MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The front wheel of a bicycle has a diameter of 50 centimetres. The bicycle rolls straight ahead, without slipping, for a distance of 100π metres. How many complete turns does the front wheel make?

1. _____

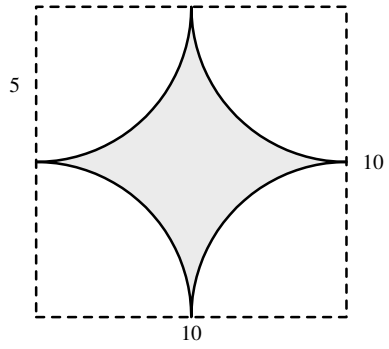
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2. Sam owns 4 different shirts, 3 different pairs of pants and 2 different hats. An outfit consists of one shirt and one pair of pants, together with either one of the hats or no hat at all. How many different outfits can Sam put together?

2. _____

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3. A square piece of card measures 10 cm by 10 cm. At each of its four corners, Mia cuts away a quarter circle of radius 5 cm whose center is that corner. What is the area, in square centimetres, of the card that is left? Express your answer in terms of π.



3. _____

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4. On a monthly calendar, the dates are arranged in rows of 7 (one row per week). Lulu outlines a 3×3 square of nine dates on it, three consecutive dates in each of three consecutive weeks, and the nine dates add up to 153. What is the smallest date inside her square?

4. _____

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5. In an arithmetic sequence, the 4th term and the 9th term add up to 50. Neither the first term nor the common difference is given. What is the sum of the first 12 terms of the sequence?

5. _____

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6. Priya lists every three-digit positive integer that is a multiple of 7, from 105 up to 994. Then she crosses out each number on her list whose units digit is even. How many numbers are left on her list?

6. _____

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7. A number exactly halfway between two choices is always rounded up. How many whole numbers round to 500 when rounded to the nearest hundred, but do not round to 500 when rounded to the nearest ten?

7. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

8. A student invents a temperature scale on which water freezes at 20 degrees and boils at 220 degrees, and the scale is linear in between and beyond. On the Fahrenheit scale, which is also linear, water freezes at 32 degrees and boils at 212 degrees. At one temperature, the student's thermometer and a Fahrenheit thermometer show the same number. What is that number?

8. _____

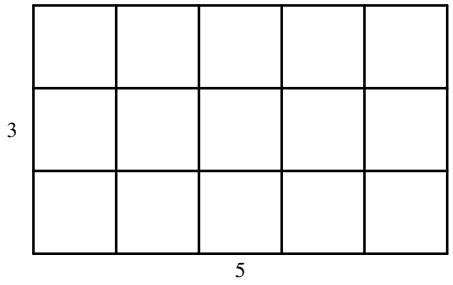
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9. Omar adds up all but one of the interior angles of a convex polygon and gets 1070°. How many degrees are in the one interior angle he left out?

9. _____

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10. A 3 by 5 rectangle is divided into 15 unit squares by grid lines. Count every rectangle, of any size, whose four sides lie along the grid lines (including the unit squares and the whole figure). How many of these rectangles are not squares?



10. _____

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11. A jar holds exactly 50 coins, each a penny or a nickel. Each penny weighs 2.5 grams, each nickel weighs 5 grams, and together the 50 coins weigh 200 grams. How many cents are the coins in the jar worth altogether?

11. _____

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12. For some prime numbers p, the number $2p + 1$ is also prime; for example, $p = 5$ gives $2 \cdot 5 + 1 = 11$, which is prime. How many prime numbers p less than 50 have the property that $2p + 1$ is also prime?

12. _____

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13. Exactly one real number x satisfies the equation $|x-2| = 2x+1$. What is it? Express your answer as a common fraction.

13. _____

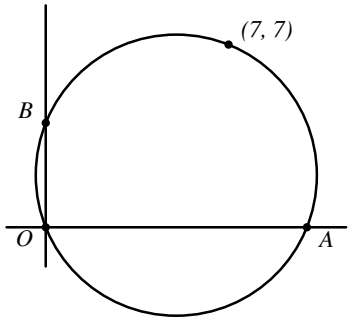
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14. Every member of a games club plays chess, checkers, or both. Exactly two thirds of the members play chess, exactly three quarters of the members play checkers, and exactly 30 members play both games. How many members play exactly one of the two games?

14. _____

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15. A circle passes through the origin O and through the point $(7, 7)$. It crosses the positive x-axis at a point A and the positive y-axis at a point B (both different from O). What is $OA + OB$?



15. _____

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16. A rectangular sheet of paper measures 6 inches by 10 inches. It can be rolled into a hollow cylinder (with no top or bottom) by bringing one pair of opposite edges together so that they just meet, and this can be done in two different ways: by joining the two 6-inch edges or by joining the two 10-inch edges. What is the volume of the larger of the two cylinders divided by the volume of the smaller one? Express your answer as a common fraction.

16. _____

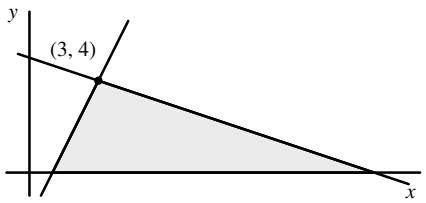
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17. A store raises the price of a coat by x percent. A month later, it lowers the new price by x percent (the same number x, now taken as a percent of the raised price). After both changes, the coat costs exactly 4 percent less than it did before the raise. What is the value of x?

17. _____

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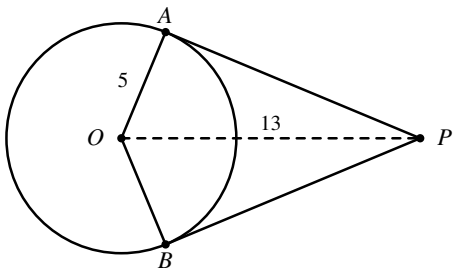
18. Two lines pass through the point $(3, 4)$. One of them has slope 2 and the other has slope $-\frac{1}{3}$. Together with the x-axis, the two lines enclose a triangle. What is the area of this triangle, in square units?



18. _____

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19. A circle has center O and radius 5. Point P is 13 units from O. The two lines through P that are tangent to the circle touch it at the points A and B. What is the area of quadrilateral $OAPB$?



19. _____

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20. In the six-digit number $\overline{34A5B8}$, the letters A and B stand for single digits (they may be equal, and either may be 0). The number is a multiple of 72. What is the sum of all six-digit numbers of this form that are multiples of 72?

20. _____

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21. A staircase has 10 steps above the floor, numbered 1 to 10 from the bottom. Mia climbs from the floor to step 10, and each stride takes her up either 1 step or 2 steps. Step 5 is broken, so she must never land on it. In how many different ways can she climb to step 10? (Two ways are different if their lists of strides are different.)

21. _____

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22. How many ordered pairs of integers (x, y) satisfy $|x - 3| + |x + 2| + |y - 1| + |y + 4| = 10$? (Here $|u|$ is the absolute value of u.)

22. _____

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23. A street lamp sits on top of a vertical post 6 metres tall on flat ground. A person 1.5 metres tall walks in a straight line directly away from the foot of the post at a steady 1.2 metres per second, so the person's shadow falls on the ground in front of them. How fast, in metres per second, is the tip of the shadow moving along the ground? Express your answer as a common fraction.

23. _____

24. The number 2026 is to be written as a sum of different positive even integers, for example $2026 = 1000 + 1026$ or $2026 = 2 + 4 + 2020$. What is the largest possible number of terms in such a sum?

24. _____

25. An isosceles triangle has two sides of length 17 and a third side (its base) of length 16. A different isosceles triangle also has two sides of length 17, and it has exactly the same area as the first one. What is the length of its base?

25. _____

26. A rectangular box has edge lengths that are whole numbers of centimetres, and its volume is 144 cubic centimetres. Two boxes count as the same if one can be turned to look exactly like the other (so a $2\times3\times24$ box is the same as a $24\times2\times3$ box). How many different boxes are possible?

26. _____

27. A bag holds 3 red marbles and 5 blue marbles. Marbles are drawn one at a time at random, without replacement, and the drawing stops as soon as all three red marbles have been drawn. What is the probability that exactly 6 marbles are drawn? Express your answer as a common fraction.

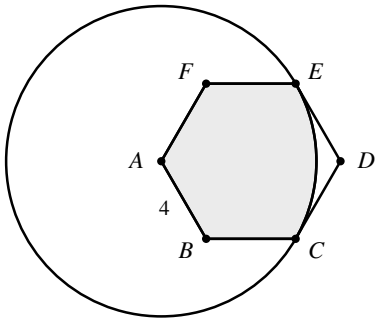
27. _____

28. A five-letter code is a string of five letters, each of which is A, B or C; for example, CABAC and BBBBB are codes. How many five-letter codes use each of the three letters A, B and C at least once? (CABAC counts, but AABBB does not.)

28. _____

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29. Regular hexagon $ABCDEF$ has sides of length 4. A circle is drawn with its center at vertex A, passing through vertex C. What is the area of the region that lies inside both the hexagon and the circle? Express your answer in simplest radical form, in terms of π.



29. _____

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30. Each of the ten digits $0, 1, 2, \ldots, 9$ is used exactly once to write five two-digit numbers (a two-digit number cannot begin with 0), and then the five numbers are added. For example, $10+32+54+76+98 = 270$. How many different values can this sum take?

30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 6

1. 200

2. 36

3. $100 - 25\pi$

4. 9

5. 300

6. 64

7. 90

8. 140

9. 10

10. 64

11. 170

12. 7

13. $\frac{1}{3}$

14. 42

15. 14

16. $\frac{5}{3}$

17. 20

18. 28

19. 60

20. 687096

21. 25

22. 36

23. $\frac{8}{5}$

24. 44

25. 30

26. 18

27. $\frac{5}{28}$

28. 150

29. $8\pi + 8\sqrt{3}$

30. 21

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Chapter Sprint Practice Contest 6

1. Answer: 200**. The idea.** In one full turn a wheel that does not slip rolls forward exactly its own circumference. The only trap is the units: the diameter is in centimetres but the distance is in metres.

Work it out. The diameter is 50 cm = 0.5 m, so the circumference is 0.5π metres. The number of turns is $\frac{100\pi}{0.5\pi} = 200$.

Check. Work in centimetres instead: the distance is 10000π cm and each turn covers 50π cm, and $10000\pi \div 50\pi = 200$. (Forgetting to convert gives the wrong answer 2.)

2. Answer: 36**. The idea.** Treat "no hat" as a third choice for the head. Then every outfit is one choice from each of three independent lists, and the counts multiply.

Work it out. There are 4 shirt choices, 3 pants choices, and 3 head choices (hat 1, hat 2, or no hat). So there are $4 \cdot 3 \cdot 3 = 36$ outfits.

Check. Count the two kinds separately: without a hat there are $4 \cdot 3 = 12$ outfits, and with a hat there are $4 \cdot 3 \cdot 2 = 24$. Together $12 + 24 = 36$. The tempting answer 24 forgets the hatless outfits.

3. Answer: $100 - 25\pi$**. The idea.** Each piece cut away is a quarter of a circle of radius 5, and there are four of them, so together they make exactly one whole circle of radius 5.

Work it out. Because the radius 5 is half the side, neighbouring quarter circles just touch at the midpoint of a side and never overlap. The cut-away area is one full circle, $\pi \cdot 5^2 = 25\pi$. What is left is $100 - 25\pi$ square centimetres.

Check. $25\pi \approx 78.5$, so about 21.5 square centimetres remain, a small four-pointed star in the middle, which is reasonable: the circle of radius 5 has almost the area of the whole square.

4. Answer: 9**. The idea.** The nine dates are balanced around the middle date: each date one step left is matched by one a step right, and each date a week earlier by one a week later. So the sum is simply 9 times the middle date.

Work it out. Call the middle date m. The nine dates are m plus each of $-8, -7, -6, -1, 0, 1, 6, 7, 8$, and these offsets add to 0. So $9m = 153$ and $m = 17$. The smallest date is the top-left one, $m - 7 - 1 = 9$.

Check. The square is $9, 10, 11/16, 17, 18/23, 24, 25$: the rows add to $30 + 51 + 72 = 153$.

5. Answer: 300**. The idea.** In an arithmetic sequence, two terms whose positions add to the same number have the same sum: moving one position right adds d, moving one position left takes d away. The positions 4 and 9 add to 13, and so do 1 and 12, 2 and 11, and so on.

Work it out. The first 12 terms split into 6 pairs, (1st, 12th), (2nd, 11th), . . ., (6th, 7th), and every pair's positions add to 13, the same as $4 + 9$. So each pair adds to 50, and the total is $6 \cdot 50 = 300$.

Check. With first term a and difference d, the 4th and 9th terms add to $2a + 11d = 50$, and the sum of 12 terms is $\frac{12}{2}(2a + 11d) = 6 \cdot 50 = 300$. For instance $a = 3$, $d = 4$ gives $15 + 35 = 50$ and $3 + 7 + \cdots + 47 = 300$.

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6. Answer: 64**. The idea.** A number has an odd units digit exactly when it is odd, and $7k$ is odd exactly when k is odd. So we want the numbers $7k$ with k odd; equivalently, the multiples of 7 that are not multiples of 14.

Work it out. The three-digit multiples of 7 are $7 \cdot 15 = 105$ up to $7 \cdot 142 = 994$. We need odd k from 15 to 141: that is $\frac{141-15}{2} + 1 = 64$ values.

Check. There are $142 - 15 + 1 = 128$ three-digit multiples of 7. The three-digit multiples of 14 run from $14 \cdot 8 = 112$ to $14 \cdot 71 = 994$, which is 64 numbers, and $128 - 64 = 64$.

7. Answer: 90**. The idea.** Find the whole block of numbers that round to 500 by hundreds, then remove the smaller block that also rounds to 500 by tens.

Work it out. Rounding to the nearest hundred gives 500 for $450, 451, \ldots, 549$ (450 is halfway and rounds up; 550 would round up to 600): that is 100 numbers. Rounding to the nearest ten gives 500 for $495, 496, \ldots, 504$: that is 10 numbers, all inside the first block. So $100 - 10 = 90$ numbers qualify.

Check. The qualifying numbers are 450 to 494 (45 numbers) and 505 to 549 (45 numbers), and $45 + 45 = 90$.

8. Answer: 140**. The idea.** Measure both scales from the same starting point, freezing water, and compare how fast each one climbs. The student's scale starts lower but climbs faster, so it catches up.

Work it out. From freezing to boiling, the student's scale climbs $220 - 20 = 200$ degrees and the Fahrenheit scale climbs $212 - 32 = 180$ degrees. So for every 10 student degrees, Fahrenheit climbs 9.

At freezing the student's scale is $32 - 20 = 12$ behind. It gains 1 degree on Fahrenheit for every 10 student degrees it climbs, so it catches up after climbing 120 student degrees above freezing: the reading is $20 + 120 = 140$.

Check. Fahrenheit climbs $\frac{9}{10} \cdot 120 = 108$ degrees over the same stretch, from 32 to 140. The same number, as required. (In Celsius this is 60°, since the student's scale reads $2C + 20$ and Fahrenheit reads $\frac{9}{5}C + 32$, and $2C + 20 = \frac{9}{5}C + 32$ gives $C = 60$.) A tempting slip is to answer 60, the Celsius temperature, rather than the number the two thermometers show.

9. Answer: 10**. The idea.** The full angle sum of a convex polygon is a multiple of 180°, namely $(n-2) \cdot 180°$, and the missing angle is strictly between 0° and 180° because the polygon is convex. So the full sum is the one multiple of 180 that is above 1070 by less than 180.

Work it out. The multiples of 180 near 1070 are 900 and 1080. The sum must be larger than 1070, and 1260 would leave an angle of 190°, which is too big. So the full sum is $1080° = (8-2) \cdot 180°$ (an octagon) and the missing angle is $1080 - 1070 = 10$ degrees.

Check. An octagon with seven angles adding to 1070° (for example six of 150° and one of 170°) and one angle of 10° is possible: every angle is less than 180° and the total is 1080°.

10. Answer: 64**. The idea.** Count all rectangles, then subtract the squares. A rectangle is set by choosing two of the vertical grid lines and two of the horizontal grid lines.

Work it out. There are 6 vertical lines and 4 horizontal lines, so there are $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$ rectangles. Squares: a 1×1 square fits in $5 \cdot 3 = 15$ places, a 2×2 in $4 \cdot 2 = 8$ places, and a

3×3 in $3 \cdot 1 = 3$ places, so there are 26 squares. That leaves $90 - 26 = 64$ rectangles that are not squares.

Check. Count by size, width w and height h: a $w \times h$ rectangle fits in $(6-w)(4-h)$ places. The widths give $5+4+3+2+1 = 15$ places across and the heights $3+2+1 = 6$ places down, so 90 in all, which matches.

11. Answer: 170**. The idea.** Start from the guess "all pennies" and see how far the weight is off. Swapping one penny for a nickel keeps the count at 50 and adds a fixed amount of weight, so the shortfall tells us the number of nickels.

Work it out. Fifty pennies would weigh $50 \cdot 2.5 = 125$ grams, which is 75 grams too light. Each swap of a penny for a nickel adds $5 - 2.5 = 2.5$ grams, so there are $75 \div 2.5 = 30$ nickels and 20 pennies. They are worth $30 \cdot 5 + 20 \cdot 1 = 170$ cents.

Check. The weight is $30 \cdot 5 + 20 \cdot 2.5 = 150 + 50 = 200$ grams. Note the question asks for money, not for the number of nickels (30).

12. Answer: 7**. The idea.** List the primes below 50 and test each one, but save work first: if p is a prime other than 3, then p leaves remainder 1 or 2 when divided by 3. When the remainder is 1, the number $2p+1$ leaves remainder $2+1 = 3$, that is, 0, so it is a multiple of 3 bigger than 3 and cannot be prime. So apart from $p = 3$, only primes that leave remainder 2 on division by 3 can work.

Work it out. $p = 3$ gives 7, prime. The primes below 50 that leave remainder 2 on division by 3 are $2, 5, 11, 17, 23, 29, 41, 47$. Test them: $2 \to 5$ (prime), $5 \to 11$ (prime), $11 \to 23$ (prime), $17 \to 35 = 5 \cdot 7$ (no), $23 \to 47$ (prime), $29 \to 59$ (prime), $41 \to 83$ (prime), $47 \to 95 = 5 \cdot 19$ (no). Together with $p = 3$ that makes 7 primes: $2, 3, 5, 11, 23, 29, 41$.

Check. The shortcut threw out $7, 13, 19, 31, 37, 43$, and indeed $15, 27, 39, 63, 75, 87$ are all multiples of 3. Testing every prime below 50 directly gives the same list of 7.

13. Answer: $\frac{1}{3}$**. The idea.** Remove the absolute value by splitting into the two cases $x - 2 \geq 0$ and $x - 2 < 0$. The catch is that each case only counts if its answer actually lies in that case; equivalently, the right side $2x + 1$ can never be negative, since it equals an absolute value.

Work it out. Case $x \geq 2$: $x - 2 = 2x + 1$ gives $x = -3$, which is not ≥ 2 (and it would make the right side -5), so it is rejected. Case $x < 2$: $2 - x = 2x + 1$ gives $3x = 1$, so $x = \frac{1}{3}$, which is less than 2 and is kept.

Check. At $x = \frac{1}{3}$: $\left|\frac{1}{3} - 2\right| = \frac{5}{3}$ and $2 \cdot \frac{1}{3} + 1 = \frac{5}{3}$. At $x = -3$: $|-5| = 5$ but $2(-3) + 1 = -5$, so -3 really fails.

14. Answer: 42**. The idea.** The fractions $\frac{2}{3}$ and $\frac{3}{4}$ add to more than 1, and the excess is exactly the fraction of members counted twice, the ones who play both. That tells us what fraction of the club the 30 members are.

Work it out. Adding the two groups counts every member once and the members who play both a second time. So the part who play both is $\frac{2}{3} + \frac{3}{4} - 1 = \frac{8+9-12}{12} = \frac{5}{12}$ of the club. Since $\frac{5}{12}$ of the club is 30 members, $\frac{1}{12}$ is 6 and the club has 72 members. Everyone who does not play both games plays exactly one, so the answer is $72 - 30 = 42$.

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Check. Chess players: $\frac{2}{3}\cdot 72 = 48$, so $48-30 = 18$ play only chess. Checkers players: $\frac{3}{4}\cdot 72 = 54$, so $54 - 30 = 24$ play only checkers. Then $18 + 24 + 30 = 72$, the whole club, and $18 + 24 = 42$. (Stopping at the club size 72 answers a different question.)

15. Answer: 14**. The idea.** The angle $\angle AOB$ is a right angle and it sits on the circle, so $\overline{AB}$ is a diameter. Then any other point P of the circle also sees $\overline{AB}$ at a right angle, which pins down how OA and OB are related.

Work it out. Let $A = (a, 0)$ and $B = (0, b)$ and $P = (7, 7)$. Since $\overline{AB}$ is a diameter, $\angle APB = 90°$, so the vectors $P - A = (7-a, 7)$ and $P - B = (7, 7-b)$ are perpendicular: $7(7-a)+7(7-b) = 0$. Dividing by 7 gives $14 - a - b = 0$, so $OA + OB = a + b = 14$.

Check. A circle through the origin has equation $x^2 + y^2 - ax - by = 0$, where a and b are exactly its intercepts on the axes (put $y = 0$ to get $x = a$, and $x = 0$ to get $y = b$). Putting in $(7, 7)$ gives $98 = 7a + 7b$, so $a + b = 14$ again. For example, $a = 10$ and $b = 4$ gives a circle through all four points. The circle itself is not fixed, but the sum is.

16. Answer: $\frac{5}{3}$**. The idea.** When you roll the sheet, the edges you join stand up as the height of the cylinder, and the other side of the sheet wraps around to become the circumference of the base. The volume depends on the circumference squared but on the height only once, so the long side should go around.

Work it out. If the circumference is C, the radius is $r = \frac{C}{2\pi}$, so the volume is $V = \pi r^2 h = \pi \cdot \frac{C^2}{4\pi^2} \cdot h = \frac{C^2 h}{4\pi}$.

Joining the 6-inch edges: height 6, circumference 10, so $V = \frac{100\cdot 6}{4\pi} = \frac{150}{\pi}$.

Joining the 10-inch edges: height 10, circumference 6, so $V = \frac{36\cdot 10}{4\pi} = \frac{90}{\pi}$.

The ratio is $\frac{150}{90} = \frac{5}{3}$.

Check. Without any π: the ratio is $\frac{C_1^2 h_1}{C_2^2 h_2} = \frac{10^2\cdot 6}{6^2\cdot 10} = \frac{10}{6} = \frac{5}{3}$, which is just the long side over the short side. The larger cylinder is the short, wide one, as predicted.

17. Answer: 20**. The idea.** A percent change multiplies the price by a factor. Going up by x percent multiplies by $1 + \frac{x}{100}$ and going down by x percent multiplies by $1 - \frac{x}{100}$. Their product is a difference of squares, so the net effect is always a loss of exactly $\left(\frac{x}{100}\right)^2$ of the original price, never a return to the start.

Work it out. Write $p = \frac{x}{100}$. The final price is the original times $(1 + p)(1 - p) = 1 - p^2$. Being 4 percent less means the factor is 0.96, so $1 - p^2 = 0.96$, $p^2 = 0.04$, and $p = 0.2$ (it is positive). So $x = 20$.

Check. Take a \$100 coat. A 20 percent raise makes it \$120; 20 percent of \$120 is \$24, so the lowered price is \$96, exactly 4 percent below \$100.

18. Answer: 28**. The idea.** The top corner of the triangle is $(3, 4)$, so its height above the x-axis is 4. The base is the stretch of the x-axis between the two x-intercepts, and each intercept is found by going down 4 units along its line: a line of slope m moves $\frac{4}{|m|}$ units sideways while dropping 4.

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Work it out. The slope-2 line rises to the right, so going down 4 units takes it $\frac{4}{2} = 2$ units to the left: it meets the axis at $x = 1$. The slope-$(-\frac{1}{3})$ line falls to the right, so going down 4 units takes it $4 \cdot 3 = 12$ units to the right: $x = 15$. The base is $15 - 1 = 14$ and the area is $\frac{1}{2} \cdot 14 \cdot 4 = 28$.

Check. The lines are $y = 2x - 2$ and $y = -\frac{1}{3}x + 5$; both give $y = 4$ at $x = 3$, and they are zero at $x = 1$ and $x = 15$. The shoelace formula on $(1, 0)$, $(15, 0)$, $(3, 4)$ gives $\frac{1}{2}|14 \cdot 4| = 28$.

19. Answer: 60**. The idea.** A radius drawn to a point of tangency is perpendicular to the tangent line. So $\angle OAP$ and $\angle OBP$ are right angles, and the quadrilateral splits along $\overline{OP}$ into two right triangles with hypotenuse $OP = 13$ and one leg 5.

Work it out. In right triangle OAP, $PA = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$, so its area is $\frac{1}{2} \cdot 5 \cdot 12 = 30$. Triangle OBP is congruent to it. The quadrilateral has area $2 \cdot 30 = 60$.

Check. The diagonals of this kite are perpendicular, so its area is also $\frac{1}{2} \cdot OP \cdot AB$. The distance from A to line OP is the height of triangle OAP onto its hypotenuse, $\frac{5 \cdot 12}{13} = \frac{60}{13}$, so $AB = \frac{120}{13}$ and the area is $\frac{1}{2} \cdot 13 \cdot \frac{120}{13} = 60$.

20. Answer: 687096**. The idea.** $72 = 8 \cdot 9$ with 8 and 9 sharing no factor, so we need both tests. The test for 8 looks only at the last three digits and involves only B; the test for 9 then fixes A.

Work it out. Divisible by 8: $\overline{5B8} = 508 + 10B$. Since $508 = 8 \cdot 63 + 4$ and $10B \equiv 2B \pmod 8$, we need $4 + 2B \equiv 0 \pmod 8$, i.e. $B \equiv 2 \pmod 4$, so $B = 2$ or $B = 6$ (check: $528 = 8 \cdot 66$, $568 = 8 \cdot 71$). Divisible by 9: the digit sum $3 + 4 + A + 5 + B + 8 = 20 + A + B$ must be a multiple of 9, so $A + B = 7$ or 16. With $B = 2$: $A = 5$ ($A = 14$ is not a digit). With $B = 6$: $A = 1$ ($A = 10$ is not a digit). The numbers are 345528 and 341568, with sum 687096.

Check. $345528 = 72 \cdot 4799$ and $341568 = 72 \cdot 4744$. Their sum is $72 \cdot 9543 = 687096$.

21. Answer: 25**. The idea.** The only way to get past step 5 without landing on it is a 2-stride from step 4 straight to step 6. So every climb splits into three independent pieces: floor to step 4, the forced jump $4 \to 6$, and step 6 to step 10. Multiply the counts.

Work it out. Let w_n be the number of ways to climb n steps with strides of 1 and 2. The last stride is either a 1 (after a climb of $n - 1$) or a 2 (after a climb of $n - 2$), so $w_n = w_{n-1} + w_{n-2}$, with $w_0 = 1$ and $w_1 = 1$. This gives $w_2 = 2$, $w_3 = 3$, $w_4 = 5$.

From the floor to step 4 is a climb of 4 steps: $w_4 = 5$ ways. The jump $4 \to 6$ is forced: 1 way (a 1-stride from step 4 would land on step 5, and a 2-stride from step 3 would too, so a route that reaches step 6 must come from step 4 by a 2-stride). From step 6 to step 10 is again a climb of 4 steps: 5 ways. In total $5 \cdot 1 \cdot 5 = 25$.

Check. Count the complement instead. With no broken step, $w_5 = 8$, $w_6 = 13$, and continuing, $w_{10} = 89$ climbs. A climb that lands on step 5 is a climb of 5 steps followed by a climb of 5 steps: $8 \cdot 8 = 64$. So $89 - 64 = 25$ climbs avoid step 5.

22. Answer: 36**. The idea.** Read $|x - 3| + |x + 2|$ as a distance: it is the distance from x to 3 plus the distance from x to -2 on the number line. That total is never less than the distance between 3 and -2, which is 5. The same holds for the y part, so a total of 10 is only possible if both parts are at their minimum.

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Work it out. If x lies between -2 and 3 (inclusive), the two distances add up to exactly 5. If x is outside that interval, the sum is 5 plus twice the distance by which x overshoots, so it is more than 5. Likewise $|y-1|+|y+4| \geq 5$, with equality exactly when $-4 \leq y \leq 1$.

Since the two parts are each at least 5 and add up to 10, both must equal 5. So $x \in \{-2,-1,0,1,2,3\}$ (6 choices) and $y \in \{-4,-3,-2,-1,0,1\}$ (6 choices), independently, for $6 \cdot 6 = 36$ pairs.

Check. Try $(x,y) = (4,0)$: $1+6+1+4 = 12 \neq 10$, so stepping x just outside its interval breaks the equation, as predicted. And $(3,-4)$ gives $0+5+5+0 = 10$, so the endpoints count.

23. Answer: $\frac{8}{5}$**. The idea.** The tip of the shadow is where the light ray over the person's head reaches the ground. The ray, the post and the person form two similar right triangles sharing the angle at the tip, so the tip's distance from the post is always a fixed multiple of the person's distance. A fixed multiple of a steady speed is a steady speed.

Work it out. Let the person be d metres from the post, with a shadow of length s. The big triangle (lamp top, foot of post, tip) has legs 6 and $d+s$; the small one (head, feet, tip) has legs 1.5 and s. Similar triangles give $\frac{s}{1.5} = \frac{d+s}{6}$, so $6s = 1.5d + 1.5s$, $4.5s = 1.5d$ and $s = \frac{d}{3}$.

The tip is at distance $d+s = \frac{4d}{3}$ from the post. The person's distance grows by 1.2 metres every second, so the tip's distance grows by $\frac{4}{3} \cdot 1.2 = 1.6 = \frac{8}{5}$ metres per second.

Check. After 5 seconds the person is 6 m from the post and the shadow is 2 m long ($\frac{1.5}{2} = \frac{6}{8}$, matching the lamp's rise over the full 8 m run), so the tip is 8 m out: 8 m in 5 s is $\frac{8}{5}$ m/s. Note the shadow itself only grows at 0.4 m/s; the tip also carries the person's own 1.2 m/s.

24. Answer: 44**. The idea.** To use many terms, the terms must be small. The smallest total that n different positive even integers can have is $2+4+\cdots+2n$, so n is too big once that minimum passes 2026. Then check that the largest n allowed really can reach exactly 2026.

Work it out. Any n different positive even integers, listed in increasing order, are at least $2, 4, \ldots, 2n$, so their sum is at least $2(1+2+\cdots+n) = n(n+1)$. Since $44 \cdot 45 = 1980 \leq 2026$ but $45 \cdot 46 = 2070 > 2026$, no sum with 45 or more terms works.

For $n = 44$: $2+4+\cdots+88 = 1980$, which is 46 short. Add 46 to the largest term, turning 88 into 134: the terms $2, 4, \ldots, 86, 134$ are still different, positive and even, and they add up to 2026. So the answer is 44.

Check. Halve every term: the question becomes the most different positive integers adding up to 1013. Since $1+2+\cdots+44 = 990 \leq 1013 < 1035 = 1+2+\cdots+45$, the answer is again 44.

25. Answer: 30**. The idea.** Cut an isosceles triangle along its height: it becomes two congruent right triangles with hypotenuse 17, and the area of the whole triangle equals the area of a rectangle made from them, half-base times height. So we need a second right triangle with hypotenuse 17 whose two legs have the same product as in the first one.

Work it out. In the first triangle the half-base is 8, so the height is $\sqrt{17^2-8^2} = \sqrt{225} = 15$, and the area is $8 \cdot 15 = 120$. Swapping the roles of the legs keeps both the hypotenuse and the product: half-base 15 and height 8. That triangle has legs 17, 17, base 30 and area $15 \cdot 8 = 120$. So the base is 30.

Check. With half-base x the area is $x\sqrt{289-x^2}$. Setting it equal to 120 and squaring gives $x^2(289-x^2) = 14400$, that is, $(x^2-64)(x^2-225) = 0$. So $x = 8$ (the first triangle) or $x = 15$ (the second), and the new base is $2 \cdot 15 = 30$.

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26. Answer: 18**. The idea.** A box is determined by its three edge lengths with order ignored, so list them as $a \le b \le c$ with $abc = 144$. Organise the list by the shortest edge a: since $a^3 \le abc = 144$, the shortest edge is at most 5, and for each a the middle edge satisfies $b^2 \le bc = \frac{144}{a}$.

Work it out. $a = 1$: $bc = 144$ with $b \le 12$, and b a divisor of 144: $b = 1, 2, 3, 4, 6, 8, 9, 12$, which is 8 boxes.

$a = 2$: $bc = 72$ with $2 \le b \le 8$: $b = 2, 3, 4, 6, 8$, which is 5 boxes.

$a = 3$: $bc = 48$ with $3 \le b \le 6$: $b = 3, 4, 6$, which is 3 boxes.

$a = 4$: $bc = 36$ with $4 \le b \le 6$: $b = 4, 6$, which is 2 boxes.

$a = 5$ does not divide 144. Total $8 + 5 + 3 + 2 = 18$.

Check. Count ordered triples instead. With $144 = 2^4 \cdot 3^2$, the four 2s can be shared among three edges in $\binom{6}{2} = 15$ ways and the two 3s in $\binom{4}{2} = 6$ ways, for 90 ordered triples. Triples with two equal edges (a, a, c) need $a^2 \mid 144$, so $a \in \{1, 2, 3, 4, 6, 12\}$: 6 of them, and no triple has all three equal (144 is not a cube). A box with three different edges appears 6 times among the ordered triples and a box with two equal edges 3 times, so there are $\frac{90-3\cdot 6}{6} + 6 = 12 + 6 = 18$ boxes.

27. Answer: $\frac{5}{28}$**. The idea.** Pretend the drawing never stops, so all 8 marbles come out in a random order. The only thing that matters is which 3 of the 8 places in that order hold the red marbles, and all $\binom{8}{3}$ choices of places are equally likely. Exactly 6 draws are needed precisely when the last red marble is in place 6.

Work it out. There are $\binom{8}{3} = 56$ equally likely sets of red places. The last red is in place 6 exactly when place 6 is red and the other two reds are among places 1 to 5: $\binom{5}{2} = 10$ sets. The probability is $\frac{10}{56} = \frac{5}{28}$.

Check. Directly: exactly 6 draws means the first 5 draws contain exactly 2 reds, and then the 6th draw is the last red. The first part has probability $\frac{\binom{3}{2}\binom{5}{3}}{\binom{8}{5}} = \frac{3\cdot 10}{56} = \frac{30}{56}$; then 3 marbles remain, one of them red, so the 6th draw is red with probability $\frac{1}{3}$. The product is $\frac{10}{56} = \frac{5}{28}$. Also, the counts for the last red in places $3, 4, \ldots, 8$ are $\binom{2}{2}, \binom{3}{2}, \ldots, \binom{7}{2} = 1, 3, 6, 10, 15, 21$, which add up to 56, as they must.

28. Answer: 150**. The idea.** Counting the good codes directly means juggling cases, so count all codes and throw away the ones that miss a letter. Be careful: a code that misses two letters gets thrown away twice.

Work it out. There are $3^5 = 243$ codes in all. The codes with no A use only B and C: $2^5 = 32$ of them, and likewise 32 with no B and 32 with no C. The codes AAAAA, BBBBB and CCCCC each miss two letters, so they were subtracted twice and must be added back once. The count is $243 - 3 \cdot 32 + 3 = 150$.

Check. Count by how often each letter appears. Either one letter appears 3 times and the others once (3 choices of that letter, and $\frac{5!}{3!\,1!\,1!} = 20$ orders: 60), or two letters appear twice and one once (3 choices of the single letter, and $\frac{5!}{2!\,2!\,1!} = 30$ orders: 90). Total $60 + 90 = 150$.

29. Answer: $8\pi + 8\sqrt{3}$**. The idea.** Find which parts of the hexagon the circle swallows whole and where its edge crosses the hexagon. The radius AC is a long diagonal of the hexagon's 30-60-90 geometry, and the key fact is that $\overline{CD}$ is tangent to the circle at C (and $\overline{DE}$ at E). So the circle crosses the boundary only at C and E, and the region splits into two triangles and one sector.

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Work it out. Each interior angle of a regular hexagon is 120°. In isosceles triangle ABC (with $AB = BC = 4$ and $\angle ABC = 120°$), the base angles are 30°, and dropping the altitude from B splits it into two 30-60-90 triangles with short leg 2, so $AC = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$. This is the radius.

Angles: $\angle CAE = 120° - 30° - 30° = 60°$ (take away $\angle BAC$ and $\angle FAE$), and $\angle ACD = 120° - 30° = 90°$, so $\overline{CD}$ is perpendicular to the radius $\overline{AC}$: it touches the circle at C and otherwise stays outside it. Likewise $\overline{DE}$ touches the circle at E. Since $AB = AF = 4 < 4\sqrt{3}$, the triangles ABC and AFE lie entirely inside the circle.

So the region is triangle ABC, plus triangle AFE, plus the 60° sector CAE. Each triangle has area $\frac{1}{2} \cdot 4\sqrt{3} \cdot 2 = 4\sqrt{3}$ (base AC, height 2 from B). The sector has area $\frac{60}{360} \cdot \pi(4\sqrt{3})^2 = \frac{48\pi}{6} = 8\pi$. The total is $8\pi + 8\sqrt{3}$.

Check. The whole hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 4^2 = 24\sqrt{3} \approx 41.57$, and our answer is about $25.13 + 13.86 = 38.99$, a bit less. The missing piece is the corner near D, between the arc and the two sides meeting at D: it is kite $ACDE$ minus the sector, $2 \cdot \frac{1}{2} \cdot 4\sqrt{3} \cdot 4 - 8\pi = 16\sqrt{3} - 8\pi \approx 2.58$, and $24\sqrt{3} - (16\sqrt{3} - 8\pi) = 8\pi + 8\sqrt{3}$.

30. Answer: 21**. The idea.** Only the choice of the five tens digits matters. If the tens digits add up to T, the units digits add up to $0 + 1 + \cdots + 9 - T = 45 - T$, so the sum is $10T + (45 - T) = 9T + 45$. Different sums come exactly from different values of T.

Work it out. The tens digits are five different digits from 1 to 9 (0 has to be a units digit), and any such choice works. The smallest T is $1 + 2 + 3 + 4 + 5 = 15$ and the largest is $5 + 6 + 7 + 8 + 9 = 35$. Every value in between also occurs: starting from $\{1, 2, 3, 4, 5\}$, raise the largest digit one step at a time up to 9, then the next largest up to 8, and so on; each step raises T by exactly 1 and ends at $\{5, 6, 7, 8, 9\}$. So T takes the $35 - 15 + 1 = 21$ values $15, \ldots, 35$, and the sum takes 21 values, from 180 to 360 in steps of 9.

Check. The example has tens digits $1, 3, 5, 7, 9$ with $T = 25$, and $9 \cdot 25 + 45 = 270$. Every possible sum is a multiple of 9, as the digit-sum test predicts: all ten digits together add to 45. A computer search over all choices finds exactly 21 different sums.

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