MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Kim starts a 250-page book. On the first day she reads 10 pages, and on each later day she reads 3 more pages than she read the day before (10 pages, then 13, then 16, and so on). On which day does she read the last page of the book? (Give the day number: the first day is day 1.)

1. _____

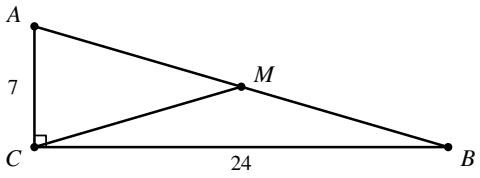
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2. At a school store, 5 pencils cost exactly as much as 3 pens, and each pen costs 40 cents more than each pencil. How many cents does one pen cost?

2. _____

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3. Triangle ABC has a right angle at C, with $CA = 7$ and $CB = 24$. Point M is the midpoint of the hypotenuse $\overline{AB}$. What is the length CM? Express your answer as a common fraction.



3. _____

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4. In a certain base b, where b is a positive integer greater than 1, the addition $35_b + 26_b = 63_b$ is correct. (Each subscript b means the number is written in base b.) What is b?

4. _____

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5. Jada's wallet holds only \$5 bills and \$10 bills, worth \$85 in all. If she had twice as many \$5 bills and half as many \$10 bills as she actually has, the bills would be worth \$80 in all. How many bills are in Jada's wallet?

5. _____

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6. Leo builds a rectangular grid of small squares out of toothpicks, using one toothpick for each side of each small square; a side shared by two squares uses just one toothpick. His grid is 4 squares tall, and he uses exactly 76 toothpicks. How many small squares are in his grid?

6. _____

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7. Two fair six-sided dice, one red and one blue, are rolled. What is the probability that the number on the red die divides the number on the blue die exactly (with no remainder)? Express your answer as a common fraction.

7. _____

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8. In triangle ABC, the bisector of $\angle ABC$ and the bisector of $\angle ACB$ meet at point I inside the triangle. The measure of $\angle BIC$ is exactly three times the measure of $\angle BAC$. What is the degree measure of $\angle BAC$?

8. _____

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9. Two different squares are chosen from an ordinary 8×8 chessboard so that they are in different rows AND in different columns (so two rooks placed on them could not attack each other). The order of the two squares does not matter. How many such pairs of squares are there?

9. _____

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10. Ali's first four test scores are 78, 85, 92 and 88. His fifth score is a whole number, and after the fifth test the mean of his five scores is exactly equal to their median. There is more than one possible fifth score. What is the sum of all the possible fifth scores?

10. _____

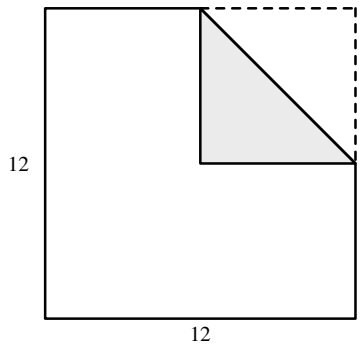
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11. The number 1000 can be written as a sum of different powers of 2 (the powers of 2 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and so on), with no power used more than once. What is the sum of the exponents of the powers used?

11. _____

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12. A 12 by 12 square sheet of paper lies flat on a table. One corner is folded over along a straight crease so that the corner lands exactly on the center of the square, and the paper is pressed flat. Seen from above, the folded paper is a pentagon. What is the area of this pentagon?



12. _____

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13. For how many positive integers n is $\dfrac{n+12}{n-3}$ an integer? (The quotient may be negative. The value $n = 3$ is not allowed, since it makes the denominator zero.)

13. _____

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14. Three consecutive powers of 3 (for example 3^2, 3^3 and 3^4, which are 9, 27 and 81) add up to 1053. What is the largest of the three powers?

14. _____

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15. Five friends, Ana, Ben, Cal, Dee and Eli, line up in a row for a photo. Ana refuses to stand in the first position, and Ben refuses to stand in the last position. In how many different orders can the five friends line up?

15. _____

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16. A map is drawn to the scale 1 : 25,000, so 1 cm on the map stands for 25,000 cm on the ground. A rectangular park appears on the map as a rectangle 6 cm long and 4 cm wide. One hectare is 10,000 square metres. What is the actual area of the park, in hectares?

16. _____

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17. A closed rectangular box has three different faces with areas 24, 32 and 48 square centimetres. What is the length, in centimetres, of a diagonal running through the inside of the box from one corner to the opposite corner? Express your answer in simplest radical form.

17. _____

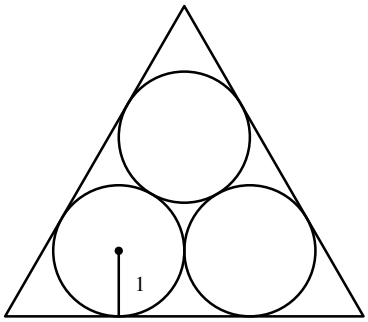
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18. A three-digit positive integer (from 100 to 999) is chosen at random, with every such integer equally likely. What is the probability that its three digits are all different AND the sum of its three digits is odd? Express your answer as a common fraction.

18. _____

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19. Three circles, each of radius 1, lie inside an equilateral triangle. Each circle is tangent to the other two circles and to two sides of the triangle. What is the side length of the triangle? Express your answer in simplest radical form.



19. _____

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20. A crew of workers, all working at the same steady rate, is scheduled to finish a job in a certain number of days. If the crew had 3 more workers, the job would take 2 fewer days. If the crew had 2 fewer workers, the job would take 3 more days. How many days would it take one of these workers, working alone, to do the whole job? 20. _____

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21. The diagonal of a rectangle is 8 units longer than the rectangle's width and 1 unit longer than the rectangle's length. What is the area of the rectangle, in square units? 21. _____

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22. When a certain two-digit number is divided by the sum of its own digits, the quotient is 7 and the remainder is 6. There is more than one such two-digit number. What is the sum of all of them? 22. _____

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23. A club has twelve members: six pairs of twins, and each pair is one boy and one girl. A committee of 4 members is to be chosen so that no two members of the committee are twins of each other, and the committee includes at least one boy and at least one girl. How many different committees are possible? 23. _____

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24. The positive integers are written in rows, in order. Row 1 is 1; row 2 is $2, 3, 4$; row 3 is $5, 6, 7, 8, 9$; and each row has two more numbers than the row before it. The rows are stacked into a triangle with equally spaced columns so that the middle numbers of all the rows $(1, 3, 7, 13, \ldots)$ lie in one vertical column. Then each row sticks out one place further on each side than the row above it, and every number has a number directly below it. What number is directly below 2000? 24. _____

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25. A right triangle has legs of lengths 6 and 8. Consider every point of the plane, inside or outside the triangle, whose distance to the nearest point of the triangle's perimeter is at most 2. What is the area of the region formed by all of these points? Express your answer in terms of π.

25. _____

26. How many ordered pairs of integers (x, y) satisfy the equation $x^2+y^2 = 2x + 4y + 60$?

26. _____

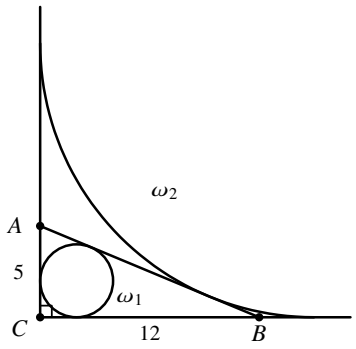
27. A certain positive integer has exactly three digits when written in base 7 and also exactly three digits when written in base 9. Its three base-9 digits are its three base-7 digits in reverse order. What is the integer, written in base 10?

27. _____

28. A straight fence 10 metres long is built by placing panels end to end in a single row. Each panel is 1, 2 or 3 metres long (there are plenty of panels of each length), and no two panels that touch each other have the same length. Two fences are different if their lists of panel lengths, read from left to right, are different. How many different fences are possible?

28. _____

29. Right triangle ABC has its right angle at C, with $CA = 5$ and $CB = 12$. Circle ω_1 is inscribed in the triangle. Circle ω_2 lies on the opposite side of line AB from C; it is tangent to segment $\overline{AB}$, to the extension of $\overline{CA}$ beyond A and to the extension of $\overline{CB}$ beyond B. What is the distance between the centers of ω_1 and ω_2? Express your answer in simplest radical form.



29. _____

30. A positive integer has exactly 9 positive divisors, and the sum of all of its positive divisors is 91. What is the integer?

30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 3

1. 11

2. 100

3. $\frac{25}{2}$

4. 8

5. 11

6. 32

7. $\frac{7}{18}$

8. 36

9. 1568

10. 179

11. 38

12. 126

13. 5

14. 729

15. 78

16. 150

17. $2\sqrt{29}$

18. $\frac{16}{45}$

19. $2 + 2\sqrt{3}$

20. 36

21. 60

22. 145

23. 210

24. 2090

25. $72 + 4\pi$

26. 16

27. 248

28. 27

29. $13\sqrt{2}$

30. 36

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Solutions — MathCounts Chapter Sprint Practice Contest 3

1. Answer: 11**. The idea.** Keep a running total. The daily amounts form an arithmetic sequence, so the total after n days is n times the average day, and we want the first n for which the total reaches 250.

Work it out. On day n she reads $10 + 3(n-1)$ pages. After n days the total is $10n + 3 \cdot \frac{n(n-1)}{2}$. After 10 days: $100 + 3 \cdot 45 = 235$ pages, still 15 short. On day 11 she reads $10 + 30 = 40$ pages, which takes her to $275 \geq 250$. So she finishes on day 11.

Check. List the days: $10, 13, 16, 19, 22, 25, 28, 31, 34, 37$ add to 235 (pair the first and last: 5 pairs of 47). Day 11 adds 40, so the book ends on day 11.

2. Answer: 100**. The idea.** Each pen is a pencil plus 40 cents. So 3 pens are 3 pencils plus 120 cents, and this has to match 5 pencils.

Work it out. From 5 pencils = 3 pencils +120 cents, the extra 2 pencils cost 120 cents, so a pencil costs 60 cents and a pen costs $60 + 40 = 100$ cents.

Check. $5 \cdot 60 = 300$ cents and $3 \cdot 100 = 300$ cents, so the two purchases really do cost the same. Careful: the question asks for the pen, not the pencil (60).

3. Answer: $\frac{25}{2}$**. The idea.** In a right triangle the midpoint of the hypotenuse is the same distance from all three vertices. One way to see it: copy the triangle, rotate the copy half a turn about M, and the two copies form a rectangle $ACBD$ whose diagonals are $\overline{AB}$ and $\overline{CD}$. The diagonals of a rectangle are equal and bisect each other, so CM is half of AB.

Work it out. By the Pythagorean theorem, $AB = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$. So $CM = \frac{25}{2}$.

Check. Use coordinates: $C = (0, 0)$, $A = (7, 0)$, $B = (0, 24)$. Then $M = \left(\frac{7}{2}, 12\right)$ and $CM^2 = \frac{49}{4} + 144 = \frac{625}{4}$, so $CM = \frac{25}{2}$.

4. Answer: 8**. The idea.** Add column by column, just as in base ten, and watch the carry. In the units column $5 + 6 = 11$, but the answer shows a units digit of 3, so there must be a carry: $11 = b + 3$.

Work it out. From the units column, $b = 11 - 3 = 8$. In the next column, $3 + 2$ plus the carried 1 is 6, which matches the digit 6 in 63_b (and $6 < 8$, so no further carry). Every digit used (2, 3, 5, 6) is less than 8, so base 8 is allowed.

Check. In base ten: $35_8 = 29$, $26_8 = 22$, and $63_8 = 48 + 3 = 51 = 29 + 22$. Without a carry the units would need $5 + 6 = 3$, which is impossible, so no other base works.

5. Answer: 11**. The idea.** Write one equation for each situation. We only want the total number of bills, and adding the two equations produces it directly: each \$5 bill contributes $5 + 10 = 15$ and each \$10 bill contributes $10 + 5 = 15$.

Work it out. Let f be the number of \$5 bills and t the number of \$10 bills. Then $5f + 10t = 85$ and $5(2f) + 10 \cdot \frac{t}{2} = 80$, that is, $10f + 5t = 80$. Adding: $15f + 15t = 165$, so $f + t = 11$.

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Check. Subtracting the equations gives $5t - 5f = 5$, so $t = f + 1$, and with $f + t = 11$ we get $f = 5$, $t = 6$. Real wallet: $25 + 60 = 85$. Changed wallet: 10 fives and 3 tens, $50 + 30 = 80$. Both match.

6. Answer: 32**. The idea.** Don't count square by square, since shared sides would be counted twice. Count the toothpicks line by line instead: the horizontal ones lie along the grid's horizontal lines, the vertical ones along its vertical lines.

Work it out. Let the grid be w squares wide. A grid 4 squares tall has 5 horizontal lines, each made of w toothpicks, and $w+1$ vertical lines, each made of 4 toothpicks. So it uses $5w+4(w+1) = 9w + 4$ toothpicks. Then $9w + 4 = 76$ gives $w = 8$, and the grid has $4 \cdot 8 = 32$ small squares.

Check. A 4 by 8 grid has $5 \cdot 8 = 40$ horizontal and $9 \cdot 4 = 36$ vertical toothpicks, 76 in all. Counting 4 toothpicks per square would give $4 \cdot 32 = 128$, far too many, because it counts every inside toothpick twice.

7. Answer: $\frac{7}{18}$**. The idea.** All 36 (red, blue) outcomes are equally likely. Fix the red number and count the blue numbers that are its multiples, from 1 to 6.

Work it out. Red 1: all 6 blue numbers work. Red 2: blue $2, 4, 6$, so 3. Red 3: blue $3, 6$, so 2. Red 4, 5, 6: only the matching number, 1 each. Total $6 + 3 + 2 + 1 + 1 + 1 = 14$, and $\frac{14}{36} = \frac{7}{18}$.

Check. Count by the blue number instead: blue $1, 2, 3, 4, 5, 6$ have $1, 2, 2, 3, 2, 4$ divisors, which also add to 14.

8. Answer: 36**. The idea.** The angle at I depends only on angle A: the bisectors take half of B and half of C, and $B + C$ is fixed once A is known.

Work it out. Let $\angle BAC = A$. In triangle BIC, the angles at B and C are $\frac{B}{2}$ and $\frac{C}{2}$, so $\angle BIC = 180^\circ - \frac{B+C}{2} = 180^\circ - \frac{180^\circ - A}{2} = 90^\circ + \frac{A}{2}$. Setting $90 + \frac{A}{2} = 3A$ gives $\frac{5A}{2} = 90$, so $A = 36$.

Check. With $A = 36^\circ$ we get $B+C = 144^\circ$, the half-angles add to 72°, and $\angle BIC = 108^\circ = 3 \cdot 36^\circ$. The formula $90^\circ + \frac{A}{2}$ also passes the sanity test that $\angle BIC$ is always obtuse.

9. Answer: 1568**. The idea.** Pick the squares one at a time: the first square rules out its whole row and column for the second. Then undo the double counting from the order.

Work it out. The first square can be any of 64. Its row and column contain $8 + 8 - 1 = 15$ squares (itself included), which leaves $64 - 15 = 49$ choices for the second. That gives $64 \cdot 49$ ordered pairs, and each unordered pair was counted twice, so there are $\frac{64 \cdot 49}{2} = 32 \cdot 49 = 1568$ pairs.

Check. Choose the two rows ($\binom{8}{2} = 28$ ways) and the two columns (28 ways). These four lines meet in a 2×2 set of squares, and exactly 2 of its pairs are diagonal: $28 \cdot 28 \cdot 2 = 1568$.

10. Answer: 179**. The idea.** The median of five numbers is the middle one, and which number sits in the middle depends on where the fifth score x lands. So split into cases by the position of x among $78, 85, 88, 92$. The four scores add to 343, so the mean is $\frac{343+x}{5}$.

Work it out. If $x \geq 88$, the sorted list is $78, 85, 88, \ldots$ and the median is 88: $\frac{343+x}{5} = 88$ gives $x = 97$, which is indeed at least 88. If $x \leq 85$, the median is 85: $343 + x = 425$ gives $x = 82$, which is indeed at most 85. If $85 < x < 88$, the median is x itself: $343 + x = 5x$ gives $x = 85.75$, not a whole number. So the possible scores are 82 and 97, with sum 179.

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Check. 78, 82, 85, 88, 92: sum 425, mean 85, median 85. 78, 85, 88, 92, 97: sum 440, mean 88, median 88.

11. Answer: 38**. The idea.** Writing a number as a sum of different powers of 2 is exactly writing it in base 2, and there is only one way to do it. To find it, repeatedly take the largest power of 2 that still fits. (If you skipped the largest power that fits, all the smaller different powers together add up to less than it, so you could never reach the total.)

Work it out. $1000 - 512 = 488$, $488 - 256 = 232$, $232 - 128 = 104$, $104 - 64 = 40$, $40 - 32 = 8$, $8 - 8 = 0$. So $1000 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$, and the exponents add to $9+8+7+6+5+3 = 38$.

Check. $1000 = 8 \cdot 125$ and $125 = 64+32+16+8+4+1 = 2^6+2^5+2^4+2^3+2^2+2^0$. Multiplying by $8 = 2^3$ adds 3 to each of the six exponents: $(6+5+4+3+2+0) + 6 \cdot 3 = 20 + 18 = 38$.

12. Answer: 126**. The idea.** Find the crease first. Every point of the crease is the same distance from the corner and from the center, so the crease is the perpendicular bisector of the segment joining them. The flap that folds over then lands on top of the paper, so the visible outline loses exactly the flap's area.

Work it out. Put the square at $(0,0)$ to $(12,12)$, the folded corner at $(0,0)$ and the center at $(6,6)$. The perpendicular bisector of that segment is the line $x + y = 6$, which meets the sides at the midpoints $(6,0)$ and $(0,6)$. The flap is the right isosceles triangle with legs 6, area $\frac{1}{2} \cdot 6 \cdot 6 = 18$. It flips onto the triangle with vertices $(6,6)$, $(6,0)$, $(0,6)$, which lies on the paper. So the outline is the square minus the flap: a pentagon of area $144 - 18 = 126$.

Check. The pentagon has vertices $(6,0)$, $(12,0)$, $(12,12)$, $(0,12)$, $(0,6)$. By the shoelace formula (or as a 12×12 square minus a corner triangle) its area is 126. The folded flap lands on the triangle $(6,6)$, $(6,0)$, $(0,6)$, whose corners $(6,0)$ and $(0,6)$ lie on the pentagon's edge and $(6,6)$ inside it, so no paper hangs outside it and the outline really is that pentagon.

13. Answer: 5**. The idea.** Pull the whole part out of the fraction: $\dfrac{n+12}{n-3} = \dfrac{(n-3)+15}{n-3} = 1 + \dfrac{15}{n-3}$. So the quotient is an integer exactly when $n - 3$ divides 15, and $n - 3$ is allowed to be negative.

Work it out. The divisors of 15 are $\pm 1, \pm 3, \pm 5, \pm 15$. Since $n \geq 1$, we need $n - 3 \geq -2$, which rules out -3, -5 and -15. The rest give $n - 3 = -1, 1, 3, 5, 15$, that is, $n = 2, 4, 6, 8, 18$. That is 5 values.

Check. $n = 2$: $14/(-1) = -14$. $n = 4$: $16/1 = 16$. $n = 6$: $18/3 = 6$. $n = 8$: $20/5 = 4$. $n = 18$: $30/15 = 2$. For $n > 18$ the fraction lies strictly between 1 and 2, so no larger n works.

14. Answer: 729**. The idea.** Don't guess and check: factor out the smallest power. If the powers are 3^k, 3^{k+1} and 3^{k+2}, their sum is $3^k(1 + 3 + 9) = 13 \cdot 3^k$.

Work it out. $13 \cdot 3^k = 1053$ gives $3^k = 1053 \div 13 = 81$. So the three powers are 81, 243 and 729, and the largest is 729.

Check. $81 + 243 + 729 = 1053$. Also, the largest power is 9 times the smallest, so it is $\frac{9}{13}$ of the total: $\frac{9}{13} \cdot 1053 = 9 \cdot 81 = 729$.

15. Answer: 78**. The idea.** Count the bad orders and subtract, but be careful: an order with Ana first AND Ben last is bad for both reasons, so subtracting both groups removes it twice. Add it back once (inclusion and exclusion).

Work it out. All orders: 5! = 120. Ana first: the other four fill the rest in 4! = 24 ways. Ben last: also 24. Ana first and Ben last: the middle three in 3! = 6 ways. Good orders: $120 - 24 - 24 + 6 = 78$.

Check. Count directly by where Ben stands. If Ben is first (1 way), Ana is not first automatically, and the others fill 4! = 24 orders. If Ben is in position 2, 3 or 4 (3 ways), Ana can take any of the 3 free spots other than the first, and the remaining three people fill 3! = 6 orders: $3 \cdot 3 \cdot 6 = 54$. Total $24 + 54 = 78$.

16. Answer: 150**. The idea.** The scale is a scale for lengths. Convert the two side lengths to real distances first and only then multiply; multiplying the map area by 25,000 would be wrong, because areas grow by the square of the scale factor.

Work it out. 6 cm on the map is $6 \cdot 25{,}000 = 150{,}000$ cm = 1500 m, and 4 cm is 100,000 cm = 1000 m. The park's area is $1500 \cdot 1000 = 1{,}500{,}000$ square metres, which is $\frac{1{,}500{,}000}{10{,}000} = 150$ hectares.

Check. Through the map area: 24 cm^2 times $25{,}000^2 = 625{,}000{,}000$ is 15,000,000,000 cm^2, and since 1 m^2 = 10,000 cm^2 that is 1,500,000 m^2 again. Multiplying 24 by 25,000 only once would give 600,000 cm^2 = 60 m^2, far too small for a park.

17. Answer: $2\sqrt{29}$**. The idea.** If the edges are a, b, c, the face areas are ab, bc, ca. Multiplying all three gives $(abc)^2$, so we get the volume at once, and dividing the volume by each face area gives back the edges.

Work it out. Let $ab = 24$, $bc = 32$, $ca = 48$. Then $(abc)^2 = 24 \cdot 32 \cdot 48 = 36864$, so $abc = 192$. Hence $c = 192/24 = 8$, $a = 192/32 = 6$, $b = 192/48 = 4$. The space diagonal is $\sqrt{a^2 + b^2 + c^2} = \sqrt{36 + 16 + 64} = \sqrt{116} = 2\sqrt{29}$.

Check. The edges 6, 4, 8 give faces $6 \cdot 4 = 24$, $4 \cdot 8 = 32$ and $6 \cdot 8 = 48$. The diagonal also comes from two right triangles: the base diagonal is $\sqrt{6^2 + 4^2} = \sqrt{52}$, then $\sqrt{52 + 64} = \sqrt{116}$.

18. Answer: $\frac{16}{45}$**. The idea.** The digit sum is odd exactly when the number of odd digits is odd: 1 or 3. Count those two cases separately, being careful that the hundreds digit cannot be 0.

Work it out. Three odd digits, all different: $5 \cdot 4 \cdot 3 = 60$ (no zeros, so no trouble). One odd digit and two different even digits: if the odd digit is in the hundreds place, there are 5 choices for it and $5 \cdot 4 = 20$ for the two even digits, giving 100. If the odd digit is in the tens or units place (2 positions, 5 choices each), the hundreds digit is a nonzero even digit (4 choices) and the remaining even digit can be any of the other 4 even digits (0 now allowed): $2 \cdot 5 \cdot 4 \cdot 4 = 160$. Total good numbers: $60 + 100 + 160 = 320$, so the probability is $\frac{320}{900} = \frac{16}{45}$.

Check. There are $9 \cdot 9 \cdot 8 = 648$ three-digit numbers with distinct digits. Those with an even digit sum have 0 or 2 odd digits: all-even gives $4 \cdot 4 \cdot 3 = 48$, and two odd digits and one even digit gives 280 (even digit in front: $4 \cdot 5 \cdot 4 = 80$; even digit in the tens or units place: $2 \cdot 5 \cdot 5 \cdot 4 = 200$), so 328. And $320 + 328 = 648$, as it should.

19. Answer: $2 + 2\sqrt{3}$**. The idea.** Two circles of the same radius cannot both be tangent to the same pair of sides and to each other, so each circle sits in its own corner, and its center lies on the bisector of that 60° angle. Then one side of the triangle splits into three pieces: from a corner to the first point of tangency, between the two points of tangency, and on to the other corner.

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Work it out. Look at the bottom side, with corners B and C; the circles in those corners have centers O_B and O_C and touch the bottom side at P and Q. The radius O_BP is perpendicular to the side and has length 1, and O_B is on the bisector of the 60° angle at B, so $\angle O_BBP = 30°$. Triangle BPO_B is a 30-60-90 triangle whose short leg is $O_BP = 1$, so $BP = \sqrt{3}$. In the same way $CQ = \sqrt{3}$. Both centers are 1 unit above the bottom side, so O_BO_CQP is a rectangle and $PQ = O_BO_C = 1 + 1 = 2$, because the two circles are tangent. The side length is $\sqrt{3} + 2 + \sqrt{3} = 2 + 2\sqrt{3}$.

Check. The three centers form an equilateral triangle of side 2, whose own inradius is $\frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$. Its sides are parallel to the big triangle's sides and 1 unit inside them, so the big triangle's inradius should be $1 + \frac{\sqrt{3}}{3}$. And indeed an equilateral triangle of side $2 + 2\sqrt{3}$ has inradius $\frac{(2+2\sqrt{3})\sqrt{3}}{6} = \frac{2\sqrt{3}+6}{6} = 1 + \frac{\sqrt{3}}{3}$. (A common slip is to take the side as $2 + 2 = 4$, forgetting the corner pieces, or to use $BP = 1$.)

20. Answer: 36**. The idea.** Measure the job in worker-days: the number of workers times the number of days is the same however the job is staffed. That turns each "what if" into an equation.

Work it out. Let the crew have n workers who need d days, so the job is nd worker-days. The first scenario gives $(n + 3)(d - 2) = nd$, so $3d - 2n = 6$. The second gives $(n - 2)(d + 3) = nd$, so $3n - 2d = 6$. Subtracting one equation from the other gives $3d - 2n - 3n + 2d = 0$, so $d = n$. Then $3n - 2n = 6$, so $n = d = 6$. The job is $6 \cdot 6 = 36$ worker-days, so one worker alone needs 36 days.

Check. 9 workers need $36/9 = 4$ days, which is 2 fewer than 6. 4 workers need $36/4 = 9$ days, which is 3 more than 6.

21. Answer: 60**. The idea.** Name one length and write the other two in terms of it; the Pythagorean theorem then gives a single equation. The squares on both sides cancel, leaving a small quadratic.

Work it out. Let the width be w. The diagonal is $w + 8$ and the length is 1 less than the diagonal, $w+7$. The width, length and diagonal form a right triangle, so $w^2+(w+7)^2 = (w+8)^2$. Expanding, $w^2 + w^2 + 14w + 49 = w^2 + 16w + 64$, so $w^2 - 2w - 15 = 0$, which factors as $(w - 5)(w + 3) = 0$. The width is positive, so $w = 5$, the length is 12, and the area is $5 \cdot 12 = 60$.

Check. The diagonal is $\sqrt{25 + 144} = 13$, which is 8 more than 5 and 1 more than 12. It is the familiar 5-12-13 right triangle.

22. Answer: 145**. The idea.** Write the number as $10a + b$. "Quotient 7, remainder 6" means $10a + b = 7(a + b) + 6$, and also that the remainder is smaller than the divisor: $6 < a + b$. The second condition is easy to forget.

Work it out. The equation simplifies to $3a - 6b = 6$, that is, $a = 2b + 2$. The digit pairs are $(a, b) = (2, 0), (4, 1), (6, 2), (8, 3)$. Their digit sums are $2, 5, 8, 11$, and a remainder of 6 needs a digit sum greater than 6, which leaves 62 and 83. Their sum is $62 + 83 = 145$.

Check. $62 = 7 \cdot 8 + 6$ and $83 = 7 \cdot 11 + 6$. On the other hand $20 = 7 \cdot 2 + 6$ and $41 = 7 \cdot 5 + 6$ are true equations, but $20 \div 2$ and $41 \div 5$ really have quotients 10 and 8 with remainders 0 and 1, so they do not count.

23. Answer: 210**. The idea.** "No two twins" means the committee uses four *different* pairs, one person from each. So choose the pairs first, then choose who represents each pair, and finally throw out the committees that are all boys or all girls.

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Work it out. Choose which 4 of the 6 pairs are represented: $\binom{6}{4} = 15$ ways. Each chosen pair sends either its boy or its girl: $2^4 = 16$ ways. That gives $15 \cdot 16 = 240$ committees with no twins together. For a fixed choice of pairs, exactly one of the 16 choices is all boys and one is all girls, so 2 of the 16 are forbidden, leaving 14. The answer is $15 \cdot 14 = 210$.

Check. Count all-boy committees directly: pick 4 of the 6 boys, $\binom{6}{4} = 15$, and they are automatically from different pairs. Same for girls. $240 - 15 - 15 = 210$. (Without the twin rule there would be $\binom{12}{4} = 495$ committees; the rule is doing a lot of work.)

24. Answer: 2090**. The idea.** Row n has $2n - 1$ numbers, so rows 1 through n hold $1 + 3 + \cdots + (2n - 1) = n^2$ numbers: row n ends with n^2. That locates 2000. Then measure its position from the middle of its row, because the middle column is the line that stays put.

Work it out. Since $44^2 = 1936$ and $45^2 = 2025$, the number 2000 is in row 45, which runs from 1937 to 2025. That row has 89 numbers, so its middle one is the 45th, $1936 + 45 = 1981$, and 2000 is 19 places to the right of the middle. Row 46 runs from 2026 to $46^2 = 2116$; its middle number is its 46th, $2025 + 46 = 2071$. The number directly below 2000 is 19 places to the right of that: $2071 + 19 = 2090$.

Check. A second route: from a number k that is jth in row n, going straight down means passing the $(2n - 1) - j$ numbers after it in row n and then $j + 1$ numbers of row $n + 1$ (that row sticks out one extra place on the left), so the number below is always $k + 2n$. For 2000 in row 45 that is $2000 + 90 = 2090$. In a small case, below 3 (row 2) is $3 + 4 = 7$, the middle of row 3, as it should be.

25. Answer: $72 + 4\pi$**. The idea.** Split the region into the part outside the triangle and the part inside. Outside, the region is a band of width 2 along the sides plus round pieces at the corners, and the corner pieces fit together into one whole circle. Inside, the surprise is that *every* point of the triangle is within 2 of a side, because 2 is exactly the radius of the inscribed circle.

Work it out. The hypotenuse is 10, the perimeter is 24 and the area is $\frac{1}{2} \cdot 6 \cdot 8 = 24$. *Outside:* along each side there is a 2-wide rectangle, with total area $2 \cdot 24 = 48$. At each corner there is a circular sector of radius 2 whose angle is 180° minus the triangle's angle there; the three sector angles add up to $3 \cdot 180^\circ - 180^\circ = 360^\circ$, so together they make one full circle of area 4π. *Inside:* the inscribed circle has radius $\frac{\text{area}}{\text{semiperimeter}} = \frac{24}{12} = 2$. For any point inside, joining it to the three corners cuts the triangle into three pieces whose areas are $\frac{1}{2} \cdot \text{side} \cdot \text{distance}$, so $24 = \frac{1}{2}(6d_1 + 8d_2 + 10d_3) \geq \frac{1}{2} \cdot 24 \cdot \min(d_1, d_2, d_3)$, which gives $\min(d_1, d_2, d_3) \leq 2$. So the whole triangle belongs to the region. The total area is $48 + 4\pi + 24 = 72 + 4\pi$.

Check. Only the incenter is exactly 2 from all three sides; every other inside point is closer to some side, so nothing inside is missed. Leaving out the inside would give $48 + 4\pi$, the common wrong answer. As a size check, $72 + 4\pi \approx 84.6$ lies between the triangle's own area 24 and the area $12 \cdot 10 = 120$ of the box $-2 \leq x \leq 8$, $-2 \leq y \leq 10$ that contains the whole region when the legs of lengths 6 and 8 lie along the positive x- and y-axes.

26. Answer: 16**. The idea.** Completing the square shows the equation is a circle centered at the lattice point $(1, 2)$; since the center is a lattice point, the question is how many ways 65 is a sum of two integer squares, counting signs and order.

Work it out. Moving terms, $x^2 - 2x + 1 + y^2 - 4y + 4 = 65$, so $(x - 1)^2 + (y - 2)^2 = 65$. Checking squares below 65: $65 = 1 + 64 = 16 + 49$, and no other pair of squares $(0, 4, 9, 25, 36$ paired with

65 minus it) works. So $(u, v) = (x - 1, y - 2)$ is one of $(\pm 1, \pm 8)$, $(\pm 8, \pm 1)$, $(\pm 4, \pm 7)$, $(\pm 7, \pm 4)$: 4 sign patterns times 4 arrangements, 16 pairs, each giving one (x, y).

Check. Take $(u, v) = (-7, 4)$, so $(x, y) = (-6, 6)$: $36 + 36 = 72$ and $2(-6) + 4(6) + 60 = 72$. It works. As a count check, the four divisors $1, 5, 13, 65$ of 65 all leave remainder 1 on division by 4, and the classical formula (4 times that number of divisors, when none leaves remainder 3) gives $4 \cdot 4 = 16$ representations.

27. Answer: 248**. The idea.** Name the digits, set the two place-value expansions equal, and let the digit bounds do the work: the resulting equation has so little room that divisibility forces everything.

Work it out. Say the number is $\overline{abc}_7 = \overline{cba}_9$. Both leading digits are nonzero, and all digits are base-7 digits, so $1 \le a, c \le 6$ and $0 \le b \le 6$. Then $49a + 7b + c = 81c + 9b + a$, which simplifies to $48a = 2b + 80c$, or $24a = b + 40c$. Now $b = 24a - 40c$ is a multiple of 8, and $0 \le b \le 6$ forces $b = 0$. Then $24a = 40c$, so $3a = 5c$; with $a, c \le 6$ the only solution is $a = 5$, $c = 3$. The number is $503_7 = 5 \cdot 49 + 3 = 248$.

Check. In base 9: $248 = 3 \cdot 81 + 0 \cdot 9 + 5$, so $248 = 305_9$, the digits $5, 0, 3$ reversed.

28. Answer: 27**. The idea.** Build the fence from left to right and keep track of two things: how long it is so far and the length of the last panel, since that is the only thing the next panel has to avoid. Let a_n, b_n, c_n be the numbers of allowed fences of total length n ending in a 1-, 2- or 3-metre panel.

Work it out. A fence of length n ending in a 1 is a fence of length $n - 1$ ending in a 2 or a 3, with a 1 added: $a_n = b_{n-1} + c_{n-1}$. Likewise $b_n = a_{n-2} + c_{n-2}$ and $c_n = a_{n-3} + b_{n-3}$; a single panel on its own is also a fence, so $a_1 = 1$, $b_2 = 1$, $c_3 = 1$ start things off. The table of (a_n, b_n, c_n) is: $n = 1$: $(1, 0, 0)$; $n = 2$: $(0, 1, 0)$; $n = 3$: $(1, 1, 1)$; $n = 4$: $(2, 0, 1)$; $n = 5$: $(1, 2, 1)$; $n = 6$: $(3, 3, 2)$; $n = 7$: $(5, 2, 2)$; $n = 8$: $(4, 5, 3)$; $n = 9$: $(8, 7, 6)$; $n = 10$: $(13, 7, 7)$. For example $a_{10} = b_9 + c_9 = 7 + 6 = 13$, $b_{10} = a_8 + c_8 = 4 + 3 = 7$ and $c_{10} = a_7 + b_7 = 5 + 2 = 7$. The answer is $13 + 7 + 7 = 27$.

Check. For length 5 the table gives $1+2+1 = 4$ fences, and indeed they are 1 3 1, 2 3, 3 2 and 2 1 2 (a fence like 1 2 2 fails because the two 2s touch). For length 10, sort by the panels used. Mixes such as five 2s, or four 1s with two 3s, cannot be arranged without two equal panels touching (too many copies of one length for the others to separate). The mixes that work: $\{2, 2, 3, 3\}$ gives 2 fences (2 3 2 3, 3 2 3 2); $\{1, 1, 1, 1, 2, 2, 2\}$ gives 1 (1 2 1 2 1 2 1); $\{1, 2, 2, 2, 3\}$ gives 2 (the 2s in places 1, 3, 5); $\{1, 1, 2, 3, 3\}$ gives $\frac{5!}{2!\,2!} - 12 - 12 + 6 = 12$ (subtract the arrangements with the 1s together or the 3s together, add back both); $\{1, 1, 1, 2, 2, 3\}$ gives 10 (arrange 2, 2, 3 as 2 3 2, 2 2 3 or 3 2 2 and put the three 1s into different gaps: $4 + 3 + 3$, since a 1 must sit between two touching 2s). Total $2 + 1 + 2 + 12 + 10 = 27$.

29. Answer: $13\sqrt{2}$**. The idea.** Both circles are tangent to the two lines through C that meet at a right angle, so each center lies on the bisector of the right angle, and a circle of radius ρ touching both lines has its center at distance $\rho\sqrt{2}$ from C (it is the far corner of a ρ-by-ρ square at C). So we only need the two radii, and equal tangent lengths give both of them.

Work it out. *Big circle.* Let its radius be R. The tangent segments from C to ω_2 along the two legs' lines both have length R. From A, the tangent along line CA has length $R - 5$, so the tangent from A to ω_2 along AB also has length $R - 5$; similarly from B it is $R - 12$. These

two tangent pieces make up all of AB, because ω_2 touches the segment AB between A and B: $(R-5)+(R-12)=13$, so $R=15$ (the semiperimeter). *Small circle.* For the incircle with radius r, the tangent lengths from A and B are $5-r$ and $12-r$, and $(5-r)+(12-r)=13$ gives $r=2$. Both centers lie on the bisector from C, at distances $15\sqrt{2}$ and $2\sqrt{2}$ from C, so they are $(15-2)\sqrt{2}=13\sqrt{2}$ apart.

Check. With $C=(0,0)$, $A=(5,0)$, $B=(0,12)$, line AB is $12x+5y=60$. A center (t,t) is at distance $\frac{|17t-60|}{13}$ from it; setting this equal to t gives $17t-60=-13t$, so $t=2$, or $17t-60=13t$, so $t=15$. The distance from $(2,2)$ to $(15,15)$ is $\sqrt{13^2+13^2}=13\sqrt{2}$. (Notice $R-r=13$ is exactly the hypotenuse.)

30. Answer: 36**. The idea.** The number of divisors pins down the shape of the prime factorization, and the divisor sum of a prime power p^k is $1+p+\cdots+p^k$, with the sums for different primes multiplying.

Work it out. If $n=p_1^{e_1}p_2^{e_2}\cdots$, then n has $(e_1+1)(e_2+1)\cdots$ divisors. For this to be $9=9$ or $3\cdot 3$, either $n=p^8$ or $n=p^2q^2$ with different primes p,q. If $n=p^8$, the divisor sum is at least $1+2+4+\cdots+256=511>91$, which is impossible. So $n=p^2q^2$, and every divisor is (a divisor of p^2) times (a divisor of q^2), so the divisor sum is $(1+p+p^2)(1+q+q^2)=91=7\cdot 13$. Each factor is at least $1+2+4=7$, so neither can be 1; hence the factors are 7 and 13. Now $1+p+p^2=7$ gives $p=2$ and $1+q+q^2=13$ gives $q=3$. So $n=2^2\cdot 3^2=36$.

Check. The divisors of 36 are $1,2,3,4,6,9,12,18,36$: nine of them, and $1+2+3+4+6+9+12+18+36=91$.

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