MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

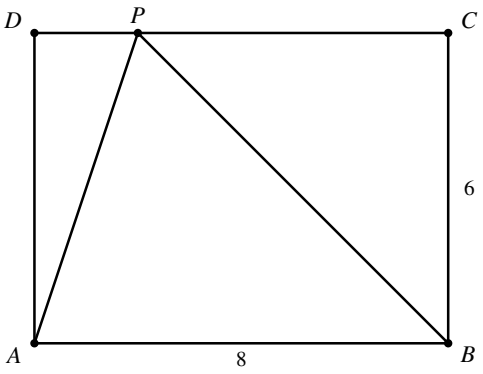
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Ana, Ben and Cal went on a trip together and agreed to split its total cost equally. Along the way Ana paid \$40, Ben paid \$65 and Cal paid \$105. Now they want to settle up so that each of them has paid exactly the same amount, and they want the total amount of money handed from one person to another to be as small as possible. What is that smallest total, in dollars?

1. _____

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2. Rectangle $ABCD$ has $AB = 8$ and $BC = 6$, with $\overline{AB}$ along the bottom and $\overline{DC}$ along the top. A point P is chosen on the top side $\overline{DC}$, and segments $\overline{PA}$ and $\overline{PB}$ are drawn, cutting the rectangle into three triangles: $\triangle APD$, $\triangle APB$ and $\triangle BPC$. The areas of $\triangle APD$ and $\triangle BPC$ are in the ratio $1 : 3$. What is the area of $\triangle APD$?



2. _____

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3. Nine players sit around a table. A deck of 2026 cards is dealt one card at a time, going around the table, with the first card going to Ana, the next to the player on her left, and so on, until the whole deck has been dealt. How many cards does Ana receive?

3. _____

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4. A solid cube with edges 1 metre long is cut into small cubes with edges 1 centimetre long, and all of the small cubes are stacked one on top of another in a single tower. How many kilometres tall is the tower?

4. _____

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5. A school has 16 students for every teacher. The school hires 10 more teachers, and its number of students stays the same. Now it has 12 students for every teacher. How many students does the school have?

5. _____

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6. A grocer stacks oranges in a triangular pyramid. The top layer is 1 orange, the layer under it is a triangle of 3 oranges, the next a triangle of 6, and so on, each layer a complete triangle with one more orange along each side than the layer above. She has 200 oranges and builds the tallest such pyramid she can, with every layer complete. How many oranges are left over?

6. _____

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7. In the expression $1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7$, each of the six boxes is filled with either a + sign or a − sign. (The 1 at the front is always positive.) In how many different ways can the six signs be chosen so that the value of the expression is exactly 0?

7. _____

8. A pizzeria offers 6 different toppings, one of which is mushrooms, and a pizza may have any combination of the toppings, each used at most once. How many different pizzas have mushrooms and an odd number of toppings in all?

8. _____

9. The number $4^{10} \cdot 5^{15} - 1$ is worked out and written in ordinary decimal form, with no exponents. What is the sum of its digits?

9. _____

10. A rectangle's length is twice its width. If its length is decreased by 3 units and its width is increased by 3 units, the new shape is a square. What is the area of the original rectangle, in square units?

10. _____

11. Three real numbers form an arithmetic sequence, so the difference between the second and the first equals the difference between the third and the second. The three numbers add up to 15 and multiply to 80. What is the largest of the three numbers?

11. _____

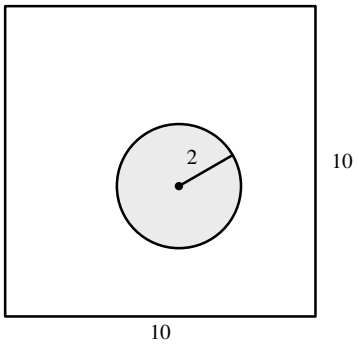
12. A sequence starts with $a_1 = 1$, and each later term is found by the rule $a_{n+1} = a_n + 2n$ for $n = 1, 2, 3, \ldots$. So $a_2 = 1 + 2 = 3$, $a_3 = 3 + 4 = 7$ and $a_4 = 7 + 6 = 13$. What is the value of a_{20}?

12. _____

13. The number $30! = 1 \cdot 2 \cdot 3 \cdots 30$ is divisible by 12^k for some positive integers k. What is the largest such integer k?

13. _____

14. A flat circular disk of radius 2 slides around inside a 10 by 10 square, always staying completely inside the square. What is the total area of the parts of the square that the disk can never cover? Express your answer in terms of π.



14. _____

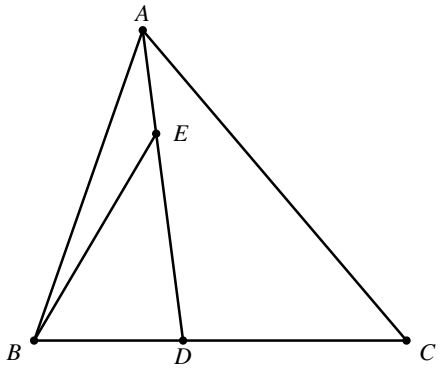
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15. The list $2, 5, 8, 11, \ldots, 2021$ contains every number from 2 to 2021 that is 2 more than a multiple of 3. How many numbers in the list are perfect squares or perfect cubes (of positive integers)?

15. _____

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16. Triangle ABC has area 60. Point D lies on side $\overline{BC}$ so that $BD : DC = 2 : 3$, and point E lies on segment $\overline{AD}$ so that $AE : ED = 1 : 2$. What is the area of triangle BED?



16. _____

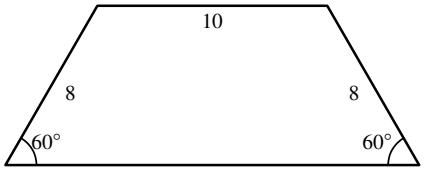
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17. A straight line passes through the point $(4, 3)$ and crosses both the positive x-axis and the positive y-axis. Together with the two axes, it encloses a triangle of area 24. What is the sum of the line's x-intercept and y-intercept?

17. _____

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18. An isosceles trapezoid has two base angles of 60° at its longer base. Each of its two legs has length 8, and its shorter base has length 10. What is the area of the trapezoid? Express your answer in simplest radical form.



18. _____

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19. The Hawks win each game they play against the Owls with probability $\frac{2}{3}$, independently of all other games, and no game ends in a tie. The two teams play a best-of-five series: they play until one team has won 3 games, and that team wins the series. What is the probability that the Hawks win the series? Express your answer as a common fraction.

19. _____

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20. Three fair six-sided dice, one red, one white and one blue, are rolled. What is the probability that the sum of the three numbers rolled is greater than their product? Express your answer as a common fraction.

20. _____

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21. The students in a class averaged 72 points on a test. The 5 students with the highest scores averaged 90 points, and if their 5 scores are removed, the remaining students average 66 points. If instead the 5 lowest scores are removed (and the top 5 kept), the remaining students average 75 points. What is the average score of the 5 students with the lowest scores?

21. _____

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22. A quadratic function $f(x) = ax^2 + bx + c$ takes the same value at $x = 2$ as at $x = 8$. Also, $f(3) = 7$ and $f(4) = 1$. What is $f(10)$?

22. _____

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23. What is the smallest three-digit positive integer that leaves a remainder of 1 when divided by 5, a remainder of 2 when divided by 7, and a remainder of 3 when divided by 9?

23. _____

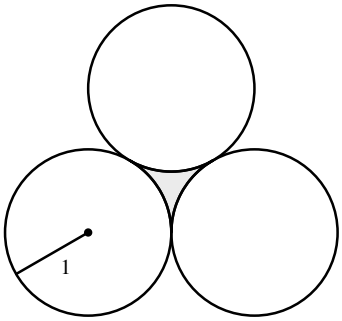
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24. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. There is exactly one positive real number x for which $x \cdot \lfloor x \rfloor = 70$. What is x? Express your answer as a common fraction.

24. _____

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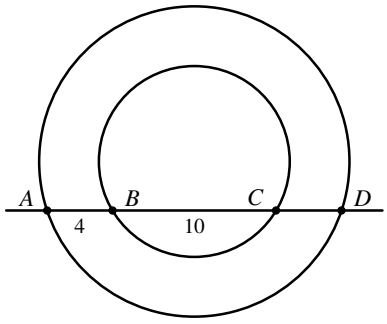
25. Three circles, each of radius 1, lie in a plane so that each circle is tangent to the other two from the outside. The three circles enclose a small curved region between them, bounded by an arc of each circle. What is the area of this region? Express your answer in simplest radical form, in terms of π.



25. _____

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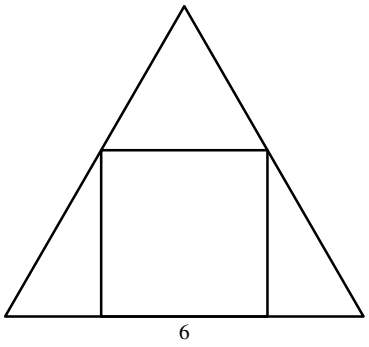
26. Two circles have the same center, and a line crosses both of them. It meets the circles at four points, A, B, C and D in that order along the line, where A and D lie on the larger circle and B and C lie on the smaller one. If $AB = 4$ cm and $BC = 10$ cm, what is the area, in square centimeters, of the ring-shaped region inside the larger circle and outside the smaller one? Express your answer in terms of π.



26. _____

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27. An equilateral triangle has side length 6. A square is drawn inside it so that one side of the square lies along the bottom side of the triangle, and the square's two top corners lie on the other two sides of the triangle. What is the side length of the square? Express your answer in simplest radical form.



27. _____

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28. Three standard six-sided dice are rolled. What is the probability that the three numbers rolled can be the side lengths of a triangle with positive area? (Repeated numbers are allowed as side lengths; for example $2, 2, 3$ works but $1, 2, 3$ does not.) Express your answer as a common fraction.

28. _____

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29. A fair six-sided die is rolled once. Then it is rolled again and again, stopping as soon as it shows a number greater than or equal to the number on the first roll. What is the expected (average) total number of rolls, including the first roll? Express your answer as a common fraction.

29. _____

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30. How many ordered pairs (m, n) of positive integers have least common multiple exactly 60? (Order matters: the pairs $(4, 15)$ and $(15, 4)$ count as two different pairs.)

30. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — MathCounts Chapter Sprint Practice Contest 5

1. 35

2. 6

3. 226

4. 10

5. 480

6. 35

7. 4

8. 16

9. 139

10. 72

11. 8

12. 381

13. 13

14. $16 - 4\pi$

15. 4

16. 16

17. 14

18. $56\sqrt{3}$

19. $\frac{64}{81}$

20. $\frac{19}{216}$

21. 63

22. 49

23. 156

24. $\frac{35}{4}$

25. $\sqrt{3} - \frac{\pi}{2}$

26. 56π

27. $12\sqrt{3} - 18$

28. $\frac{37}{72}$

29. $\frac{69}{20}$

30. 45

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Solutions — MathCounts Chapter Sprint Practice Contest 5

1. Answer: 35**. The idea.** Work out what each person owes or is owed. Money should only flow from people who paid too little to people who paid too much; passing it through a third person only adds to the total.

Work it out. The trip cost $40 + 65 + 105 = 210$ dollars, so each share is \$70. Ana is \$30 short, Ben is \$5 short and Cal is \$35 over. Ana hands Cal \$30 and Ben hands Cal \$5, for a total of $30 + 5 = 35$ dollars.

Check. It cannot be done with less: Cal must receive \$35 more than he hands out, so at least \$35 must reach him. Routing money through Ben (Ana gives Ben \$30, Ben gives Cal \$35) also works, but moves \$65 in all.

2. Answer: 6**. The idea.** The middle triangle APB has base $AB = 8$ and height 6 no matter where P is, so it is always half the rectangle. The two side triangles share the other half.

Work it out. The rectangle has area 48 and $\triangle APB$ has area $\frac{1}{2} \cdot 8 \cdot 6 = 24$. So $\triangle APD$ and $\triangle BPC$ together have area 24. Splitting 24 in the ratio $1 : 3$ gives 6 and 18, so $\triangle APD$ has area 6.

Check. $\triangle APD$ has legs $AD = 6$ and DP, so $3 \cdot DP = 6$ gives $DP = 2$ and $PC = 6$; then $\triangle BPC$ has area $\frac{1}{2} \cdot 6 \cdot 6 = 18$, three times 6.

3. Answer: 226**. The idea.** One trip around the table deals 9 cards, one to each player. Divide 2026 by 9 and then look carefully at who gets the leftover cards.

Work it out. $2026 = 9 \cdot 225 + 1$. After 225 full rounds every player has 225 cards and one card is left. The next card always goes to Ana, who starts each round, so she gets it too: $225 + 1 = 226$.

Check. Ana gets cards number $1, 10, 19, \ldots$, the numbers one more than a multiple of 9. The last of these is $2026 = 9 \cdot 225 + 1$, and from 1 up to 2026 in steps of 9 there are $225 + 1 = 226$ numbers. (Answering 225 forgets the leftover card.)

4. Answer: 10**. The idea.** First count the small cubes: 1 metre is 100 centimetres, so each edge of the big cube holds 100 small cubes, and the cube is 100 by 100 by 100 of them. Each one adds 1 cm to the tower.

Work it out. There are $100^3 = 1{,}000{,}000$ small cubes, so the tower is 1,000,000 cm tall. That is 10,000 m, which is 10 km.

Check. One layer of the big cube has $100 \cdot 100 = 10{,}000$ small cubes and there are 100 layers. Forgetting the third dimension would give only 10,000 cubes, a tower of 100 m = 0.1 km.

5. Answer: 480**. The idea.** Compare the two ratios through the number of students S: the teacher counts are $\frac{S}{16}$ before and $\frac{S}{12}$ after, and they differ by 10.

Work it out. $\frac{S}{12} - \frac{S}{16} = 10$. With the common denominator 48: $\frac{4S-3S}{48} = \frac{S}{48} = 10$, so $S = 480$.

Check. 480 students with 16 per teacher means 30 teachers. With 40 teachers there are $480 \div 40 = 12$ per teacher, and $40 - 30 = 10$ new teachers.

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6. Answer: 35**. The idea.** The layers are the triangular numbers $1, 3, 6, 10, \ldots$, so the pyramid totals are running sums of triangular numbers. Keep a running total until it passes 200.

Work it out. The triangular numbers are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$. The running totals are $1, 4, 10, 20, 35, 56, 84, 120, 165, 220, \ldots$.

A 9-layer pyramid uses 165 oranges, and a 10-layer pyramid would need $220 > 200$. So she builds 9 layers and has $200 - 165 = 35$ oranges left.

Check. The totals are the tetrahedral numbers $\frac{n(n+1)(n+2)}{6}$: for $n = 9$ this is $\frac{9 \cdot 10 \cdot 11}{6} = 165$, and for $n = 10$ it is $\frac{10 \cdot 11 \cdot 12}{6} = 220$. Also, 35 oranges would build a whole extra 5-layer pyramid, but it cannot add a tenth layer, which needs 55.

7. Answer: 4**. The idea.** Split the numbers into a plus group and a minus group. The two groups must have the same total, and all seven numbers add up to $1 + 2 + \cdots + 7 = 28$, so each group must add up to 14. The 1 is in the plus group, so the other numbers in the plus group must add up to 13.

Work it out. We need the sets of different numbers from $\{2, 3, 4, 5, 6, 7\}$ that add up to 13. Four numbers are too many, since even the smallest four give $2 + 3 + 4 + 5 = 14 > 13$. Two numbers: only $6 + 7$. Three numbers: go by the largest one. With 7 the other two add to 6, which gives only $\{2, 4\}$. With largest 6 the other two add to 7 and are at most 5, which gives $\{2, 5\}$ and $\{3, 4\}$. With largest 5 the other two add to 8 and are at most 4, which is impossible. So the plus groups are $\{6, 7\}$, $\{2, 4, 7\}$, $\{2, 5, 6\}$ and $\{3, 4, 6\}$, which is 4 ways.

Check. Take the plus group $\{2, 4, 7\}$: $1 + 2 - 3 + 4 - 5 - 6 + 7 = 0$. Each plus group fixes every sign, so different groups give different sign patterns.

8. Answer: 16**. The idea.** Once mushrooms are on, the pizza is decided by which of the other 5 toppings it has, and the total is odd exactly when an even number of those 5 are chosen. Pair up the choices to see that exactly half of them are even.

Work it out. There are $2^5 = 32$ ways to choose from the other 5 toppings. Pair each choice with the one that differs only in whether it has peppers (say). In each pair one choice has an even number of toppings and the other an odd number, so $32 \div 2 = 16$ choices are even. Adding mushrooms makes each of these an odd total, so there are 16 pizzas.

Check. Count directly: choosing 0, 2 or 4 of the other five toppings gives $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$ pizzas, with 1, 3 or 5 toppings in all.

9. Answer: 139**. The idea.** Every factor $2 \cdot 5$ makes a factor of 10. Pair up the 2s with the 5s, and the product turns into a short number followed by a string of zeros. Subtracting 1 from such a number turns all those zeros into 9s.

Work it out. Since $4 = 2^2$, we have $4^{10} = 2^{20}$. Then $2^{20} \cdot 5^{15} = 2^5 \cdot (2^{15} \cdot 5^{15}) = 32 \cdot 10^{15}$, which is 32 followed by 15 zeros.

Subtracting 1 borrows through all 15 zeros: the result is 31 followed by 15 nines, 31,999,999,999,999,999.

Its digit sum is $3 + 1 + 15 \cdot 9 = 4 + 135 = 139$.

Check. Small case of the same shape: $32 \cdot 10^2 - 1 = 3199$, digit sum $3 + 1 + 9 + 9 = 22 = 4 + 2 \cdot 9$. Common slips: pairing only ten 2s with ten 5s (there are fifteen 5s, so fifteen pairs), or forgetting that the leading 32 becomes 31 (which gives 140).

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10. Answer: 72**. The idea.** Call the width w, so the length is $2w$. After the change the two sides are $2w - 3$ and $w + 3$, and "the new shape is a square" says exactly that these are equal.

Work it out. $2w - 3 = w + 3$ gives $w = 6$, so the rectangle is 12 by 6 and its area is 72 square units.

Check. After the change the sides are $12 - 3 = 9$ and $6 + 3 = 9$, a square indeed. Note that the question asks for the original rectangle; the square's area, 81, is a tempting wrong answer.

11. Answer: 8**. The idea.** Name the numbers around the middle one: $m - d$, m, $m + d$. Then the sum is $3m$, which pins down the middle number at once.

Work it out. $3m = 15$ gives $m = 5$. The product is $(5 - d) \cdot 5 \cdot (5 + d) = 5(25 - d^2) = 80$, so $25 - d^2 = 16$ and $d^2 = 9$. Whether $d = 3$ or $d = -3$, the three numbers are 2, 5 and 8 (in one order or the other). The largest is 8.

Check. $2 + 5 + 8 = 15$, $2 \cdot 5 \cdot 8 = 80$, and the gaps are 3 and 3.

12. Answer: 381**. The idea.** Do not compute twenty terms one at a time. Each step adds $2n$, so a_{20} is the starting value plus all the steps taken to get there, and there are exactly 19 of them.

Work it out. Going from a_1 to a_{20} uses the steps for $n = 1, 2, \ldots, 19$: $a_{20} = 1+2(1+2+\cdots+19) = 1 + 2 \cdot \frac{19 \cdot 20}{2} = 1 + 380 = 381$.

Check. The same reasoning gives $a_n = 1 + (n - 1)n$, and this matches the given terms: $a_2 = 1 + 2 = 3$, $a_3 = 1 + 6 = 7$, $a_4 = 1 + 12 = 13$. Then $a_{20} = 1 + 19 \cdot 20 = 381$. (Summing the steps up to $n = 20$ is the trap; it gives 421.)

13. Answer: 13**. The idea.** Since $12 = 2^2 \cdot 3$, the power $12^k = 2^{2k} \cdot 3^k$ uses up two 2s for every 3. Count how many 2s and 3s are inside 30! and see which one runs out first. It is not necessarily the rarer prime!

Work it out. Factors of 2 in 30!: $\lfloor 30/2 \rfloor + \lfloor 30/4 \rfloor + \lfloor 30/8 \rfloor + \lfloor 30/16 \rfloor = 15 + 7 + 3 + 1 = 26$. Factors of 3: $\lfloor 30/3 \rfloor + \lfloor 30/9 \rfloor + \lfloor 30/27 \rfloor = 10 + 3 + 1 = 14$.

We need $2k \le 26$ and $k \le 14$, so $k \le 13$, and $k = 13$ works. The 2s run out first, even though 30! has more 2s than 3s.

Check. $12^{13} = 2^{26} \cdot 3^{13}$ divides 30!, because 30! contains 2^{26} and 3^{14}, while $12^{14} = 2^{28} \cdot 3^{14}$ would need 28 factors of 2, and 30! has only 26.

14. Answer: $16 - 4\pi$**. The idea.** Follow the center of the disk. It can be anywhere at least 2 from every side, which is the 6 by 6 square in the middle. A point of the big square is covered exactly when some allowed center is within 2 of it. Along the sides this is easy, and the only trouble is in the corners.

Work it out. Any point within 2 of a side, but not in a corner 2 by 2 square, can be covered: slide the disk against that side until the point is under it. Every point of the middle is covered too. Now take the corner 2 by 2 square at the bottom left, and put the origin at the corner of the big square. The closest the center can get is the point $(2, 2)$, so a corner point is covered only if it lies within 2 of $(2, 2)$. The quarter disk of radius 2 about $(2, 2)$ is covered, and the rest, of area $4 - \pi$, is not. With four corners, the uncovered area is $4(4 - \pi) = 16 - 4\pi$.

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Check. Picture the disk pushed into a corner, touching both walls: the bit of the corner square outside its quarter arc is plainly out of reach. Each such piece is $4 - \pi \approx 0.86$, a small sliver, which is sensible. The total is about 3.43 out of 100.

15. Answer: 4. **The idea.** Look at remainders when dividing by 3. Every whole number k has the form $3m$, $3m+1$ or $3m+2$, and that decides the remainder of k^2 and of k^3.

Work it out. Squares: $(3m)^2 = 9m^2$ leaves remainder 0; $(3m+1)^2 = 9m^2 + 6m + 1$ leaves 1; $(3m+2)^2 = 9m^2 + 12m + 4$ leaves 1. So no square leaves remainder 2, and no square is in the list.

Cubes: $(3m)^3$ leaves 0, $(3m+1)^3$ leaves 1 (it is 1 plus multiples of 3), and $(3m+2)^3$ leaves the same remainder as $2^3 = 8$, which is 2. So a cube k^3 is in the list exactly when k is 2 more than a multiple of 3 and $k^3 \le 2021$. Since $12^3 = 1728$ and $13^3 = 2197$, we need $k \le 12$, so $k = 2, 5, 8, 11$. The list contains 4 cubes: 8, 125, 512 and 1331.

Check. $8 = 3 \cdot 2 + 2$, $125 = 3 \cdot 41 + 2$, $512 = 3 \cdot 170 + 2$ and $1331 = 3 \cdot 443 + 2$, all in the list. For squares, the nearby ones 4, 16, 25, 36, 49 leave remainders 1, 1, 1, 0, 1, never 2.

16. Answer: 16. **The idea.** Two triangles that share a vertex and have their bases on the same line have the same height, so their areas are in the ratio of their bases. Use this twice: once with the bases on $\overline{BC}$, once with the bases on $\overline{AD}$.

Work it out. Triangles ABD and ABC both have apex A and bases on line BC, so $[ABD] = \frac{BD}{BC}[ABC] = \frac{2}{5} \cdot 60 = 24$. Now look at triangle ABD from the vertex B: triangles BED and BAD have apex B and bases ED and AD on line AD. Since $AE : ED = 1 : 2$, $ED = \frac{2}{3}AD$, so $[BED] = \frac{2}{3} \cdot 24 = 16$.

Check. The other piece of ABD is $[ABE] = \frac{1}{3} \cdot 24 = 8$, and $8 + 16 = 24$. With coordinates $B = (0,0)$, $C = (10,0)$, $A = (3,12)$ (area 60): $D = (4,0)$, $E = A + \frac{1}{3}(D - A) = (\frac{10}{3}, 8)$, and triangle BED has base $BD = 4$ and height 8, area 16.

17. Answer: 14. **The idea.** Write the line in intercept form $\frac{x}{a} + \frac{y}{b} = 1$. The point gives one equation in a and b, the area gives another, and multiplying out the first turns the system into a single quadratic.

Work it out. The triangle has legs a and b, so $\frac{ab}{2} = 24$ and $ab = 48$. The point $(4,3)$ gives $\frac{4}{a} + \frac{3}{b} = 1$; multiply by ab: $4b + 3a = ab = 48$. Substitute $a = \frac{48}{b}$: $4b + \frac{144}{b} = 48$, so $b^2 - 12b + 36 = 0$, that is $(b-6)^2 = 0$. Hence $b = 6$, $a = 8$, and $a + b = 14$.

Check. The line $\frac{x}{8} + \frac{y}{6} = 1$ passes through $(4,3)$ since $\frac{1}{2} + \frac{1}{2} = 1$, and its triangle has area $\frac{8 \cdot 6}{2} = 24$. The double root is no accident: $(4,3)$ is the midpoint of the segment from $(8,0)$ to $(0,6)$, and 24 is the smallest area any line through $(4,3)$ can cut off, so only one line achieves it.

18. Answer: $56\sqrt{3}$. **The idea.** Drop perpendiculars from the two ends of the shorter base to the longer base. They cut off a rectangle in the middle and two congruent right triangles at the ends, and each of those is a 30-60-90 triangle whose hypotenuse is a leg of the trapezoid. That one triangle gives both the height and the length of the longer base.

Work it out. In a 30-60-90 triangle with hypotenuse 8, the side opposite the 30° angle is half the hypotenuse, 4, and the side opposite the 60° angle is $4\sqrt{3}$. At a 60° base angle, the horizontal piece along the longer base is the side next to that angle, 4, and the height is $4\sqrt{3}$. So the longer base is $4 + 10 + 4 = 18$, and the area is $\frac{10+18}{2} \cdot 4\sqrt{3} = 14 \cdot 4\sqrt{3} = 56\sqrt{3}$.

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Check. Cut the trapezoid into the $10 \times 4\sqrt{3}$ rectangle (area $40\sqrt{3}$) and two right triangles with legs 4 and $4\sqrt{3}$ (area $8\sqrt{3}$ each): $40\sqrt{3} + 16\sqrt{3} = 56\sqrt{3}$. A common slip is to swap the legs of the 30-60-90 triangle, which would give a height of 4 and a longer base of $10 + 8\sqrt{3}$.

19. Answer: $\frac{64}{81}$. **The idea.** A series stops early, which makes its cases awkward (3 games, 4 games or 5 games). Instead, pretend all five games are always played, even after the series is decided. The Hawks win the series exactly when they win at least 3 of the 5 imagined games, because once a team reaches 3 wins the other team can no longer reach 3.

Work it out. With $p = \frac{2}{3}$ and $q = \frac{1}{3}$, the chance of exactly k wins in 5 games is $\binom{5}{k} p^k q^{5-k} = \binom{5}{k} \frac{2^k}{243}$. For $k = 3, 4, 5$: $\frac{10 \cdot 8 + 5 \cdot 16 + 1 \cdot 32}{243} = \frac{80+80+32}{243} = \frac{192}{243} = \frac{64}{81}$.

Check. Count the real series directly. Win in 3: $p^3 = \frac{8}{27}$. Win in 4: the Hawks lose exactly one of the first three and win game 4: $3p^3q = \frac{24}{81} = \frac{8}{27}$. Win in 5: they split the first four 2-2 and win game 5: $6p^3q^2 = \frac{48}{243} = \frac{16}{81}$. Total $\frac{24+24+16}{81} = \frac{64}{81}$.

20. Answer: $\frac{19}{216}$. **The idea.** Products grow much faster than sums, so the sum can only win when the dice are very small. Sort the three numbers as $a \le b \le c$ and pin down exactly which small cases work, then count the orders in which each can appear.

Work it out. If the smallest die a is at least 2, then $abc \ge 2 \cdot 2 \cdot c = 4c > 3c \ge a + b + c$, so the product wins. Hence $a = 1$, and we need $1 + b + c > bc$, that is, $bc - b - c + 1 < 2$, or $(b-1)(c-1) < 2$.

Since $b - 1$ and $c - 1$ are whole numbers, $(b-1)(c-1)$ is 0 or 1: either $b = 1$ (and c is anything from 1 to 6), or $b = c = 2$. So the sorted rolls that work are $\{1, 1, 1\}$, $\{1, 1, c\}$ for $c = 2, \ldots, 6$, and $\{1, 2, 2\}$.

Count the ordered outcomes on the three coloured dice: $\{1, 1, 1\}$ gives 1; each $\{1, 1, c\}$ gives 3, for $5 \cdot 3 = 15$; $\{1, 2, 2\}$ gives 3. Total $1 + 15 + 3 = 19$ out of $6^3 = 216$, so the probability is $\frac{19}{216}$.

Check. Test the edges: $(1, 2, 2)$ gives $5 > 4$, yes; $(1, 2, 3)$ gives $6 = 6$, a tie, so no; $(1, 1, 6)$ gives $8 > 6$, yes; $(2, 2, 2)$ gives $6 < 8$, no. Forgetting that the dice are distinguishable (counting $\{1, 1, c\}$ once instead of 3 times) would give a wrong total of 7.

21. Answer: 63. **The idea.** First find how many students there are by balancing distances from the class average 72: the amount the top group sits above 72 must be cancelled by the amount everyone else sits below it. Once the class size is known, the second removal gives the total of the bottom 5.

Work it out. Each of the top 5 is on average 18 points above 72, for a surplus of $5 \cdot 18 = 90$ points. The other students are on average 6 points below 72, so there must be $90/6 = 15$ of them, and the class has 20 students. The class total is $20 \cdot 72 = 1440$. Removing the bottom 5 leaves 15 students with total $15 \cdot 75 = 1125$, so the bottom 5 scored $1440 - 1125 = 315$ points, an average of $\frac{315}{5} = 63$.

Check. Top 5: 450 points; the other 15: $15 \cdot 66 = 990$; $450 + 990 = 1440$. The bottom 5 (315) plus the middle 10 ($990 - 315 = 675$, average 67.5) plus the top 5 (450) is again 1440, and the averages $63 < 67.5 < 90$ are in a sensible order, so such a class can exist.

22. Answer: 49. **The idea.** The graph of a quadratic is a parabola, symmetric about a vertical line through its vertex. Since $f(2) = f(8)$, that line is halfway between, at $x = 5$. So write

$f(x) = a(x-5)^2 + k$, which leaves only two unknowns for the two given values.

Work it out. $f(3) = a(-2)^2 + k = 4a + k = 7$ and $f(4) = a(-1)^2 + k = a + k = 1$. Subtracting, $3a = 6$, so $a = 2$ and $k = -1$. Then $f(10) = 2 \cdot 5^2 - 1 = 49$.

Check. Expanded, $f(x) = 2x^2 - 20x + 49$; indeed $f(2) = 8 - 40 + 49 = 17$ and $f(8) = 128 - 160 + 49 = 17$, $f(3) = 18 - 60 + 49 = 7$, $f(4) = 32 - 80 + 49 = 1$. By symmetry $f(10) = f(0) = c = 49$, which matches.

23. Answer: 156**. The idea.** Satisfy the conditions one at a time. Once a number works for two of the divisors, adding the least common multiple of those two divisors keeps it working for both, so the candidates form a short list that steps by that multiple.

Work it out. Numbers leaving remainder 3 when divided by 9: $3, 12, 21, 30, \ldots$. Test these for remainder 2 on division by 7: $3 \to 3$, $12 \to 5$, $21 \to 0$, $30 \to 2$. So 30 works for both, and since $\operatorname{lcm}(9, 7) = 63$, all the numbers that work for both are $30, 93, 156, 219, \ldots$. A remainder of 1 on division by 5 means the last digit is 1 or 6: 30 no, 93 no, 156 yes. The next numbers that work for all three are $156 + 315k$, and $156 - 315 < 0$, so 156 is the smallest positive one and it has three digits.

Check. $156 = 5 \cdot 31 + 1 = 7 \cdot 22 + 2 = 9 \cdot 17 + 3$. The only smaller number meeting all three conditions would be $156 - 315$, which is negative, so no three-digit number below 156 works.

24. Answer: $\frac{35}{4}$**. The idea.** Name the integer part: if $\lfloor x \rfloor = n$, then $x = 70/n$, and the only question is which n is consistent with $n \le x < n + 1$.

Work it out. Since $x > 0$ and $x \cdot n = 70 > 0$, n is a positive integer. We need $n \le \frac{70}{n} < n + 1$, that is, $n^2 \le 70 < n^2 + n$. The first inequality gives $n \le 8$ (as $8^2 = 64 \le 70 < 81$). The second needs $n^2 + n > 70$: $7^2 + 7 = 56$ is too small but $8^2 + 8 = 72$ works. So $n = 8$ and $x = \frac{70}{8} = \frac{35}{4}$.

Check. $\frac{35}{4} = 8.75$, whose floor is 8, and $8.75 \cdot 8 = 70$. Note that $x\lfloor x \rfloor$ increases with x (it jumps from about $n^2 + n$ up to $(n+1)^2$ at each integer), so it cannot hit 70 twice.

25. Answer: $\sqrt{3} - \frac{\pi}{2}$**. The idea.** Join the three centers. Two tangent circles of radius 1 have centers 2 apart, so the centers form an equilateral triangle of side 2. The triangle is made of the curved gap plus three pie slices, one from each circle, and each slice has the triangle's 60° angle at its center.

Work it out. The equilateral triangle of side 2 has area $\frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}$. Each slice is a 60° sector of a circle of radius 1, that is $\frac{60}{360} = \frac{1}{6}$ of a disk of area π, so the three slices together have area $3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$. The gap is what is left: $\sqrt{3} - \frac{\pi}{2}$.

Check. Numerically $\sqrt{3} \approx 1.732$ and $\frac{\pi}{2} \approx 1.571$, so the gap is about 0.16, a small positive area, as the picture of three touching coins suggests. The three sectors together are exactly half a coin, since their angles add to 180°.

26. Answer: 56π**. The idea.** The ring's area is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$, so we never need the radii themselves, or even how far the line is from the center: only the difference of the squares of the radii. Drop a perpendicular from the center to the line; it bisects both chords at once, and two right triangles sharing a leg hand us $R^2 - r^2$.

Work it out. Let O be the center and M the foot of the perpendicular from O to the line. A perpendicular from the center bisects every chord, so M is the midpoint of $\overline{BC}$ and also of $\overline{AD}$.

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Hence $MB = MC = 5$, and then $CD = AB = 4$, so $MA = MD = 5 + 4 = 9$. With $d = OM$, the right triangles OMA and OMB give $R^2 = d^2 + 9^2$ and $r^2 = d^2 + 5^2$. Subtracting, $R^2 - r^2 = 81 - 25 = 56$, whatever d is. The ring's area is 56π square cm.

Check. Power of a point: B is inside the large circle at distance r from its center, so $BA \cdot BD = R^2 - OB^2 = R^2 - r^2$, and $BA \cdot BD = 4 \cdot 14 = 56$. Or take the line through the center ($d = 0$): then $r = 5$, $R = 9$ and $81\pi - 25\pi = 56\pi$.

27. Answer: $12\sqrt{3} - 18$. **The idea.** The top side of the square is parallel to the bottom of the triangle, so it cuts off a smaller triangle at the top that is also equilateral. That small triangle has side s, the side of the square. Stacking heights, the height of the big triangle equals the square's height plus the small triangle's height.

Work it out. An equilateral triangle of side t has height $\frac{\sqrt{3}}{2}t$. The big triangle has height $3\sqrt{3}$, the small one $\frac{\sqrt{3}}{2}s$, so $s + \frac{\sqrt{3}}{2}s = 3\sqrt{3}$, that is $s \cdot \frac{2+\sqrt{3}}{2} = 3\sqrt{3}$ and $s = \frac{6\sqrt{3}}{2+\sqrt{3}}$. Multiply top and bottom by $2 - \sqrt{3}$ (and note $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$): $s = 6\sqrt{3}(2 - \sqrt{3}) = 12\sqrt{3} - 18$.

Check. Work along the bottom instead. Each corner triangle beside the square is a 30-60-90 triangle with the 60° angle at the triangle's corner, legs $\frac{s}{\sqrt{3}}$ (along the bottom) and s (the square's side). So $\frac{s}{\sqrt{3}} + s + \frac{s}{\sqrt{3}} = 6$, giving $s(\sqrt{3} + 2) = 6\sqrt{3}$, the same equation. Numerically $s \approx 20.785 - 18 = 2.785$, less than the height 5.196, as it must be.

28. Answer: $\frac{37}{72}$. **The idea.** Three lengths form a triangle with positive area exactly when the largest is *less* than the sum of the other two. So sort by the largest number c and, for each c, count the sorted pairs $a \le b \le c$ with $a + b > c$. Then weight each sorted triple by how many ordered rolls produce it: 6 if all three differ, 3 if exactly two are equal, 1 if all equal.

Work it out. $c = 1$: $(1,1,1)$, weight 1. $c = 2$: $(1,2,2), (2,2,2)$: $3 + 1 = 4$. $c = 3$: $(1,3,3), (2,2,3), (2,3,3), (3,3,3)$: 3+3+3+1 = 10. $c = 4$: $(1,4,4), (2,3,4), (2,4,4), (3,3,4), (3,4,4), (4,4,4)$: 3+6+3+3+3+1 = 19. $c = 5$: $(1,5,5), (2,4,5), (2,5,5), (3,3,5), (3,4,5), (3,5,5), (4,4,5), (4,5,5), (5,5,5)$: 3+6+3+3+6+3+3+3+1 = 31. $c = 6$: $(1,6,6), (2,5,6), (2,6,6), (3,4,6), (3,5,6), (3,6,6), (4,4,6), (4,5,6), (4,6$ $3 + 6 + 3 + 6 + 6 + 3 + 3 + 6 + 3 + 3 + 3 + 1 = 46$. Total $1 + 4 + 10 + 19 + 31 + 46 = 111$ good rolls out of 216, so the probability is $\frac{111}{216} = \frac{37}{72}$.

Check. Count the failures instead: $a + b \le c$. For largest value c the failing sorted pairs (a, b) with $a + b \le c$, $a \le b$: $c = 2$: $(1,1)$; $c = 3$: $(1,1), (1,2)$; $c = 4$: $(1,1), (1,2), (1,3), (2,2)$; and so on. Weighted, the failures come to $0 + 3 + 9 + 18 + 30 + 45 = 105$, and $111 + 105 = 216$. Also each row total plus its failures equals $c^3 - (c-1)^3$ (e.g. $46 + 45 = 91 = 216 - 125$).

29. Answer: $\frac{69}{20}$. **The idea.** Split on the first roll v. Once v is known, every later roll is a fresh try that ends the game with the same probability $p = \frac{7-v}{6}$ (it must show one of the $7 - v$ numbers $v, v+1, \ldots, 6$). Waiting for an event of probability p takes $\frac{1}{p}$ tries on average: if W is that average, the next try is always made and fails with probability $1 - p$, sending us back to the start, so $W = 1 + (1 - p)W$ and $W = \frac{1}{p}$.

Work it out. After a first roll of v, the expected number of further rolls is $\frac{6}{7-v}$: for $v = 1, 2, 3, 4, 5, 6$ that is $\frac{6}{6}, \frac{6}{5}, \frac{6}{4}, \frac{6}{3}, \frac{6}{2}, \frac{6}{1}$. Each first roll has probability $\frac{1}{6}$, so the expected number of further rolls is $\frac{1}{6} \cdot 6\left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right) = \frac{10+12+15+20+30+60}{60} = \frac{147}{60} = \frac{49}{20}$. Adding the first roll, the expected total is $1 + \frac{49}{20} = \frac{69}{20}$.

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Check. The answer 3.45 is sensible: a first roll of 1 always stops after 2 rolls, while a first roll of 6 needs on average 6 more rolls, and that bad case pulls the average well above 2. A tempting shortcut is wrong: the average stopping chance is $\frac{1}{6}\left(\frac{6+5+4+3+2+1}{6}\right) = \frac{7}{12}$, but $1 + \frac{12}{7} = \frac{19}{7}$ is too small, because the average of $\frac{1}{p}$ is not $\frac{1}{\text{average of } p}$ (the rare first roll of 6 costs a lot).

30. Answer: 45**. The idea.** Both numbers must divide $60 = 2^2 \cdot 3 \cdot 5$, so each is $2^x 3^y 5^z$ with small exponents. The least common multiple takes, for each prime, the larger of the two exponents. So the condition splits into one separate condition for each prime, and the choices multiply.

Work it out. For the prime 2: the two exponents are each 0, 1 or 2 and the larger must be 2. Of the $3 \cdot 3 = 9$ pairs of exponents, the $2 \cdot 2 = 4$ with both exponents at most 1 fail, leaving 5. For the prime 3: exponents 0 or 1 with larger exponent 1: $4 - 1 = 3$ pairs. For the prime 5: likewise 3. In all, $5 \cdot 3 \cdot 3 = 45$ ordered pairs.

Check. Try the same method on $12 = 2^2 \cdot 3$: it predicts $5 \cdot 3 = 15$ pairs. Listing them: $(12, d)$ and $(d, 12)$ for the six divisors d of 12 give 11 pairs, and the pairs without a 12 need a 4 and a multiple of 3 other than 12: $(4, 3)$, $(4, 6)$, $(3, 4)$, $(6, 4)$. That is 15. For 60 a computer count over all pairs up to 60 also gives 45.

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