangements, times 2 for the order inside the block (AB or BA): 240 seatings with A, B adjacent. Among these, glue C and D as well: 5 units give $4! = 24$ arrangements, times $2 \cdot 2$ for the orders inside the two blocks: 96. So $240 - 96 = 144$ seatings have A, B together and C, D apart.

Check. Fix A's seat. B sits on one of A's two sides, and 5 seats remain in a row (the circle is cut open by the AB block). C and D fill 2 of those 5 seats in $5 \cdot 4 = 20$ ordered ways, of which $4 \cdot 2 = 8$ are adjacent, leaving 12. The other three people fill the rest in $3! = 6$ ways: $2 \cdot 12 \cdot 6 = 144$.

27. Answer: 8**. The idea.** Measure all three points by their distance from B along $\overline{BC}$. The altitude foot comes from two Pythagorean equations sharing the height, the bisector foot from the Angle Bisector Theorem ($BD : DC = AB : AC$), and the midpoint is free.

Work it out. Let $BH = x$ and $AH = h$. Then $x^2 + h^2 = 25^2$ and $(17-x)^2 + h^2 = 26^2$. Subtracting, $(17-x)^2 - x^2 = 676 - 625 = 51$, so $289 - 34x = 51$ and $x = 7$ (and $h = 24$). By the Angle Bisector Theorem, $BD = 17 \cdot \frac{25}{25+26} = \frac{25}{3}$. The midpoint has $BM = \frac{17}{2}$. So the order along the side is H, D, M, with $HD = \frac{25}{3} - 7 = \frac{4}{3}$ and $DM = \frac{17}{2} - \frac{25}{3} = \frac{1}{6}$. The ratio is $\frac{4/3}{1/6} = 8$.

Check. Coordinates: $B = (0,0)$, $C = (17,0)$, $A = (7,24)$ give $AB = \sqrt{49+576} = 25$ and $AC = \sqrt{100+576} = 26$, as required. Then $DC = 17 - \frac{25}{3} = \frac{26}{3}$, and $BD : DC = 25 : 26$, as the bisector theorem demands. It also fits the fact that the bisector foot always lies between the altitude foot and the midpoint.

28. Answer: 103**. The idea.** Fill the board with path counts, row by row: the number of ways to reach a square is the sum of the numbers on the (at most two) squares diagonally below it. The board's edges are what make this more than 2^7: a square in column 1 or 8 can be reached from only one square below.

Work it out. Write the counts in columns 1 to 8 (blank squares cannot be reached, by a colour argument). Row 1: a 1 in column 4. Row 2: columns 3, 5 get 1, 1. Row 3: columns 2, 4, 6 get 1, 2, 1. Row 4: columns 1, 3, 5, 7 get 1, 3, 3, 1. Row 5: columns 2, 4, 6, 8 get $1 + 3 = 4$, $3 + 3 = 6$,

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$3 + 1 = 4$, and 1 (from column 7 only). Row 6: columns $1, 3, 5, 7$ get 4, $4 + 6 = 10$, $6 + 4 = 10$, $4 + 1 = 5$. Row 7: columns $2, 4, 6, 8$ get $14, 20, 15, 5$. Row 8: columns $1, 3, 5, 7$ get 14, 34, 35, 20. The total is $14 + 34 + 35 + 20 = 103$.

Check. Without edges there would be $2^7 = 128$ sequences; count the ones that fall off. Falling off the left means reaching column 0, which needs 4 more left moves than right moves at some moment. By the reflection trick, such sequences are counted by twice the number of sequences ending at column -1 or beyond (at most 1 right move): $2(1 + 7) = 16$. Falling off the right means reaching column 9, 5 more right than left moves: sequences with at least 6 right moves (8 of them) plus those with 7 (1), 9 in all. No sequence can do both. So $128 - 16 - 9 = 103$.

29. Answer: 675**. The idea.** Divisors come in pairs d and $\frac{N}{d}$, so the three largest divisors are N divided by the three smallest. First pin down the three smallest divisors; then the three largest are known multiples of N.

Work it out. The smallest divisor is 1, so the next two add to 8: they are 2 and 6, or 3 and 5. The pair $2, 6$ is impossible: if 6 divides N, so does 3, and 3 would come before 6. So the three smallest divisors are $1, 3, 5$ (this also says N is odd). The three largest are then N, $\frac{N}{3}$ and $\frac{N}{5}$, with sum $N\left(1 + \frac{1}{3} + \frac{1}{5}\right) = \frac{23N}{15}$. Setting $\frac{23N}{15} = 1035$ gives $N = 15 \cdot 45 = 675$.

Check. $675 = 3^3 \cdot 5^2$ is odd, and its divisors are $1, 3, 5, 9, 15, 25, 27, 45, 75, 135, 225, 675$. The three smallest add to 9, and the three largest add to $675 + 225 + 135 = 1035$.

30. Answer: 824**. The idea.** Complete the square. Then "a square equals another square minus a constant" becomes a difference of squares, which factors, so the problem turns into listing factor pairs of the constant.

Work it out. $n^2+20n-1924 = (n+10)^2-2024$. If this equals m^2 with $m \geq 0$, then $(n+10)^2-m^2 = 2024$, so $(n + 10 - m)(n + 10 + m) = 2024$. The two factors are positive (their product and their sum $2(n + 10)$ are positive), and they have the same parity because they differ by $2m$. Since 2024 is even, both must be even: write them as $2d$ and $2e$ with $d \leq e$ and $de = 506 = 2 \cdot 11 \cdot 23$. Then $n+10 = d+e$. The pairs are $(1, 506)$, $(2, 253)$, $(11, 46)$, $(22, 23)$, giving $n+10 = 507, 255, 57, 45$, so $n = 497, 245, 47, 35$. All four are positive, and the sum is $497 + 245 + 47 + 35 = 824$.

Check. $35^2+700-1924 = 1$, $47^2+940-1924 = 1225 = 35^2$, $245^2+4900-1924 = 63{,}001 = 251^2$ and $497^2 + 9940 - 1924 = 255{,}025 = 505^2$. No other n works, because every solution comes from a factor pair of 506, and a computer search of n up to 20,000 finds only these four.

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