MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

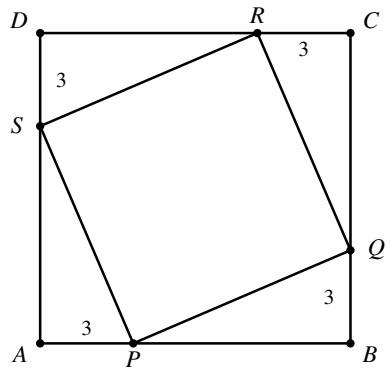
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A stationery store has a deal on notebooks: every second notebook in a purchase costs half price. So in one purchase the 1st, 3rd, 5th, ... notebooks cost the full price, and the 2nd, 4th, 6th, ... notebooks cost half of the full price. Mia buys 7 notebooks in one purchase and pays \$33 in all. What is the full price of one notebook, in dollars?

1. _____

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2. Square $ABCD$ has side length 10, with its vertices labelled in order around it. Points P, Q, R and S lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, respectively, so that $AP = BQ = CR = DS = 3$. What is the area of quadrilateral $PQRS$, in square units?



2. _____

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3. Two different numbers x and y, with $y \neq 0$, satisfy $\dfrac{2x+y}{x+2y} = \dfrac{5}{4}$. What is the value of $\dfrac{x+y}{x-y}$?

3. _____

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4. A square garden has an area of 36 square meters. The fence around it is taken down and all of it is used, with nothing left over, to fence an equilateral triangular garden instead. What is the area of the triangular garden, in square meters? Express your answer in simplest radical form.

4. _____

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5. The year 2031 is written with four different digits. How many even years from 2000 to 2999, inclusive, are written with four different digits?

5. _____

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6. Kai walks at 4 km per hour and runs at 10 km per hour. One morning he covers exactly 21 km in exactly 3 hours, walking for part of the time and running for the rest. For how many minutes does he run?

6. _____

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7. Three boxes are weighed two at a time, once for each of the three possible pairs. The three pair weights are 17, 21 and 24 kilograms. How many kilograms does the heaviest box weigh? 7. _____

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8. An old clock chimes on every hour as many times as the hour shows (once at 1:00, twice at 2:00, and so on, with 12 chimes at noon and at midnight). It also chimes once at every half hour (12:30, 1:30, and so on). The clock is switched on at 12:01 a.m. on Monday and switched off at 11:59 p.m. on Wednesday. How many times does it chime in all while it is on? 8. _____

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9. At a party, every guest came with his or her spouse, and nobody else attended. Each person shook hands exactly once with every other person except his or her own spouse (spouses did not shake hands with each other). Altogether there were exactly 60 handshakes. How many married couples were at the party? 9. _____

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10. Two positive integers have a product of 360, and their greatest common divisor is 6. Neither of the two numbers is equal to 6. What is the sum of the two numbers? 10. _____

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11. A chocolate maker melts down a solid chocolate rabbit that is 20 cm tall. With no chocolate lost, she recasts all of it into small solid chocolate rabbits, each with exactly the same shape as the big one but only 5 cm tall. How many small rabbits does she make? 11. _____

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12. The integers from 1 to 20 are written one after another, in order, to form the 31-digit number 12345678910111213 . . . 1920. What is the remainder when this number is divided by 9? 12. _____

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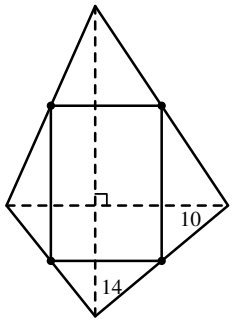
13. What is the smallest positive integer whose digits have a product of 2700? (For example, the digits of 326 have a product of $3 \cdot 2 \cdot 6 = 36$.) 13. _____

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14. Sam and Lily both start with no savings. Sam saves \$1 in the first week, \$2 in the second week, \$3 in the third week, and so on, always \$1 more than the week before. Lily saves \$20 every week. At the end of which week is Sam's total savings greater than Lily's for the first time? Give the week number. 14. _____

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15. A convex quadrilateral has diagonals of lengths 10 and 14 that are perpendicular to each other. The midpoints of its four sides are joined in order to form a second quadrilateral. What is the area of this second quadrilateral, in square units?



15. _____

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16. The equation $x^2 - 7x + k = 0$ has two different real roots. When the smaller root is squared and subtracted from the square of the larger root, the result is 21. What is the value of k?

16. _____

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17. A rectangular garden has a perimeter of 34 meters; its length and width are not given. A path of the same width all the way around is built along the outside of the garden, so that the garden and the path together form a larger rectangle. The area of the path alone is 138 square meters. How many meters wide is the path?

17. _____

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18. Five letters are written to five different people, and an envelope is addressed to each person. The letters are put into the envelopes at random, one letter per envelope, with all arrangements equally likely. What is the probability that exactly two of the letters end up in their correct envelopes? Express your answer as a common fraction.

18. _____

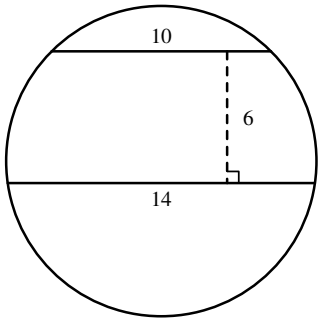
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19. Two different integers are chosen at random from the integers 1 through 20, inclusive, with every pair of different integers equally likely to be chosen. What is the probability that the sum of the two chosen integers is a multiple of 4? Express your answer as a common fraction.

19. _____

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20. Two parallel chords of a circle have lengths 10 and 14, and the distance between the two chords is 6. What is the radius of the circle? Express your answer in simplest radical form.



20. _____

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21. Let $f(x) = x^2 - 1$ and $g(x) = 2x + 3$. For some real numbers x, applying g first and then f gives the same result as applying f first and then g; that is, $f(g(x)) = g(f(x))$. What is the sum of all real numbers x with this property?

21. _____

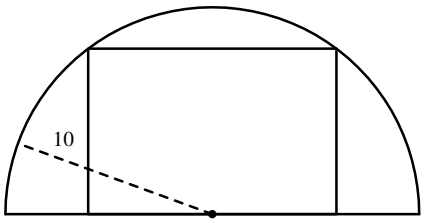
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22. A flag is a 2-by-2 grid of four squares. Each square is painted one of 4 available colors. Not every color has to be used, but any two squares that share a side must be painted different colors. Flags are never turned or flipped, so two flags are different whenever some square of the grid has a different color. How many different flags can be made?

22. _____

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23. A semicircle has radius 10. A rectangle is drawn inside it so that one side of the rectangle lies along the diameter of the semicircle and the rectangle's other two vertices lie on the curved arc. What is the largest possible area of such a rectangle?



23. _____

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24. The grass in a pasture grows at a steady rate every day, and every cow eats the same amount of grass per day. A herd of 12 cows eats the pasture bare in exactly 24 days, while a herd of 18 cows, starting from the same amount of grass, eats it bare in exactly 12 days. Starting again from the same amount of grass, how many days would it take 30 cows to eat the pasture bare?

24. _____

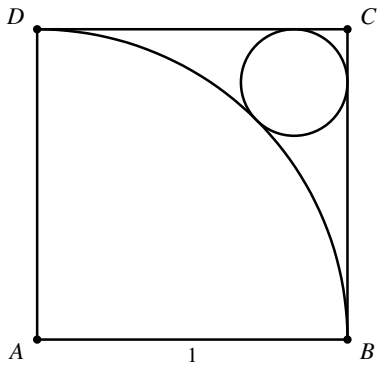
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25. What is the largest positive integer less than 1000 that can be written as the sum of 7 consecutive positive integers and can also be written as the sum of 8 consecutive positive integers?

25. _____

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26. Square $ABCD$ has side length 1. A quarter circle with center A and radius 1 is drawn inside the square, from B to D. A small circle lies in the region of the square outside the quarter circle, near corner C; it is tangent to side $\overline{BC}$, to side $\overline{CD}$, and to the arc of the quarter circle. What is the radius of the small circle? Express your answer in simplest radical form.



26. _____

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27. A sequence starts with $a_1 = 5$. Each term after the first is found by multiplying the term before it by 3 and then subtracting 4, so $a_2 = 11$ and $a_3 = 29$. What is the sum $a_1 + a_2 + \cdots + a_{10}$ of the first ten terms?

27. _____

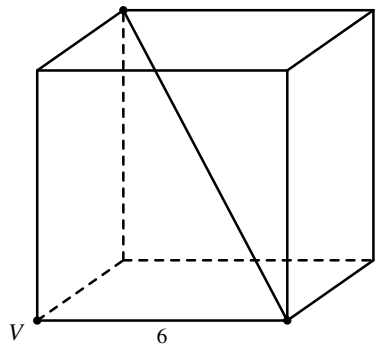
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28. The twelve hour marks $12, 1, 2, \ldots, 11$ are equally spaced around a clock face. Two marks are called neighbors if they are next to each other around the circle; for example, 11 and 12 are neighbors, and so are 12 and 1. In how many ways can Ana circle exactly three of the twelve marks so that no two of the circled marks are neighbors?

28. _____

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29. A cube has edge length 6. A space diagonal of the cube is a segment joining two opposite vertices (vertices that do not lie on a common face). Let V be a vertex of the cube, and choose a space diagonal that does not pass through V. What is the distance from V to that space diagonal? Express your answer in simplest radical form.



29. _____

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30. A positive divisor of 10! = 3,628,800 is called *special* if it is a perfect square, a perfect cube, or both. (The divisor 1 counts, since $1 = 1^2 = 1^3$.) How many special divisors does 10! have?

30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 2

1. 6

2. 58

3. 3

4. $16\sqrt{3}$

5. 224

6. 90

7. 14

8. 528

9. 6

10. 42

11. 64

12. 3

13. 25569

14. 40

15. 35

16. 10

17. 3

18. $\frac{1}{6}$

19. $\frac{9}{38}$

20. $5\sqrt{2}$

21. -6

22. 84

23. 100

24. 6

25. 980

26. $3 - 2\sqrt{2}$

27. 88592

28. 112

29. $2\sqrt{6}$

30. 34

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Chapter Sprint Practice Contest 2

1. Answer: 6**. The idea.** Count how many notebooks are charged at full price and how many at half price. With an odd number of notebooks the last one has no partner, so it is not "$3\frac{1}{2}$ pairs".

Work it out. Of the 7 notebooks, numbers $1, 3, 5, 7$ are full price and numbers $2, 4, 6$ are half price. If the full price is p dollars, Mia pays $4p + 3 \cdot \frac{p}{2} = 5.5p$. So $5.5p = 33$ and $p = 6$.

Check. Four notebooks at \$6 cost \$24 and three at \$3 cost \$9; $24 + 9 = 33$. Treating the purchase as 3.5 pairs at $1.5p$ each would give $5.25p = 33$, which is not even a whole number of cents.

2. Answer: 58**. The idea.** Instead of measuring $PQRS$ directly, cut away what lies outside it: four right triangles in the corners of the big square, all the same size.

Work it out. At corner B, the triangle PBQ has legs $PB = 10 - 3 = 7$ and $BQ = 3$, so its area is $\frac{1}{2} \cdot 7 \cdot 3 = \frac{21}{2}$. The other three corners give the same triangle turned around the square. The four together have area $4 \cdot \frac{21}{2} = 42$, so $PQRS$ has area $100 - 42 = 58$.

Check. Turning the whole picture a quarter turn about the center carries P to Q, Q to R, and so on, so $PQRS$ is a square. Its side is the hypotenuse of a 3 by 7 right triangle, so its area is $3^2 + 7^2 = 58$, matching.

3. Answer: 3**. The idea.** The equation does not pin down x and y separately, only their ratio. Find that ratio, and the second fraction follows, since it too depends only on the ratio.

Work it out. Cross-multiply: $4(2x + y) = 5(x + 2y)$, so $8x + 4y = 5x + 10y$, which gives $3x = 6y$ and $x = 2y$. Then $\dfrac{x + y}{x - y} = \dfrac{2y + y}{2y - y} = \dfrac{3y}{y} = 3$.

Check. Take $y = 1$, $x = 2$: $\frac{2 \cdot 2 + 1}{2 + 2 \cdot 1} = \frac{5}{4}$, as required, and $\frac{2+1}{2-1} = 3$. Any other choice, such as $y = 5$, $x = 10$, gives $\frac{25}{20} = \frac{5}{4}$ and $\frac{15}{5} = 3$ again.

4. Answer: $16\sqrt{3}$**. The idea.** The fence length (the perimeter) is what the two gardens share, not the area. Go from area to side to perimeter, then back down to the triangle's side and area.

Work it out. The square has side $\sqrt{36} = 6$, so its perimeter is $4 \cdot 6 = 24$ meters. The triangle's three equal sides are then $24 \div 3 = 8$ meters each. An equilateral triangle of side s has height $\frac{\sqrt{3}}{2}s$, so its area is $\frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$. With $s = 8$: $\frac{\sqrt{3}}{4} \cdot 64 = 16\sqrt{3}$.

Check. The height is $4\sqrt{3}$ (half the side is 4, and $4^2 + (4\sqrt{3})^2 = 16 + 48 = 64 = 8^2$), so the area is $\frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3} \approx 27.7$, a little less than the square's 36, as expected: for a fixed perimeter the square encloses more than the triangle.

5. Answer: 224**. The idea.** Fill the most restricted places first. The thousands digit is fixed as 2, and the units digit must be even and different from that 2; after those, the two middle digits are free choices from what is left.

Work it out. The thousands digit is 2. The units digit must be even but not 2: one of $0, 4, 6, 8$, so 4 choices. That uses two different digits, leaving 8 digits for the hundreds place and then 7 for the tens place. The count is $4 \cdot 8 \cdot 7 = 224$.

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Check. Count all years 2000 to 2999 with different digits: $9 \cdot 8 \cdot 7 = 504$ (the last three places are an ordered choice from the 9 digits other than 2). Among those 9 digits, 4 are even and 5 odd, and by symmetry each digit is equally likely to be the units digit, so the even ones are $\frac{4}{9} \cdot 504 = 224$. A common slip is to allow 5 even units digits, which counts years like 2312 with a repeated 2.

6. Answer: 90**. The idea.** Imagine first that Kai walks the whole 3 hours, then trade walking time for running time. Every hour switched from walking to running adds $10 - 4 = 6$ km, so the missing distance tells us how long he ran.

Work it out. Walking for all 3 hours would cover $3 \cdot 4 = 12$ km, which is $21 - 12 = 9$ km short. Each hour of running instead of walking gains 6 km, so he runs for $\frac{9}{6} = \frac{3}{2}$ hours, which is 90 minutes.

Check. He walks $1\frac{1}{2}$ hours, covering 6 km, and runs $1\frac{1}{2}$ hours, covering 15 km: $6 + 15 = 21$ km in 3 hours. With equations: $w + r = 3$ and $4w + 10r = 21$ give $6r = 9$, the same.

7. Answer: 14**. The idea.** Every box appears in exactly two of the three pairs. So adding the three pair weights counts each box twice, and gives twice the total weight of all three boxes.

Work it out. $17 + 21 + 24 = 62$, so the three boxes together weigh $62 \div 2 = 31$ kilograms. The heaviest box is the one left out of the lightest pair, so it weighs $31 - 17 = 14$ kilograms. (The other two weigh $31 - 21 = 10$ and $31 - 24 = 7$.)

Check. The boxes weigh 7, 10 and 14: $7 + 10 = 17$, $7 + 14 = 21$ and $10 + 14 = 24$, the three given pair weights. A common slip is to take the heaviest box from the heaviest pair, as $31 - 24 = 7$, which is actually the lightest box.

8. Answer: 528**. The idea.** In any full day, every hour from 1 to 12 comes around exactly twice, and there are 24 half hours. Then watch the ends of the time window: which midnight chimes does the clock actually hear?

Work it out. Hourly chimes in one day: $2(1 + 2 + \cdots + 12) = 2 \cdot 78 = 156$. Half-hour chimes: 24. So a day that includes its own 12:00 a.m. strike has $156 + 24 = 180$ chimes.

Three such days (Monday, Tuesday, Wednesday, each starting at its own midnight) would give $3 \times 180 = 540$. But the clock is switched on at 12:01 a.m. Monday, one minute too late to hear Monday's midnight strike of 12. It does hear the midnight strikes that start Tuesday and Wednesday, and it is off before Thursday's. So the total is $540 - 12 = 528$.

Check. Count by kind: the clock hears the midnight strike only twice (Tuesday and Wednesday), the noon strike three times, and every other hour 1 to 11 six times (a.m. and p.m. on three days): $6 \cdot 66 + 3 \cdot 12 + 2 \cdot 12 = 396 + 36 + 24 = 456$, plus $3 \cdot 24 = 72$ half-hour chimes, for 528.

9. Answer: 6**. The idea.** Count all possible pairs of people, then remove the pairs that are couples.

Work it out. With n couples there are $2n$ people and $\binom{2n}{2} = n(2n - 1)$ pairs. Exactly n of those pairs are spouses, so the handshakes number $n(2n - 1) - n = 2n^2 - 2n = 2n(n - 1)$. Setting $2n(n - 1) = 60$ gives $n(n - 1) = 30$, so $n = 6$.

Check. Count per person: each of the 12 people shakes the 10 people other than themself and their spouse, giving $12 \cdot 10 = 120$ handshakes counted twice, so 60. It works for 6 couples.

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10. Answer: 42**. The idea.** Both numbers are multiples of 6, so write them as $6x$ and $6y$. Because 6 is the greatest common divisor, x and y can share no factor bigger than 1; otherwise the two numbers would have a larger common divisor.

Work it out. From $6x \cdot 6y = 360$ we get $xy = 10$. The ways to split 10 into two factors with no common factor are $1 \cdot 10$ and $2 \cdot 5$. The split $1 \cdot 10$ gives the numbers 6 and 60, which is ruled out, so the split is $2 \cdot 5$: the numbers are 12 and 30, and their sum is 42.

Check. $12 \cdot 30 = 360$, and the greatest common divisor of $12 = 2^2 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ is $2 \cdot 3 = 6$. A pair like 18 and 20 also has product 360, but its greatest common divisor is 2.

11. Answer: 64**. The idea.** The amount of chocolate is a volume, and volume grows with length times width times height. A rabbit 4 times as tall is also 4 times as wide and 4 times as deep, so it holds $4 \cdot 4 \cdot 4$ times the chocolate, not 4 times.

Work it out. Every length of the big rabbit is $\frac{20}{5} = 4$ times the matching length of a small one, so its volume is $4^3 = 64$ times a small rabbit's volume. All of that chocolate is reused, so she makes 64 small rabbits.

Check. Picture the big rabbit cut into tiny cubes of edge 4 mm. The small rabbit uses the same pattern of cubes with edge 1 mm, and one 4 mm cube is a $4 \times 4 \times 4$ stack of 64 cubes of edge 1 mm. The tempting answer 4 scales only the height.

12. Answer: 3**. The idea.** A number leaves the same remainder when divided by 9 as the sum of its digits. So only the total of all the digits matters, and the digits of each written integer add up to something with the same remainder as that integer itself.

Work it out. The digit sum of the long number is the sum of the digit sums of $1, 2, \ldots, 20$. Each of those digit sums has the same remainder mod 9 as the integer itself (for example, $17 \to 1+7 = 8$, and $17 = 9 + 8$). So the long number has the same remainder as $1 + 2 + \cdots + 20 = \frac{20 \cdot 21}{2} = 210$. Since $210 = 9 \cdot 23 + 3$, the remainder is 3.

Check. Add the digits directly: 1 through 9 give 45; 10 through 19 give ten 1s plus $0 + 1 + \cdots + 9$, which is $10 + 45 = 55$; 20 gives 2. The total is $45 + 55 + 2 = 102 = 9 \cdot 11 + 3$, again remainder 3.

13. Answer: 25569**. The idea.** A number with fewer digits is always smaller, so first find the fewest digits possible, then arrange the best choice of digits in increasing order.

Work it out. Factor: $2700 = 2^2 \cdot 3^3 \cdot 5^2$. No digit can be 0, and the only nonzero digit divisible by 5 is 5 itself, so the number has exactly two digits equal to 5. The other digits multiply to $2^2 \cdot 3^3 = 108$.

Two digits multiply to at most $9 \cdot 9 = 81 < 108$, so 108 needs at least three digits, and the number has at least 5 digits. Three digits with product 108: a digit 1 would leave 108 for two digits, impossible, so list the options with smallest digit 2 or more: $2 \cdot 6 \cdot 9$, $3 \cdot 4 \cdot 9$, $3 \cdot 6 \cdot 6$.

For the smallest 5-digit number we want the smallest possible leading digit, which is 2 (from $\{2, 6, 9\}$). Writing $2, 5, 5, 6, 9$ in increasing order gives 25569.

Check. $2 \cdot 5 \cdot 5 \cdot 6 \cdot 9 = 50 \cdot 54 = 2700$. The other two digit sets give 34559 and 35566, both larger. A common slip is to break 108 greedily as $9 \cdot 4 \cdot 3$ (biggest digits first), which gives 34559; the leading digit matters most.

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14. Answer: 40**. The idea.** After n weeks Sam has $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ dollars and Lily has $20n$ dollars. Compare the two totals, and watch for the week when they are exactly equal, which is not yet "greater".

Work it out. Sam is ahead when $\frac{n(n+1)}{2} > 20n$. Dividing by the positive number $\frac{n}{2}$ gives $n + 1 > 40$, so $n > 39$. At $n = 39$ the totals are equal: $\frac{39 \cdot 40}{2} = 780 = 20 \cdot 39$. The first week Sam is strictly ahead is week 40.

Check. After week 40 Sam has $\frac{40 \cdot 41}{2} = 820$ dollars and Lily has 800. Another view: Sam's average weekly saving over n weeks is $\frac{n+1}{2}$ dollars, and it must beat Lily's steady \$20, so $\frac{n+1}{2} > 20$.

15. Answer: 35**. The idea.** In a triangle, the segment joining the midpoints of two sides is parallel to the third side and half as long. Each side of the midpoint quadrilateral sits in a triangle cut off by one diagonal, so it is parallel to that diagonal and half its length.

Work it out. Call the quadrilateral $ABCD$. In triangle ABC, the midpoints of $\overline{AB}$ and $\overline{BC}$ are joined by a segment parallel to $\overline{AC}$ of length $\frac{14}{2} = 7$ (say $AC = 14$); the same holds on the other side of $\overline{AC}$ in triangle ADC. Likewise the other two sides are parallel to $\overline{BD}$ with length $\frac{10}{2} = 5$. So the midpoint quadrilateral has opposite sides parallel, and since the diagonals are perpendicular, adjacent sides are perpendicular: it is a 5 by 7 rectangle, with area 35.

Check. A quadrilateral with perpendicular diagonals d_1, d_2 has area $\frac{1}{2}d_1d_2 = \frac{1}{2} \cdot 10 \cdot 14 = 70$, and the midpoint quadrilateral always has half the area of the original: 35. With coordinates $A(-3, 0)$, $B(0, -4)$, $C(7, 0)$, $D(0, 10)$, the midpoints are $(-1.5, -2)$, $(3.5, -2)$, $(3.5, 5)$, $(-1.5, 5)$: a 5 by 7 rectangle.

16. Answer: 10**. The idea.** Factor the difference of squares: $r^2 - s^2 = (r - s)(r + s)$. The sum $r + s$ is known from the equation, so this tells us the difference $r - s$, and then the roots themselves.

Work it out. Let the roots be $r > s$. By Vieta, $r + s = 7$ and $rs = k$. Then $21 = r^2 - s^2 = (r - s)(r + s) = 7(r - s)$, so $r - s = 3$. With $r + s = 7$ this gives $r = 5$, $s = 2$, and $k = rs = 10$.

Check. $x^2 - 7x + 10 = (x - 5)(x - 2)$ has roots 5 and 2, and $25 - 4 = 21$. Also $(r - s)^2 = (r + s)^2 - 4rs = 49 - 4k = 9$ gives $k = 10$ again.

17. Answer: 3**. The idea.** The garden's length and width are not known, but they are not needed: the area of a path around a rectangle depends only on the rectangle's perimeter and the path's width. Split the path into pieces to see this.

Work it out. Let the path have width w. Cut the path into four strips running along the four sides of the garden and four w-by-w squares at the corners. The strips have total length equal to the garden's perimeter, 34, and width w, so together they have area $34w$; the corner squares add $4w^2$. So $4w^2 + 34w = 138$, that is $2w^2 + 17w - 69 = 0$, which factors as $(w - 3)(2w + 23) = 0$. The width is positive, so $w = 3$ meters.

Check. Take a 10-by-7 garden (perimeter 34). With a 3-meter path the outer rectangle is 16 by 13, and $16 \cdot 13 - 10 \cdot 7 = 208 - 70 = 138$. A 12-by-5 garden gives $18 \cdot 11 - 60 = 198 - 60 = 138$ as well.

18. Answer: $\frac{1}{6}$**. The idea.** Choose which two letters are right, and then make sure that each of the other three is wrong. Three letters that are all in wrong envelopes must form a cycle, and there are exactly 2 ways to do that.

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Work it out. There are $5! = 120$ equally likely arrangements. Choose the two correct letters in $\binom{5}{2} = 10$ ways. Call the other three letters A, B, C and their envelopes a, b, c. No letter may go in its own envelope, so A goes in b or c, and after that everything is forced: $A \to b, B \to c, C \to a$ or $A \to c, C \to b, B \to a$. That makes $10 \cdot 2 = 20$ good arrangements, and the probability is $20/120 = 1/6$.

Check. Sort all 120 arrangements by the number of correct letters: 44 with none, 45 with one, 20 with two, 10 with three, 0 with exactly four (the fifth letter would have to be right too), and 1 with five. These add up to $44 + 45 + 20 + 10 + 0 + 1 = 120$.

19. Answer: $\frac{9}{38}$. The idea. Whether a sum is a multiple of 4 depends only on the remainders of the two numbers when divided by 4. Sort 1 through 20 by remainder; each remainder class has exactly five members, and the good pairs come from only three combinations of classes.

Work it out. The classes are remainder 0: $\{4, 8, 12, 16, 20\}$, remainder 1: $\{1, 5, 9, 13, 17\}$, remainder 2: $\{2, 6, 10, 14, 18\}$, remainder 3: $\{3, 7, 11, 15, 19\}$. Two remainders add to a multiple of 4 exactly when they are $0 + 0$, $2 + 2$, or $1 + 3$. Two numbers from class 0: $\binom{5}{2} = 10$ pairs. Two from class 2: 10 pairs. One from class 1 and one from class 3: $5 \cdot 5 = 25$ pairs. That is $10 + 10 + 25 = 45$ good pairs out of $\binom{20}{2} = 190$, so the probability is $\frac{45}{190} = \frac{9}{38}$.

Check. Count from the first number's point of view: whatever the first number is, the second must lie in one particular class (the same class for remainders 0 and 2, the partner class for 1 and 3). That class has 4 other numbers when it is the first number's own class and 5 otherwise, so the chance is $\frac{1}{2} \cdot \frac{4}{19} + \frac{1}{2} \cdot \frac{5}{19} = \frac{9}{38}$.

20. Answer: $5\sqrt{2}$. The idea. The perpendicular from the center to a chord cuts the chord in half. So if the center is at distance d_1 from the chord of length 10 and d_2 from the chord of length 14, then $r^2 = d_1^2 + 5^2 = d_2^2 + 7^2$. The one thing to decide is whether the chords lie on the same side of the center or on opposite sides.

Work it out. From the two expressions for r^2, $d_1^2 - d_2^2 = 49 - 25 = 24$, that is $(d_1 - d_2)(d_1 + d_2) = 24$. If the chords were on the same side of the center, then $d_1 - d_2 = 6$ and $d_1 + d_2 = 4$, which would make d_2 negative, so that is impossible. So they are on opposite sides: $d_1 + d_2 = 6$ and $d_1 - d_2 = 4$, giving $d_1 = 5$ and $d_2 = 1$. Then $r^2 = 5^2 + 5^2 = 50$, so $r = 5\sqrt{2}$.

Check. With the chord of length 14 at distance 1: $1^2 + 7^2 = 50$ as well, and $5 + 1 = 6$ is the distance between the chords. The longer chord is the one closer to the center, as it should be.

21. Answer: -6. The idea. Write out both compositions carefully (the order matters: $f(g(x))$ squares $2x + 3$, while $g(f(x))$ doubles $x^2 - 1$), set them equal, and get a quadratic. Before using the sum-of-roots shortcut, make sure the quadratic really has two real roots.

Work it out. $f(g(x)) = (2x + 3)^2 - 1 = 4x^2 + 12x + 8$ and $g(f(x)) = 2(x^2 - 1) + 3 = 2x^2 + 1$. Setting them equal: $4x^2 + 12x + 8 = 2x^2 + 1$, so $2x^2 + 12x + 7 = 0$. Its discriminant is $12^2 - 4 \cdot 2 \cdot 7 = 144 - 56 = 88 > 0$, so there are two different real solutions, and their sum is $-\frac{12}{2} = -6$.

Check. The quadratic formula gives $x = \frac{-12 \pm \sqrt{88}}{4} = -3 \pm \frac{\sqrt{22}}{2}$, and these add to -6. Taking one of them, $x \approx -0.655$: then $g(x) \approx 1.690$ and $f(g(x)) \approx 1.857$, while $f(x) \approx -0.571$ and $g(f(x)) \approx 1.857$.

22. Answer: 84. **The idea.** In a 2-by-2 grid each square shares a side with two others, and the only

pairs that do not share a side are the two diagonal pairs. So the only freedom is whether each diagonal pair is painted one color or two. Split into cases on that.

Work it out. Call the diagonal pairs $\{P, Q\}$ (top left and bottom right) and $\{R, S\}$ (top right and bottom left). Every square of one pair shares a side with every square of the other pair.

Case 1: both pairs are single-colored. Pick the color of $\{P, Q\}$ (4 ways) and a different color for $\{R, S\}$ (3 ways): 12 flags.

Case 2: exactly one pair is single-colored. Choose which pair (2 ways) and its color (4 ways); the other two squares must use two different colors, both different from the first: $3 \cdot 2 = 6$ ways. That is $2 \cdot 4 \cdot 6 = 48$ flags.

Case 3: neither pair is single-colored. All four squares are then different colors: $4 \cdot 3 \cdot 2 \cdot 1 = 24$ flags.

In all, $12 + 48 + 24 = 84$.

Check. Split instead on whether the top left and bottom right squares match. If they match (4 colors), each of the other two squares may be any of the other 3 colors: $4 \cdot 3 \cdot 3 = 36$. If they differ ($4 \cdot 3 = 12$ ways), each of the other two squares must avoid both of those colors, leaving 2 choices each: $12 \cdot 2 \cdot 2 = 48$. Total $36 + 48 = 84$.

23. Answer: 100**. The idea.** By symmetry the rectangle is centered on the center O of the diameter. A top vertex is a point on the circle, so its horizontal and vertical distances from O satisfy the Pythagorean theorem. The area is then $2xy$, and the inequality $(x-y)^2 \geq 0$ compares $2xy$ directly with $x^2 + y^2$, which is fixed.

Work it out. Let the rectangle have half-width x and height y. Its top right vertex is at distance 10 from O, so $x^2 + y^2 = 100$. The area is $(2x) \cdot y = 2xy$. Since $(x-y)^2 \geq 0$, we get $x^2 + y^2 \geq 2xy$, so the area is at most 100. Equality holds when $x = y$, that is $2x^2 = 100$, $x = y = 5\sqrt{2}$: a rectangle $10\sqrt{2}$ wide and $5\sqrt{2}$ tall, with area $10\sqrt{2} \cdot 5\sqrt{2} = 100$. So the largest area is 100.

Check. Try other shapes: $x = 6$, $y = 8$ gives area $2 \cdot 6 \cdot 8 = 96$; $x = 8$, $y = 6$ gives 96 too; both are less than 100. A second view: reflect the rectangle across the diameter to get a rectangle inscribed in the full circle, with diagonal 20. That rectangle has area 2·(our area), and a rectangle with diagonal 20 has area at most $\frac{20^2}{2} = 200$ (a square), so our area is at most 100.

24. Answer: 6**. The idea.** Measure grass in "cow-days" (what one cow eats in a day). The grass eaten equals the starting grass plus everything that grew meanwhile. Comparing the two herds removes the unknown starting amount and reveals the daily growth.

Work it out. Let G be the starting grass and g the daily growth. The 12 cows eat $12 \cdot 24 = 288$ cow-days, so $G + 24g = 288$. The 18 cows eat $18 \cdot 12 = 216$, so $G + 12g = 216$. Subtracting, $12g = 72$, so $g = 6$ and $G = 144$. With 30 cows, 6 of them just keep up with the growth and the other 24 eat the starting 144: that takes $144/24 = 6$ days.

Check. In 6 days, 30 cows eat 180 cow-days, and the pasture supplies $144 + 6 \cdot 6 = 180$. The trend is also sensible: more cows ($12 \to 18 \to 30$) finish faster ($24 \to 12 \to 6$ days), but not in proportion, because of the regrowth.

25. Answer: 980**. The idea.** Describe each kind of sum by a divisibility condition. Seven consecutive integers are balanced around the middle one, so their sum is 7 times the middle number.

Eight consecutive integers have no middle number, and their sum always lands 4 past a multiple of 8. Combine the two conditions.

Work it out. Seven in a row: $(m-3)+\cdots+(m+3) = 7m$ with $m \geq 4$, so the possible sums are the multiples of 7 from 28 on. Eight in a row: $a+(a+1)+\cdots+(a+7) = 8a+28 = 8(a+3)+4$ with $a \geq 1$, so the possible sums are the numbers from 36 on that leave remainder 4 when divided by 8. We need N to be a multiple of 7 and 4 more than a multiple of 8. Among $4, 12, 20, 28, \ldots$ the first multiple of 7 is 28, and the pattern repeats every $\operatorname{lcm}(7,8) = 56$, so $N = 28, 84, 140, \ldots$ (and 28 itself is too small for eight positive integers, whose sum is at least 36). The largest below 1000 is $28 + 56 \cdot 17 = 980$.

Check. $980 = 7{\cdot}140 = 137{+}138{+}\cdots{+}143$, and $980 = 8a{+}28$ gives $a = 119$: $119{+}120{+}\cdots{+}126 = 8 \cdot 122.5 = 980$. The next candidate, 1036, is over 1000.

26. Answer: $3 - 2\sqrt{2}$. **The idea.** Tangency to two circles means the distance between centers is the sum of the radii. The small circle's center sits on the diagonal $\overline{AC}$ (it is equally far from the two sides at C), so everything happens along that diagonal.

Work it out. Put $A = (0,0)$ and $C = (1,1)$. A circle of radius r tangent to the lines $x = 1$ and $y = 1$, inside the square, has center $(1-r, 1-r)$, at distance $\sqrt{2}(1-r)$ from A. It is outside the quarter circle and tangent to it, so this distance is $1 + r$: $\sqrt{2} - \sqrt{2}r = 1 + r$, so $r = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$.

Check. Along the diagonal, the distance from A to C is $\sqrt{2} \approx 1.414$. The quarter circle uses 1 of it, leaving about 0.414, and the small circle takes up its diameter $2r$ plus the extra stretch $\sqrt{2}r - r$ into the corner: $r(1+\sqrt{2}) = (3-2\sqrt{2})(1+\sqrt{2}) = \sqrt{2} - 1 \approx 0.414$. It fits exactly.

27. Answer: 88592**. The idea.** The terms grow fast and irregularly, but look at how far each term is above 2: $5-2=3$, $11-2=9$, $29-2=27$. The number 2 is special because the rule leaves it unchanged ($3 \cdot 2 - 4 = 2$), and the distance from 2 is exactly tripled at every step. So each term is 2 plus a power of 3, and the sum splits into a geometric series plus a constant.

Work it out. If $a_n = 2 + d$, then $a_{n+1} = 3(2+d) - 4 = 2 + 3d$: the excess over 2 triples. It starts at $a_1 - 2 = 3$, so $a_n = 2 + 3^n$. The sum of the first ten terms is $10 \cdot 2 + (3 + 3^2 + \cdots + 3^{10})$. For the geometric part, let $S = 3 + 3^2 + \cdots + 3^{10}$; then $3S - S = 3^{11} - 3$, so $S = \frac{177147-3}{2} = 88572$. The total is $20 + 88572 = 88592$.

Check. Listing the terms directly: $5, 11, 29, 83, 245, 731, 2189, 6563, 19685, 59051$, each 2 more than $3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049$. Adding: $5 + 11 + 29 + 83 + 245 = 373$, $731 + 2189 + 6563 = 9483$, $19685 + 59051 = 78736$, and $373 + 9483 + 78736 = 88592$.

28. Answer: 112**. The idea.** The circle makes direct counting awkward, because the first and last marks are also neighbors. Fix one circled mark to break the circle into a straight row, count there, and then correct for having fixed a mark.

Work it out. First count the choices that include mark 12. Then 11 and 1 may not be circled, and the other two circled marks come from the row $2, 3, \ldots, 10$ of 9 marks, with no two next to each other. Choosing 2 non-adjacent marks from a row of 9: of the $\binom{9}{2} = 36$ pairs, 8 are adjacent, leaving 28. By symmetry each of the 12 marks is in 28 good choices, giving $12 \cdot 28 = 336$; but every choice contains 3 circled marks and so was counted 3 times. The answer is $336/3 = 112$.

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Check. Count the bad choices instead. Of $\binom{12}{3} = 220$ choices, there are 12 with all three marks in a row (one for each starting mark). Those with exactly one neighboring pair: 12 pairs of neighbors, and the third mark avoids the pair and its two outside neighbors, so $12 - 4 = 8$ choices each, giving 96. Then $220 - 12 - 96 = 112$.

29. Answer: $2\sqrt{6}$. **The idea.** Join V to the two ends of the diagonal. One end is joined to V by an edge and the other by a face diagonal, and these two are perpendicular. So V and the diagonal form a right triangle whose hypotenuse is the space diagonal, and the distance we want is the altitude to the hypotenuse, found by computing the area two ways.

Work it out. Every space diagonal not through V has one end U adjacent to V (joined by an edge) and the other end W on a face diagonal from V. For example, with $V = (0,0,0)$, $U = (6,0,0)$ and $W = (0,6,6)$. Then $VU = 6$, $VW = 6\sqrt{2}$, $UW = 6\sqrt{3}$, and since $6^2 + (6\sqrt{2})^2 = 36 + 72 = 108 = (6\sqrt{3})^2$, the angle at V is a right angle. The area of $\triangle VUW$ is $\frac{1}{2} \cdot 6 \cdot 6\sqrt{2} = 18\sqrt{2}$, and also $\frac{1}{2} \cdot 6\sqrt{3} \cdot h$. So $h = \frac{36\sqrt{2}}{6\sqrt{3}} = \frac{6\sqrt{2}}{\sqrt{3}} = 2\sqrt{6}$.

Check. Let t run along the diagonal: the point $(6 - 6t, 6t, 6t)$ goes from U to W. Its squared distance from V is $36(1-t)^2 + 72t^2 = 36(3t^2 - 2t + 1)$, which is smallest at $t = \frac{1}{3}$, where it equals $36 \cdot \frac{2}{3} = 24$. So the distance is $\sqrt{24} = 2\sqrt{6}$, reached at a point inside the diagonal.

30. Answer: 34. **The idea.** Factor 10! into primes. A divisor is a square exactly when every exponent is even, and a cube exactly when every exponent is a multiple of 3. Count each kind, then use inclusion-exclusion to remove the divisors counted twice.

Work it out. By counting factors of each prime in $1, \ldots, 10$: $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. Squares: the exponent of 2 is one of $0, 2, 4, 6, 8$ (5 choices), of 3 is $0, 2, 4$ (3), of 5 is $0, 2$ (2), of 7 is 0 (1), for 30 in all. Cubes: exponent of 2 is $0, 3, 6$ (3), of 3 is $0, 3$ (2), and of 5 and of 7 only 0: 6 cubes. Both square and cube means a sixth power: exponent of 2 is 0 or 6 and every other exponent is 0, so there are 2 (1 and 64). The total is $30 + 6 - 2 = 34$.

Check. The six cube divisors are $1, 8, 64, 27, 216, 1728$, and exactly 1 and 64 are also squares. So the cubes add 4 new divisors to the 30 squares: 34.

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