MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

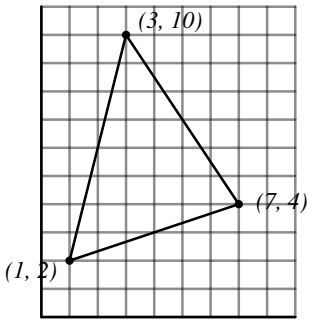
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. At a certain moment between noon and midnight of the same day, the time that has passed since noon is exactly one third of the time that remains until midnight. How many minutes after noon is it?

1. _____

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2. On a coordinate grid where each unit is one meter, a triangular garden bed has its corners at $(1, 2)$, $(7, 4)$ and $(3, 10)$. What is the area of the garden bed, in square meters?



2. _____

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3. At a supermarket, identical shopping carts are pushed together in a straight row, each cart nesting into the one in front of it. A single cart is 1.2 m long, and every additional cart in the row adds 0.3 m to the length of the row. One row of carts is exactly 9 m long. How many carts are in this row?

3. _____

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4. A club has 7 members, one of whom is Ana. The club chooses a committee of 3 of its members and names one of those three as the committee's chair. Ana has one condition: if she is on the committee, then she must be its chair. In how many ways can the committee and its chair be chosen?

4. _____

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5. On a calculator with a very long display, Jordan divides two whole numbers and sees $0.1363636\ldots$, where the block 36 repeats forever after the first digit 1. What is this number? Express your answer as a common fraction in lowest terms.

5. _____

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6. A square, a regular hexagon and a regular polygon with n sides are laid flat so that each of them has a vertex at the same point P. Together, their three angles at P fill the whole space around P exactly, with no gaps and no overlaps. What is n?

6. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

7. The eleven consecutive integers from 2016 to 2026 are multiplied together: $2016 \times 2017 \times 2018 \times \cdots \times 2026$. How many zeros are at the end of this product? 7. _____

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8. Some real numbers x make the two sides of the equation $(x-3)(x+5) = (x-3)(2x-1)$ equal. What is the sum of all such real numbers x? 8. _____

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9. Ann, Bo and Cy share a prize of \$200. Bo gets exactly twice as much as Cy, and Ann gets exactly \$10 more than Bo. All of the prize money is handed out. How many dollars does Ann get? 9. _____

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10. Find the sum of all prime numbers p for which both $p+10$ and $p+14$ are also prime. 10. _____

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11. During a storm, 2.5 cm of rain falls evenly on a flat rectangular roof that measures 12 m by 16 m. All of this water runs into an empty tank shaped like a rectangular box whose base measures 2 m by 3 m, and no water is lost. How many centimeters deep is the water in the tank? 11. _____

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12. A perfect square is the square of a whole number, such as 1, 4 or 9. How many perfect squares from 1 to 1000, inclusive, are multiples of 12? 12. _____

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13. A ribbon 62 cm long is cut into three pieces. Twice the length of the first piece, three times the length of the second piece and five times the length of the third piece are all equal. How many centimeters longer is the first piece than the third piece? 13. _____

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14. A bug starts at 0 on a number line. Every second it moves exactly 1 unit, either left or right, each with probability $\frac{1}{2}$, independently of its earlier moves. What is the probability that the bug is at 0 again at the end of 6 seconds? Express your answer as a common fraction. 14. _____

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15. A digital clock shows hours and minutes in 24-hour form, from 00:00 up to 23:59, always with four digits (for example 07:05 or 19:40). During how many of the 1440 minutes of a day does the display contain at least one digit 7? 15. _____

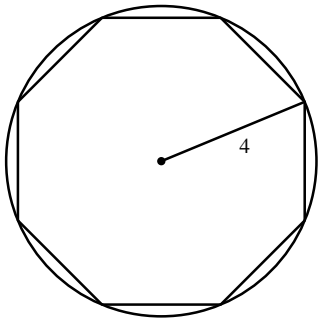
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16. Let $x = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}$ and $y = \sqrt{20 - \sqrt{20 - \sqrt{20 - \cdots}}}$, where each pattern of nested square roots goes on forever. (Each is the number that the finite versions $\sqrt{20}$, $\sqrt{20 \pm \sqrt{20}}$, $\sqrt{20 \pm \sqrt{20 \pm \sqrt{20}}}$, ... get closer and closer to.) What is the value of $x + y$?

16. _____

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17. A regular octagon is drawn so that all eight of its vertices lie on a circle of radius 4 units. What is the area of the octagon, in square units? Express your answer in simplest radical form.



17. _____

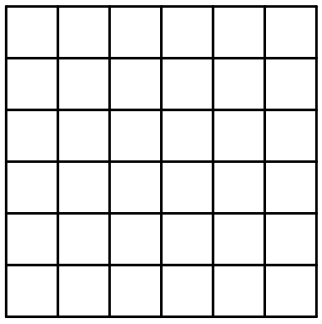
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18. Consider every fraction $\frac{a}{b}$ where a and b are whole numbers with $1 \leq a < b \leq 10$. There are 45 such fractions, and fractions with equal values are all included separately (for example, both $\frac{1}{2}$ and $\frac{2}{4}$ are in the list). What is the sum of all 45 fractions? Express your answer as a common fraction.

18. _____

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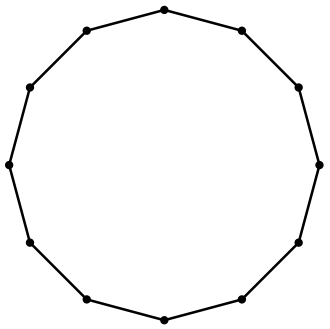
19. A 6 by 6 grid of unit squares is drawn. How many different rectangles with area exactly 12 square units can be traced along the lines of the grid? (Rectangles in different positions count as different, and squares count as rectangles.)



19. _____

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20. Consider the 12 vertices of a regular 12-sided polygon. How many rectangles that are not squares have all four of their vertices among these 12 points?



20. _____

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21. Consider the equation $(x^2 - 6x + 7)^2 = 4$. What is the sum of all the different real numbers x that satisfy it? (A solution that appears more than once is counted only once.)

21. _____

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22. The number $360^2 = 129{,}600$ has many positive divisors, including 1, 2, 360 and 129,600 itself. How many of the positive divisors of 360^2 are less than 360?

22. _____

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23. The product of four consecutive positive integers is 57,120. For example, if the integers were $2, 3, 4, 5$, the product would be 120. What is the sum of the four integers?

23. _____

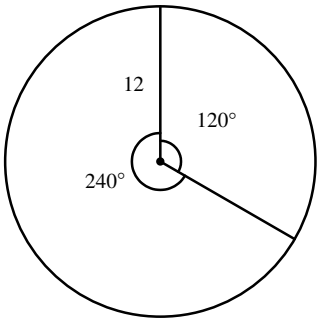
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24. A right circular cylinder is inscribed in a sphere of radius 5 cm, so that both circular edges of the cylinder lie on the sphere. The curved (lateral) surface of the cylinder has area 48π square centimeters, and the cylinder is taller than it is wide (its height is greater than its diameter). What is the volume of the cylinder, in cubic centimeters? Express your answer in terms of π.

24. _____

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25. A paper circle of radius 12 is cut along two radii into two sectors: one with a central angle of 240° and one with a central angle of 120°. Each sector is rolled into the curved surface of a cone (with no base) by joining its two straight edges, without overlap. What is the volume of the larger cone divided by the volume of the smaller cone? Express your answer in simplest radical form.



25. _____

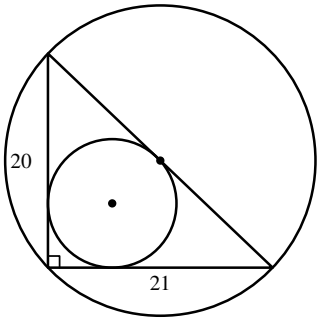
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26. The pages of a book are numbered 1, 2, 3, . . . up to the last page. Each sheet of paper carries two consecutive page numbers: an odd number on its front and the next even number on its back (so the sheets hold pages 1 and 2, pages 3 and 4, and so on). One sheet, with both of its pages numbered, is torn out of the book. The page numbers still in the book add up to 2026. What is the page number on the front of the torn sheet?

26. _____

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27. A right triangle has legs of lengths 20 and 21. What is the distance between the center of the circle inscribed in the triangle and the center of the circle that passes through all three of its vertices? Express your answer in simplest radical form.



27. _____

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28. Three friends each choose a whole number from 1 to 10 inclusive, independently and uniformly at random; they may choose the same number. When the three numbers are listed from smallest to largest, the middle one is the median. What is the probability that the median is exactly 5? Express your answer as a common fraction.

28. _____

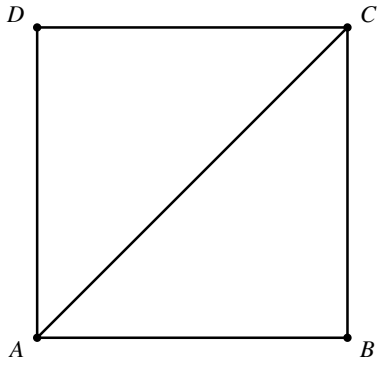
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29. Eight friends sit evenly spaced around a round table, and each one flips a fair coin once. A friend is called lonely if his or her coin shows a different face from the coins of both neighbors at the table. What is the probability that no one at the table is lonely? Express your answer as a common fraction.

29. _____

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30. A point P is chosen uniformly at random from the interior of square $ABCD$. What is the probability that P is closer to side $\overline{AB}$ than to the diagonal $\overline{AC}$? Express your answer as a common fraction in simplest radical form.



30. _____

Answer Key — MathCounts Chapter Sprint Practice Contest 4

1. 180

2. 22

3. 27

4. 75

5. $\frac{3}{22}$

6. 12

7. 3

8. 9

9. 86

10. 3

11. 80

12. 5

13. 18

14. $\frac{5}{16}$

15. 252

16. 9

17. $32\sqrt{2}$

18. $\frac{45}{2}$

19. 34

20. 12

21. 9

22. 52

23. 62

24. 72π

25. $\sqrt{10}$

26. 59

27. $\frac{\sqrt{145}}{2}$

28. $\frac{37}{250}$

29. $\frac{23}{128}$

30. $\frac{\sqrt{2}-1}{2}$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Chapter Sprint Practice Contest 4

1. Answer: 180**. The idea.** The trap is to cut the 12 hours into thirds. "One third of what remains" means the time passed is 1 part and the time remaining is 3 parts, so the 12 hours split into 1+3 = 4 equal parts.

Work it out. Let x be the number of hours since noon; then $12 - x$ hours remain, and $x = \frac{12-x}{3}$. Multiply by 3: $3x = 12 - x$, so $4x = 12$ and $x = 3$ hours, which is $3 \cdot 60 = 180$ minutes.

Check. At 3:00 p.m., 3 hours have passed and 9 hours remain until midnight, and 3 is one third of 9. (At 4:00 p.m. the ratio would be 4 to 8, a half, not a third.)

2. Answer: 22**. The idea.** A slanted triangle is hard to measure directly, but it fits snugly inside a rectangle with sides along the grid lines; the leftover pieces are right triangles whose areas are easy.

Work it out. The x-values run from 1 to 7 and the y-values from 2 to 10, so the enclosing rectangle is 6×8, area 48. Cut off three right triangles: between $(1, 2)$ and $(7, 4)$: legs 6 and 2, area 6; between $(7, 4)$ and $(3, 10)$: legs 4 and 6, area 12; between $(3, 10)$ and $(1, 2)$: legs 2 and 8, area 8. The garden is $48 - 6 - 12 - 8 = 22$ square meters.

Check. The shoelace formula: $\frac{1}{2}\,|1(4 - 10) + 7(10 - 2) + 3(2 - 4)| = \frac{1}{2}|-6 + 56 - 6| = 22$.

3. Answer: 27**. The idea.** The first cart is different from the others: it contributes its full 1.2 m, while every cart after it adds only the 0.3 m that sticks out behind the cart it nests into. So set the first cart aside, then count how many 0.3 m steps fit in what is left.

Work it out. Beyond the first cart the row extends $9 - 1.2 = 7.8$ m. Each extra cart adds 0.3 m, so there are $7.8 \div 0.3 = 26$ extra carts. With the first cart, the row has $26 + 1 = 27$ carts.

Check. A row of n carts is $1.2 + 0.3(n - 1)$ m long; for $n = 27$ this is $1.2 + 0.3 \cdot 26 = 1.2 + 7.8 = 9$ m. The tempting $9 \div 0.3 = 30$ forgets that the first cart is longer, and $9 \div 1.2$ ignores the nesting altogether.

4. Answer: 75**. The idea.** Split on whether Ana is on the committee. Without her there is no restriction; with her, the chair is already decided.

Work it out. Committees without Ana: choose 3 of the other 6 members in $\binom{6}{3} = 20$ ways, then choose any of the 3 as chair: $20 \cdot 3 = 60$. Committees with Ana: she is the chair, and the other two members come from the remaining 6 in $\binom{6}{2} = 15$ ways. Total: $60 + 15 = 75$.

Check. Without the condition there are $\binom{7}{3} \cdot 3 = 105$ choices. The forbidden ones have Ana on the committee but someone else as chair: 15 committees containing Ana, each with 2 wrong chairs, giving 30. And $105 - 30 = 75$.

5. Answer: $\frac{3}{22}$**. The idea.** Shift the decimal so that the repeating parts line up, then subtract to kill the endless tail.

Work it out. Let $x = 0.1363636\ldots$. Then $10x = 1.363636\ldots$ and $1000x = 136.363636\ldots$. Both have the same tail $.3636\ldots$, so $1000x - 10x = 136 - 1 = 135$, that is $990x = 135$ and $x = \frac{135}{990} = \frac{3}{22}$.

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Check. Split it instead: $0.1 + 0.0363636\ldots = \frac{1}{10} + \frac{36}{990} = \frac{1}{10} + \frac{2}{55} = \frac{11}{110} + \frac{4}{110} = \frac{15}{110} = \frac{3}{22}$. And long division $3 \div 22 = 0.13636\ldots$ confirms it.

6. Answer: 12**. The idea.** The angles around a point add up to 360°, so the unknown polygon's interior angle is whatever the square and hexagon leave over. Then the exterior angle (which is 180° minus the interior angle, and $360° \div n$ for a regular n-gon) gives n quickly.

Work it out. The square's angle is 90° and the regular hexagon's is 120°, so the third angle is $360° - 90° - 120° = 150°$. Its exterior angle is $180° - 150° = 30°$, and the exterior angles of a regular polygon add up to 360°, so $n = 360 \div 30 = 12$.

Check. A regular 12-gon has angle sum $(12-2)\cdot 180° = 1800°$, so each angle is $1800° \div 12 = 150°$, and $90 + 120 + 150 = 360$.

7. Answer: 3**. The idea.** Each zero at the end is a factor of $10 = 2 \cdot 5$. Among these numbers there are far more factors of 2 than of 5, so the number of zeros equals the number of factors of 5 in the product.

Work it out. The only multiples of 5 from 2016 to 2026 are 2020 and 2025. Now $2020 = 5 \cdot 404$ contributes one 5, and $2025 = 25 \cdot 81 = 5^2 \cdot 81$ contributes two. That is 3 factors of 5. The even numbers $2016, 2018, \ldots, 2026$ supply many more than 3 factors of 2 (already $2016 = 2^5 \cdot 63$). So the product ends in exactly 3 zeros.

Check. The trap is to count only the two multiples of 5 and answer 2; the square $2025 = 45^2$ carries an extra 5.

8. Answer: 9**. The idea.** It is tempting to divide both sides by $x - 3$, but that is only allowed when $x - 3 \neq 0$, and it silently throws away the solution $x = 3$. Instead, move everything to one side and factor.

Work it out. $(x-3)(x+5) - (x-3)(2x-1) = 0$ becomes $(x-3)\big[(x+5) - (2x-1)\big] = 0$, that is $(x-3)(6-x) = 0$. So $x = 3$ or $x = 6$, and the sum is $3 + 6 = 9$.

Check. At $x = 3$ both sides are 0. At $x = 6$ the left side is $3 \cdot 11 = 33$ and the right side is $3 \cdot 11 = 33$. Someone who divided by $x - 3$ would find only $x = 6$ and answer 6.

9. Answer: 86**. The idea.** Everything is described in terms of Cy's share, so measure the whole prize in "Cy-shares". Bo's share is two of them and Ann's is two of them plus \$10.

Work it out. If Cy gets c dollars, Bo gets $2c$ and Ann gets $2c + 10$. Together $c + 2c + 2c + 10 = 5c + 10 = 200$, so $5c = 190$ and $c = 38$. Then Bo gets 76 and Ann gets $76 + 10 = 86$ dollars.

Check. $86+76+38 = 200$, Bo's 76 is twice Cy's 38, and Ann's 86 is 10 more than Bo's. Splitting \$200 in the ratio 2 : 2 : 1 first and adding the \$10 afterwards would overspend the prize.

10. Answer: 3**. The idea.** Look at remainders when dividing by 3. Since 10 leaves remainder 1 and 14 leaves remainder 2, the numbers p, $p + 10$, $p + 14$ leave the three remainders p, $p + 1$, $p + 2$ (mod 3) in some order, and one of those is 0.

Work it out. So one of the three numbers is a multiple of 3. A prime multiple of 3 must be 3 itself. Since $p + 10 \geq 12$ and $p + 14 \geq 16$, the only candidate is $p = 3$, and indeed 3, 13, 17 are all prime. So $p = 3$ is the only such prime, and the sum is 3.

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Check. Try a few others: $p = 7$ gives 21 (divisible by 3); $p = 13$ gives 27; $p = 17$ gives 27. Each time one of the three is a multiple of 3, as the idea predicts.

11. Answer: 80**. The idea.** The same volume of water is spread over two different areas. The roof is much bigger than the tank's base, so the water in the tank is deeper by exactly the ratio of the two areas, and no unit conversion of the volume is needed.

Work it out. Roof area: $12 \cdot 16 = 192$ square meters. Tank base: $2 \cdot 3 = 6$ square meters. The roof is $192 \div 6 = 32$ times as large, so the water stands 32 times as deep in the tank: $32 \cdot 2.5 = 80$ cm.

Check. Work in meters: the rain is 0.025 m deep, so its volume is $192 \cdot 0.025 = 4.8$ cubic meters. Spread over 6 square meters it is $4.8 \div 6 = 0.8$ m = 80 cm deep.

12. Answer: 5**. The idea.** Write the square as k^2. Since $12 = 2^2 \cdot 3$, we need $2^2 \cdot 3$ to divide k^2. Primes appear in a square an even number of times, so $3 \mid k^2$ forces $3 \mid k$ (and then $9 \mid k^2$), and $4 \mid k^2$ forces $2 \mid k$. So k^2 is a multiple of 12 exactly when k is a multiple of 6, not of 12.

Work it out. We need $k = 6, 12, 18, 24, 30, \ldots$ with $k^2 \le 1000$. Since $31^2 = 961 \le 1000 < 32^2$, we need $k \le 31$, so $k = 6, 12, 18, 24, 30$. The squares are $36, 144, 324, 576, 900$: that is 5 of them.

Check. $36 = 12 \cdot 3$, $324 = 12 \cdot 27$, $900 = 12 \cdot 75$. The trap answer 2 comes from requiring $12 \mid k$, which misses 36, 324 and 900.

13. Answer: 18**. The idea.** Three unknowns look like a lot, but the chain of equal products says the pieces are in a fixed ratio. Give the common value a name that 2, 3 and 5 all divide, say $30t$, and every piece becomes a whole multiple of t.

Work it out. Let the pieces be a, b and c cm with $2a = 3b = 5c = 30t$. Then $a = 15t$, $b = 10t$ and $c = 6t$. Together they make the whole ribbon: $15t + 10t + 6t = 31t = 62$, so $t = 2$. The pieces are 30, 20 and 12 cm, and the first is $30 - 12 = 18$ cm longer than the third.

Check. $30 + 20 + 12 = 62$, and $2 \cdot 30 = 3 \cdot 20 = 5 \cdot 12 = 60$. A common slip is to split 62 in the ratio $2 : 3 : 5$; that gives the longest piece to the third piece, the wrong way round, because a piece that must be multiplied by more to reach the common value is shorter.

14. Answer: $\frac{5}{16}$**. The idea.** The bug's position depends only on how many of its moves go left and how many go right, not on their order. To be back at 0 after 6 moves it must go left exactly 3 times and right exactly 3 times.

Work it out. There are $2^6 = 64$ equally likely lists of six moves. The good lists choose which 3 of the 6 seconds are left moves: $\binom{6}{3} = 20$. The probability is $\frac{20}{64} = \frac{5}{16}$.

Check. Count from small cases. After 2 seconds the chance of being at 0 is $\frac{2}{4} = \frac{1}{2}$, after 4 seconds it is $\frac{\binom{4}{2}}{16} = \frac{6}{16} = \frac{3}{8}$, and after 6 it is $\frac{20}{64} = \frac{5}{16}$: the chance shrinks slowly, as it should when the bug has more room to wander. (Note the bug may pass through 0 earlier, and that is allowed.)

15. Answer: 252**. The idea.** Counting displays with a 7 directly means juggling overlaps (a 7 in the hour, in the minutes, or both). It is much easier to count the displays with NO 7 and subtract from 1440.

Work it out. Hours with no 7: of the 24 hours 00 to 23, only 07 and 17 contain a 7, leaving 22. Minutes with no 7: the tens digit is 0 to 5 (6 choices, none of them 7) and the units digit is any of the 9 digits other than 7, so $6 \cdot 9 = 54$. Displays with no 7: $22 \cdot 54 = 1188$. Displays with at least one 7: $1440 - 1188 = 252$.

Check. Count directly. During hours 07 and 17 every minute counts: $2 \cdot 60 = 120$. In each of the other 22 hours, the minutes with a 7 are the 6 minutes ending in 7 (07, 17, . . . , 57), since no minute starts with 7: $22 \cdot 6 = 132$. Total $120 + 132 = 252$.

16. Answer: 9**. The idea.** An endless expression contains a full copy of itself. Under the first square root sign in x sits all of x again, so $x = \sqrt{20 + x}$. In the same way, $y = \sqrt{20 - y}$. Each of these is a quadratic equation, and we keep the positive root, since a square root can't be negative.

Work it out. Squaring gives $x^2 = 20 + x$, so $x^2 - x - 20 = (x - 5)(x + 4) = 0$ and $x = 5$. Also $y^2 = 20 - y$, so $y^2 + y - 20 = (y + 5)(y - 4) = 0$ and $y = 4$. Therefore $x + y = 9$.

Check. Try the finite versions: $\sqrt{20} \approx 4.47$, then $\sqrt{24.47} \approx 4.95$, then $\sqrt{24.95} \approx 4.995$, which climbs toward 5. For y: $\sqrt{20} \approx 4.47$, then $\sqrt{15.53} \approx 3.94$, then $\sqrt{16.06} \approx 4.007$, which swings closer and closer to 4.

17. Answer: $32\sqrt{2}$**. The idea.** We do not know the octagon's side length, but we do know the distance from the center to every vertex. So cut the octagon from the center into 8 identical triangles. Each has two sides of length 4 (radii) with a $\frac{360^\circ}{8} = 45^\circ$ angle between them, and its area is easy to find by using one radius as the base.

Work it out. In one of these triangles, take a radius $\overline{OA}$ of length 4 as the base. The other radius $\overline{OB}$ makes a 45° angle with it, so the height from B to line OA is a leg of a 45-45-90 triangle with hypotenuse 4: the height is $\frac{4}{\sqrt{2}} = 2\sqrt{2}$. The triangle's area is $\frac{1}{2} \cdot 4 \cdot 2\sqrt{2} = 4\sqrt{2}$, and the octagon's area is $8 \cdot 4\sqrt{2} = 32\sqrt{2}$.

Check. Split the octagon differently. Every other vertex forms a square whose diagonals are diameters of length 8, so the square has area $\frac{8 \cdot 8}{2} = 32$ and side $4\sqrt{2}$. Its sides are $2\sqrt{2}$ from the center, so each of the four leftover triangles has base $4\sqrt{2}$ and height $4 - 2\sqrt{2}$, with area $\frac{1}{2} \cdot 4\sqrt{2}\,(4 - 2\sqrt{2}) = 8\sqrt{2} - 8$. The total is $32 + 4(8\sqrt{2} - 8) = 32\sqrt{2} \approx 45.3$, a bit less than the circle's $16\pi \approx 50.3$.

18. Answer: $\frac{45}{2}$**. The idea.** Adding 45 fractions with different denominators head-on is a mess. Group them by denominator instead: fractions with the same denominator add easily, and each group turns out to be worth a simple amount.

Work it out. For a fixed denominator b, the numerators are $1, 2, \ldots, b - 1$, which add to $\frac{(b-1)b}{2}$. Dividing by b, the group is worth $\frac{b-1}{2}$. So the whole sum is $\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \cdots + \frac{9}{2}$ (for $b = 2, 3, \ldots, 10$), which is $\frac{1+2+\cdots+9}{2} = \frac{45}{2}$.

Check. Pair each fraction $\frac{a}{b}$ with $\frac{b-a}{b}$ (it is also in the list). Each pair adds to 1, so the average fraction in the list is $\frac{1}{2}$ (a fraction $\frac{a}{b}$ with $a = b - a$ pairs with itself and equals $\frac{1}{2}$). Then the sum is $45 \cdot \frac{1}{2} = \frac{45}{2}$.

19. Answer: 34**. The idea.** First list the shapes that can have area 12 and still fit inside a 6 by 6 grid; then, for each shape, count the places it can sit. A rectangle w units wide fits in $7 - w$ horizontal positions, because its left edge can be on any of the grid lines $0, 1, \ldots, 6 - w$.

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Work it out. The factor pairs of 12 are 1×12, 2×6 and 3×4. A side of 12 does not fit, so the shapes are 2×6, 6×2, 3×4 and 4×3 (width by height), and there is no square since 12 is not a perfect square. Positions: 2×6: $5 \cdot 1 = 5$; 6×2: $1 \cdot 5 = 5$; 3×4: $4 \cdot 3 = 12$; 4×3: $3 \cdot 4 = 12$. The total is $5 + 5 + 12 + 12 = 34$.

Check. Count by choosing the rectangle's vertical sides (two of the 7 vertical lines, a distance w apart) and its horizontal sides (a distance $12/w$ apart). For $w = 2, 3, 4, 6$ the numbers of pairs of lines are 5, 4, 3, 1 and for the matching heights 6, 4, 3, 2 they are 1, 3, 4, 5. The products 5, 12, 12, 5 again add to 34.

20. Answer: 12**. The idea.** All 12 points lie on one circle, so any rectangle using them is inscribed in that circle. The diagonals of a rectangle are equal and bisect each other, so their common midpoint is the same distance from all four vertices: it is the center of the circle, and both diagonals are diameters. Conversely, any two different diameters are the diagonals of a rectangle. So rectangles correspond exactly to pairs of diameters.

Work it out. The 12 vertices form 6 diameters (each vertex with the vertex directly opposite). Any 2 of these give a rectangle: $\binom{6}{2} = 15$ rectangles. The rectangle is a square exactly when its diagonals are perpendicular. The diameters point in the directions $0°, 30°, 60°, 90°, 120°, 150°$, and the perpendicular pairs are $0°$ with $90°$, $30°$ with $120°$, and $60°$ with $150°$: 3 squares. That leaves $15 - 3 = 12$ rectangles that are not squares.

Check. Number the vertices 1 to 12 around the polygon. The squares are $\{1, 4, 7, 10\}$, $\{2, 5, 8, 11\}$ and $\{3, 6, 9, 12\}$. A non-square example is $\{1, 2, 7, 8\}$: its diagonals 1–7 and 2–8 are diameters meeting at a $30°$ angle, so its sides are unequal.

21. Answer: 9**. The idea.** Do not multiply out the fourth-degree polynomial. Something squared equals 4 exactly when that something is 2 or -2, so the equation splits into two quadratics. Solve each one, then watch for a solution that shows up twice, because the question asks for different numbers.

Work it out. Case $x^2 - 6x + 7 = 2$: then $x^2 - 6x + 5 = (x - 1)(x - 5) = 0$, so $x = 1$ or $x = 5$. Case $x^2 - 6x + 7 = -2$: then $x^2 - 6x + 9 = (x - 3)^2 = 0$, so $x = 3$, and only 3 (it is a double root). The different real solutions are 1, 3 and 5, and their sum is 9.

Check. At $x = 3$: $9 - 18 + 7 = -2$ and $(-2)^2 = 4$. At $x = 5$: $25 - 30 + 7 = 2$ and $2^2 = 4$. Two tempting wrong answers: 6, from forgetting the -2 case; and 12, from adding all four roots of the expanded quartic $x^4 - 12x^3 + \cdots$, which counts the double root 3 twice.

22. Answer: 52**. The idea.** Divisors of a number N come in pairs d and $\frac{N}{d}$ whose product is N. When $N = 360^2$, one member of each pair is below 360 and the other is above it, except for the single divisor 360, which pairs with itself. So the divisors below 360 are exactly half of the rest.

Work it out. $360 = 2^3 \cdot 3^2 \cdot 5$, so $360^2 = 2^6 \cdot 3^4 \cdot 5^2$, which has $(6 + 1)(4 + 1)(2 + 1) = 105$ positive divisors. Removing 360 itself leaves 104 divisors, split evenly between those below 360 and those above. The answer is $\frac{104}{2} = 52$.

Check. If $d < 360$ then $\frac{360^2}{d} > \frac{360^2}{360} = 360$, so the pairing really does send small divisors to large ones and back. The count 105 is odd, as it must be for a perfect square, which is why exactly one divisor (360) is left unpaired.

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23. Answer: 62. **The idea.** Factor 57,120 into primes and look for a large prime factor: it must be one of the four integers (or divide one of them), and the four integers are all close to each other, near the fourth root of the product.

Work it out. $57{,}120 = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 17$. Since $15^4 = 50{,}625$ and $16^4 = 65{,}536$, the four numbers are near 15 or 16, so the factor 17 must be 17 itself, one of the four numbers. The run could be 14, 15, 16, 17 or 15, 16, 17, 18 or later ones; the first has product $14{\cdot}15{\cdot}16{\cdot}17 = 210{\cdot}272 = 57{,}120$. The sum is $14 + 15 + 16 + 17 = 62$.

Check. The neighbouring runs miss: $13 \cdot 14 \cdot 15 \cdot 16 = 43{,}680$ is too small and $15 \cdot 16 \cdot 17 \cdot 18 = 73{,}440$ is too large. And $14 \cdot 15 \cdot 16 \cdot 17 = (2 \cdot 7)(3 \cdot 5)(2^4)(17) = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 17$, matching the factorization.

24. Answer: 72π. **The idea.** Cut the picture through the axis of the cylinder. The cylinder becomes a rectangle inscribed in a circle of radius 5, so half the height k and the radius r are the legs of a right triangle with hypotenuse 5: $r^2 + k^2 = 25$. The lateral area gives the product rk, and a sum of squares together with a product pins down $r + k$ and $r - k$ without solving any quadratic.

Work it out. The lateral area is $2\pi rh = 2\pi r(2k) = 4\pi rk = 48\pi$, so $rk = 12$. Then $(r + k)^2 = r^2 + k^2 + 2rk = 25 + 24 = 49$ and $(r - k)^2 = 25 - 24 = 1$, so $r + k = 7$ and $r - k = \pm 1$. That gives $\{r, k\} = \{3, 4\}$. The two cylinders are $r = 4, h = 6$ (wider than tall) and $r = 3, h = 8$ (taller than wide). The condition picks $r = 3$, $h = 8$, and the volume is $\pi \cdot 3^2 \cdot 8 = 72\pi$.

Check. The rim of the cylinder is 4 above the center and 3 out from the axis, and $3^2 + 4^2 = 25 = 5^2$, so the rim lies on the sphere. The lateral area is $2\pi \cdot 3 \cdot 8 = 48\pi$, and the height 8 exceeds the diameter 6.

25. Answer: $\sqrt{10}$. **The idea.** When a sector is rolled up, its radius becomes the cone's slant height and its arc becomes the circumference of the cone's base. That gives the base radius, and the Pythagorean theorem gives the height.

Work it out. Both cones have slant height 12. The 240° arc has length $\frac{2}{3} \cdot 24\pi = 16\pi$, so the base radius is 8 and the height is $\sqrt{144 - 64} = \sqrt{80} = 4\sqrt{5}$; its volume is $\frac{1}{3}\pi \cdot 64 \cdot 4\sqrt{5} = \frac{256\sqrt{5}}{3}\pi$. The 120° arc has length 8π, so the base radius is 4 and the height is $\sqrt{144 - 16} = \sqrt{128} = 8\sqrt{2}$; its volume is $\frac{1}{3}\pi \cdot 16 \cdot 8\sqrt{2} = \frac{128\sqrt{2}}{3}\pi$. The ratio is $\frac{256\sqrt{5}}{128\sqrt{2}} = \frac{2\sqrt{5}}{\sqrt{2}} = \sqrt{10}$.

Check. The ratio is $\frac{r_1^2 h_1}{r_2^2 h_2} = \frac{64}{16} \cdot \frac{\sqrt{80}}{\sqrt{128}} = 4\sqrt{\frac{5}{8}} = \sqrt{10} \approx 3.16$. It makes sense that the ratio is less than 4: the larger cone is wider but flatter.

26. Answer: 59. **The idea.** If the book has n pages, all its page numbers add to $\frac{n(n+1)}{2}$, and the torn sheet removes $(2k - 1) + 2k = 4k - 1$ from that. Two facts pin everything down: the removed amount is odd, and it is at most $(n - 1) + n = 2n - 1$, which is small compared with the total.

Work it out. Since $\frac{63 \cdot 64}{2} = 2016 < 2026$, the book has at least 64 pages. For $n = 64$ the total is 2080 and the removed amount would be 54, which is even, so that fails. For $n = 65$ the total is 2145 and the removed amount is $119 = 59 + 60$, a real sheet. For $n = 66$ the removed amount would be $2211 - 2026 = 185$, more than the largest possible sheet sum $65 + 66 = 131$; and each extra page raises the total by far more than it raises that limit, so larger n fail too. The torn sheet holds pages 59 and 60, and the front page is 59.

Check. $1 + 2 + \cdots + 65 = 2145$ and $2145 - 59 - 60 = 2026$.

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27. Answer: $\frac{\sqrt{145}}{2}$**. The idea.** Put the right angle at the origin with the legs along the axes, and find each center as a point. The circle through all three vertices of a right triangle has the hypotenuse as a diameter, so its center is the midpoint of the hypotenuse. The inscribed circle touches both legs, so its center is (r, r), where r is its radius; and r comes from the equal tangent lengths.

Work it out. The hypotenuse is $\sqrt{20^2 + 21^2} = \sqrt{841} = 29$. Put $C = (0, 0)$, $A = (21, 0)$ and $B = (0, 20)$. From each vertex, the two tangent segments to the inscribed circle are equal, and from the right angle both have length r. So the legs are $r + (21 - r)$ and $r + (20 - r)$, and the hypotenuse is $(21 - r) + (20 - r) = 41 - 2r$. Setting $41 - 2r = 29$ gives $r = 6$, so the inscribed center is $I = (6, 6)$. The other center is the midpoint of $\overline{AB}$: $O = (10.5, 10)$. Then $IO = \sqrt{4.5^2 + 4^2} = \sqrt{20.25 + 16} = \sqrt{36.25} = \sqrt{\frac{145}{4}} = \frac{\sqrt{145}}{2}$.

Check. Area: $\frac{1}{2} \cdot 20 \cdot 21 = 210$, and also area $= r \cdot \frac{20+21+29}{2} = 6 \cdot 35 = 210$, so $r = 6$ is right. And $O = (10.5, 10)$ is $\sqrt{10.5^2 + 10^2} = \sqrt{210.25} = 14.5$ from C, half of 29, as it should be. The answer is about 6.02.

28. Answer: $\frac{37}{250}$**. The idea.** Counting "median exactly 5" directly means juggling many cases of repeats. It is much easier to count "median at most k": the median is at most k exactly when at least two of the three numbers are at most k. Then $P(\text{median} = 5) = P(\text{median} \le 5) - P(\text{median} \le 4)$.

Work it out. If each number is at most k with probability $p = k/10$, then at least two of three are at most k with probability $3p^2(1 - p) + p^3$. For $k = 5$, $p = \frac{1}{2}$: $3 \cdot \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} = \frac{1}{2}$. For $k = 4$, $p = \frac{2}{5}$: $3 \cdot \frac{4}{25} \cdot \frac{3}{5} + \frac{8}{125} = \frac{36}{125} + \frac{8}{125} = \frac{44}{125}$. The difference is $\frac{1}{2} - \frac{44}{125} = \frac{125-88}{250} = \frac{37}{250}$.

Check. Count the 1000 ordered outcomes directly. Exactly one 5, one number below (4 choices) and one above (5 choices), in $3! = 6$ orders: 120. Exactly two 5s and one other number (9 choices, 3 positions): 27. Three 5s: 1. Total $148 = \frac{37}{250} \cdot 1000$.

29. Answer: $\frac{23}{128}$**. The idea.** Go around the table and group neighbors whose coins match into blocks (a block of heads, then a block of tails, and so on). A friend is lonely exactly when he or she forms a block of size 1. So we need every block to have at least 2 people. Because the table is round, heads-blocks and tails-blocks alternate all the way around, so there is an even number of blocks (or just one block, when all eight coins match).

Work it out. There are $2^8 = 256$ equally likely outcomes. One block: all heads or all tails, 2 outcomes. Two blocks: the heads form one unbroken arc of 2 to 6 seats and the tails fill the rest (also an arc of at least 2). An arc of a given length can start at any of the 8 seats, so there are $5 \cdot 8 = 40$ outcomes. Four blocks: four blocks of at least 2 seats in 8 seats must all have exactly 2, giving the pattern HHTTHHTT around the table; it can be rotated into 4 different positions (rotating by 4 seats gives the same outcome back), so 4 outcomes. Six or more blocks would need at least 12 seats. The total is $2 + 40 + 4 = 46$, and the probability is $\frac{46}{256} = \frac{23}{128}$.

Check. Try the same count with 4 friends: one block gives 2, two blocks of 2 give 4 (the heads-pair can start at any of 4 seats), total 6 of the 16 outcomes. Listing them directly: HHHH, TTTT, HHTT, THHT, TTHH, HTTH, and indeed no other arrangement of 4 avoids a lonely friend. A common slip in the 8-person case is to forget the four-block pattern HHTTHHTT and answer $\frac{42}{256}$.

30. Answer: $\frac{\sqrt{2}-1}{2}$**. The idea.** The diagonal $\overline{AC}$ cuts the square into two triangles, ABC (touching side $\overline{AB}$) and ACD. In triangle ACD the diagonal always wins, since it stands between P and

$\overline{AB}$. In triangle ABC, the points equally far from the two lines AB and AC form the bisector of angle BAC, and the points closer to $\overline{AB}$ are the ones on its side of that bisector.

Work it out. Let the square have side 1 (the probability does not depend on the size). If P is in triangle ACD, the perpendicular from P down to $\overline{AB}$ crosses $\overline{AC}$ at some point Q on the way, so the distance from P to the diagonal is at most PQ, which is less than the distance from P to $\overline{AB}$. So none of those points count. In triangle ABC, angle BAC is 45°; let its bisector meet $\overline{BC}$ at E. The points closer to $\overline{AB}$ fill triangle ABE. By the angle bisector theorem, $\frac{BE}{EC} = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$, and $BE + EC = 1$, so $BE = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. Triangle ABE has a right angle at B, so its area is $\frac{1}{2} \cdot 1 \cdot (\sqrt{2} - 1)$. Dividing by the square's area 1, the probability is $\frac{\sqrt{2}-1}{2}$.

Check. Find BE another way. Reflect B across the bisector $\overline{AE}$: it lands at the point B' on $\overline{AC}$ with $AB' = AB = 1$, and $EB' = EB$ with a right angle at B'. Since $AC = \sqrt{2}$, we get $B'C = \sqrt{2}-1$, and triangle $EB'C$ has a 45° angle at C and a right angle at B', so $EB' = B'C = \sqrt{2} - 1$. Size check: $\frac{\sqrt{2}-1}{2} \approx 0.207$, a bit less than the $\frac{1}{4}$ that would come from half of triangle ABC, which fits, because the bisector meets $\overline{BC}$ below its midpoint ($BE \approx 0.414 < 0.5$).

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