MathCastle

CHAPTER COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A rectangular field measures 36 m by 20 m. A fence runs all the way around it. There is a post at each of the four corners, and further posts are placed so that neighbouring posts are exactly 4 m apart along each 36 m side and exactly 5 m apart along each 20 m side. How many posts are there in all?

1. _____

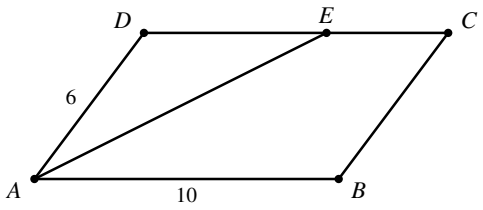
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2. Ann and Bo each have some money. If Ann gives Bo \$5, they will have the same amount. If instead Bo gives Ann \$5, Ann will have exactly twice as much as Bo. How many dollars do Ann and Bo have altogether?

2. _____

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3. Parallelogram $ABCD$ has $AB = 10$ and $AD = 6$, where $\overline{AB}$ and $\overline{CD}$ are the longer sides. The bisector of $\angle DAB$ meets side $\overline{CD}$ at the point E. What is the length CE?



3. _____

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4. The four digits of 2026 (two 2s, a 0 and a 6) are rearranged in every possible way to form four-digit numbers; a number may not begin with 0. Counting each different number once, how many of these numbers are divisible by 4?

4. _____

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5. A parking meter takes only nickels, dimes and quarters, and Maya wants to put in exactly 75 cents. She may use any number of each kind of coin, including none of a kind. Two payments count as the same if they use the same number of nickels, the same number of dimes and the same number of quarters (the order the coins go in does not matter). In how many different ways can Maya pay?

5. _____

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6. Every student in a class is either a girl or a boy. Three fifths of the girls is the same number of students as one half of the boys. What fraction of the whole class are girls? Express your answer as a common fraction.

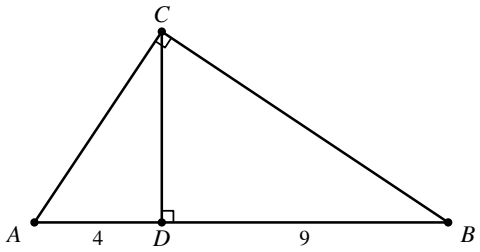
6. _____

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7. A double-six set of dominoes has 28 tiles: one tile for each unordered pair of numbers from 0 to 6, with the doubles 0-0, 1-1, . . ., 6-6 included (the tile 2-5 is the same as 5-2). How many of the 28 tiles have an even total number of pips, adding both ends?

7. _____

8. In right triangle ABC the right angle is at C. The altitude from C meets the hypotenuse $\overline{AB}$ at D, with $AD = 4$ and $DB = 9$. What is the area of triangle ABC?



8. _____

9. Two sequences go on forever. The first is $7, 11, 15, 19, \ldots$, going up by 4 each time, and the second is $2, 9, 16, 23, \ldots$, going up by 7 each time. How many numbers less than 2026 appear in both sequences?

9. _____

10. Let $S = 1! + 2! + 3! + \cdots + 100!$, where $n!$ is the product $1 \cdot 2 \cdot 3 \cdots n$. What is the remainder when S is divided by 36?

10. _____

11. A cake mix is made of flour, sugar and butter in the ratio 5 : 2 : 3 by weight, and the ratio must be kept exactly. Mia has 450 g of flour, 200 g of sugar and as much butter as she needs. She does not have to use all of her flour or sugar. What is the greatest total weight of mix, in grams, that she can make?

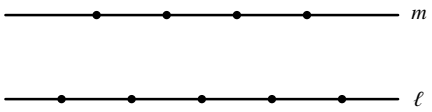
11. _____

12. The number 7^{2026} is written out in full in base ten. What two-digit number do its last two digits form? (For example, the last two digits of $7^3 = 343$ form the number 43.)

12. _____

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13. Five different points are marked on a line ℓ, and four different points are marked on a second line m that is parallel to ℓ. How many triangles have all three of their vertices among these nine points?



13. _____

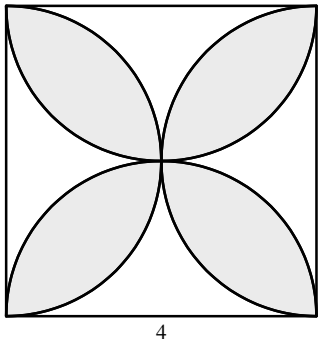
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14. Two runners start together from the same point on a circular track and run in the same direction, each at a steady speed. Pat completes a lap every 60 seconds and Quinn completes a lap every 84 seconds. After how many seconds does Pat first catch up with Quinn from behind, having run exactly one more lap than Quinn?

14. _____

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15. A square has side length 4. On each of its four sides, a semicircle is drawn inside the square with that side as its diameter. Adjacent semicircles overlap, and the overlaps form four petal-shaped regions, each running from a corner of the square to its center. What is the total area of the four petals? Express your answer in terms of π.



15. _____

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16. Seven numbers are written in a row. The average of the first four numbers is 6, the average of the last four numbers is 11, and the average of all seven numbers is 8. What is the fourth number in the row?

16. _____

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17. A function has the form $f(x) = ax + b$, where a and b are constants and $a > 0$. Applying f twice gives $f(f(x)) = 4x + 9$ for every real number x. What is the value of $f(5)$?

17. _____

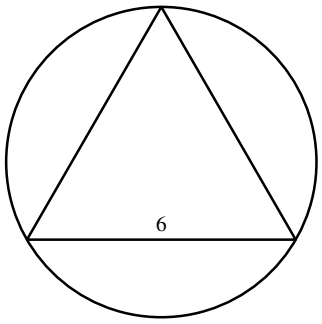
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18. Using each of the digits $1, 3, 5, 7, 9$ exactly once, Mia writes down a three-digit number and a two-digit number and multiplies them together. What is the largest product she can get?

18. _____

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19. An equilateral triangle with side length 6 has all three of its vertices on a circle. What is the area of the region that lies inside the circle but outside the triangle? Express your answer in terms of π and in simplest radical form.



19. _____

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20. A group of friends agreed to rent a boat for \$120 and to split the cost equally. On the day of the trip, two of the friends could not come, so the \$120 was split equally among the friends who did come, and each of them paid \$5 more than had been planned. How many friends went on the boat trip?

20. _____

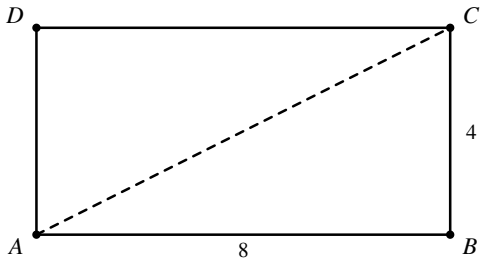
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21. A pair of fair six-sided dice is rolled again and again, and each time the two numbers are added. The rolling stops as soon as the sum is 7 or 8. What is the probability that the rolling stops on a sum of 7? Express your answer as a common fraction.

21. _____

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22. A rectangular sheet of paper $ABCD$ has $AB = 8$ and $BC = 4$. It is folded along the diagonal $\overline{AC}$, so that triangle ACD flips over onto the other half of the sheet, and the paper is pressed flat. What is the area of the region where the folded paper is two layers thick?



22. _____

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23. A geometric sequence of real numbers has first term a and common ratio r. The sum of its first 2 terms is 7, and the sum of its first 6 terms is 91. What is the sum of its first 4 terms?

23. _____

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24. A five-digit palindrome is a number of the form $\overline{abcba}$, with digits a, b, c and $a \neq 0$; for example, 37273. How many five-digit palindromes are divisible by 11?

24. _____

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25. Six coins lie in a row, all showing heads. A move consists of choosing two coins that are next to each other and turning both of them over. After some number of moves (possibly none), how many different patterns of heads and tails can the row of six coins show?

25. _____

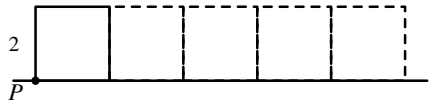
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26. When each of the numbers 2023, 2146 and 2310 is divided by the same whole number d, where d is greater than 1, all three divisions leave the same remainder r. What is r?

26. _____

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27. A square with side length 2 rests on a horizontal line. It is rolled to the right along the line: each roll turns the square 90° clockwise about its bottom-right corner, until the next side lies flat on the line. After four rolls the square rests on the line in its original position, moved 8 units to the right. Point P is the vertex of the square that was at the bottom-left before the first roll. What is the total length of the path traced by P during the four rolls? Express your answer in terms of π and in simplest radical form.



27. _____

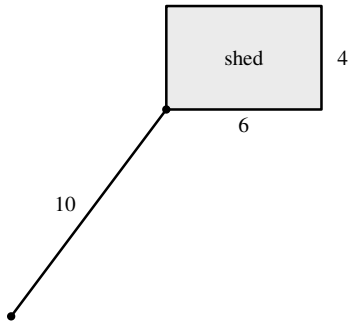
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28. A three-digit positive integer is exactly 5 times the product of its three digits. What is this integer?

28. _____

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29. A shed with a rectangular floor, 6 m by 4 m, stands in an open, grassy field. A goat is tied to one corner of the shed with a rope 10 m long. The goat cannot enter the shed or pass through its walls, but the rope can bend around the shed's corners. How many square metres of grass can the goat reach? Express your answer in terms of π.



29. _____

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30. Four standard six-sided dice (faces 1 through 6) are rolled. What is the probability that the largest number rolled is exactly 3 more than the smallest number rolled? Express your answer as a common fraction.

30. _____

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Answer Key — MathCounts Chapter Sprint Practice Contest 1

1. 26

2. 60

3. 4

4. 3

5. 18

6. $\frac{5}{11}$

7. 16

8. 39

9. 72

10. 9

11. 900

12. 49

13. 70

14. 210

15. $8\pi - 16$

16. 12

17. 13

18. 69843

19. $12\pi - 9\sqrt{3}$

20. 6

21. $\frac{6}{11}$

22. 10

23. 28

24. 82

25. 32

26. 14

27. $(2 + \sqrt{2})\pi$

28. 175

29. 88π

30. $\frac{55}{216}$

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Solutions — MathCounts Chapter Sprint Practice Contest 1

1. **Answer:** 26. **The idea.** On a straight fence with both ends open, there is one more post than there are gaps. But this fence is a closed loop: walking all the way around, every gap is followed by exactly one new post, and the last gap brings you back to the first post. So the number of posts equals the number of gaps, and the corners take care of themselves.

   Work it out. Each 36 m side has $36 \div 4 = 9$ gaps and each 20 m side has $20 \div 5 = 4$ gaps. Around the loop there are $9 + 4 + 9 + 4 = 26$ gaps, so there are 26 posts.

   Check. Count side by side instead. A 36 m side with posts at both ends has 10 posts, and a 20 m side has 5. That gives $10 + 5 + 10 + 5 = 30$, but each of the 4 corner posts was counted on two sides, so the true count is $30 - 4 = 26$.

2. **Answer:** 60. **The idea.** The first fact tells you the gap between them; the second then pins down the amounts.

   Work it out. Let Ann have A dollars and Bo B. After Ann gives \$5, $A - 5 = B + 5$, so $A = B + 10$. If instead Bo gives \$5, $A + 5 = 2(B - 5)$. Substitute: $B + 15 = 2B - 10$, so $B = 25$ and $A = 35$. Together they have $25 + 35 = 60$ dollars.

   Check. Ann gives \$5: both have \$30. Bo gives \$5: Ann has \$40 and Bo \$20, and $40 = 2 \cdot 20$.

3. **Answer:** 4. **The idea.** Because $\overline{AB}$ and $\overline{DC}$ are parallel, the bisector crosses $\overline{DC}$ at the same angle at which it leaves $\overline{AB}$. That makes triangle ADE isosceles, and the answer does not depend on the angles of the parallelogram at all.

   Work it out. Let the bisector split $\angle DAB$ into two equal angles x, so $\angle DAE = \angle EAB = x$. Since $\overline{DC} \parallel \overline{AB}$, the alternate angles $\angle DEA$ and $\angle EAB$ are equal, so $\angle DEA = x$. In triangle ADE the angles at A and E are both x, so the sides opposite them are equal: $DE = AD = 6$. Since $DC = AB = 10$, we get $CE = DC - DE = 10 - 6 = 4$.

   Check. Try a rectangle 10 by 6: the bisector of the right angle at A is a 45° line, which rises 6 while moving 6 across, so it meets the top side 6 from D, leaving $10 - 6 = 4$ to C.

4. **Answer:** 3. **The idea.** A number is divisible by 4 exactly when the number formed by its last two digits is, because 100 is a multiple of 4. So test the possible two-digit endings first, and only then fill in the front.

   Work it out. The last two digits come from $\{2, 2, 0, 6\}$. The possible endings are 20, 22, 26, 02, 06, 60, 62. Of these only 20 and 60 are multiples of 4 (22, 26, 62 are 2 more than a multiple of 4, and 02 = 2, 06 = 6 are not multiples of 4).

   Ending 20 leaves the digits 2 and 6 for the front: 2620 and 6220. Ending 60 leaves 2 and 2: only 2260. The 0 is used up in each ending, so no number begins with 0. That makes 3 numbers.

   Check. $2620 = 4 \cdot 655$, $6220 = 4 \cdot 1555$, $2260 = 4 \cdot 565$. Every other arrangement ends in 22, 26, 62, 02 or 06, none of which is a multiple of 4.

5. **Answer:** 18. **The idea.** Organize by the largest coin. Once the number of quarters is fixed, the rest of the money is made of dimes and nickels, and every allowed number of dimes leaves an amount the nickels fill in exactly one way.

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Work it out. With 0 quarters, 75 cents remain, and the dimes can number $0, 1, \ldots, 7$ (at most 70 cents): 8 ways. With 1 quarter, 50 cents remain: 0 to 5 dimes, 6 ways. With 2 quarters, 25 cents remain: 0 to 2 dimes, 3 ways. With 3 quarters nothing remains: 1 way. Total $8 + 6 + 3 + 1 = 18$.

Check. Every leftover amount is a multiple of 5, so after choosing quarters and dimes the nickels are forced and never negative as long as the dimes do not overshoot; the counts $\lfloor 75/10 \rfloor + 1 = 8$, $\lfloor 50/10 \rfloor + 1 = 6$, $\lfloor 25/10 \rfloor + 1 = 3$, 1 agree. So there are 18 ways.

6. Answer: $\frac{5}{11}$**. The idea.** The condition gives the ratio of girls to boys, not the fraction of the class. Turn the ratio into parts, then compare the girls with the whole.

Work it out. Let there be g girls and b boys. Then $\frac{3}{5}g = \frac{1}{2}b$. Multiply by 10: $6g = 5b$, so $g : b = 5 : 6$. If the girls are 5 parts, the boys are 6 parts and the class is 11 parts, so girls make up $\frac{5}{11}$ of the class. (The tempting answer $\frac{5}{6}$ compares girls with boys, not with the class.)

Check. Take 10 girls and 12 boys: three fifths of 10 is 6 and one half of 12 is 6. The class has 22 students, and $\frac{10}{22} = \frac{5}{11}$.

7. Answer: 16**. The idea.** A sum of two whole numbers is even exactly when both are even or both are odd. So split the numbers 0 to 6 by parity and count the tiles inside each group, doubles included.

Work it out. The even numbers are $0, 2, 4, 6$ (four of them). Tiles with two different even ends: $\binom{4}{2} = 6$; even doubles: 4. That is 10 tiles. The odd numbers are $1, 3, 5$ (three of them). Tiles with two different odd ends: $\binom{3}{2} = 3$; odd doubles: 3. That is 6 tiles. In all, $10 + 6 = 16$ tiles have an even total.

Check. Count the odd totals instead: those tiles have one even end and one odd end, and there are $4 \cdot 3 = 12$ of them (no double can be odd-total). Then $28 - 12 = 16$.

8. Answer: 39**. The idea.** The altitude to the hypotenuse cuts the triangle into two smaller triangles that are similar to each other (and to ABC). That similarity gives the altitude, and the altitude with the hypotenuse gives the area.

Work it out. Let $h = CD$. Triangles ADC and CDB are similar (each has a right angle at D, and $\angle ACD = \angle CBD$ because both complement $\angle A$). Matching the legs, $\frac{h}{4} = \frac{9}{h}$, so $h^2 = 36$ and $h = 6$. The hypotenuse is $4 + 9 = 13$, so the area is $\frac{1}{2} \cdot 13 \cdot 6 = 39$.

Check. The legs are $AC = \sqrt{4^2 + 6^2} = \sqrt{52}$ and $BC = \sqrt{9^2 + 6^2} = \sqrt{117}$. Then $AC^2 + BC^2 = 169 = 13^2$, and the area $\frac{1}{2}\sqrt{52 \cdot 117} = \frac{1}{2}\sqrt{6084} = \frac{78}{2} = 39$.

9. Answer: 72**. The idea.** A number is in the first sequence exactly when it is 3 more than a multiple of 4 (and at least 7), and in the second exactly when it is 2 more than a multiple of 7 (and at least 2). Find the first common number; after that the common numbers go up by the least common multiple 28, because stepping by 28 keeps a number in both sequences, and any smaller step does not.

Work it out. List the second sequence until a term leaves remainder 3 on division by 4: 2, 9, 16, 23, and $23 = 4 \cdot 5 + 3$ works (it is the 5th term of the first sequence). So the common numbers are $23, 51, 79, \ldots$, that is, $23 + 28k$ for $k = 0, 1, 2, \ldots$. We need $23 + 28k < 2026$, so $28k \leq 2002$ and $k \leq 71.5$, giving $k = 0, 1, \ldots, 71$: that is 72 numbers.

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Check. The largest is $23+28\cdot71 = 2011$: $2011-7 = 2004 = 4\cdot501$ and $2011-2 = 2009 = 7\cdot287$, so it is in both, while the next one, 2039, is too big. No common number is smaller than 23: the second sequence's terms before it are $2, 9, 16$, which leave remainders $2, 1, 0$ on division by 4.

10. Answer: 9**. The idea.** Factorials soon become multiples of 36, and after that every term adds 0 to the remainder. Find where that happens, and add up only the terms before it.

Work it out. $36 = 4 \cdot 9$. The factorial $6! = 720 = 36 \cdot 20$ is a multiple of 36, and every later factorial $n!$ (with $n \geq 6$) is $6!$ times a whole number, so it is a multiple of 36 too.

But $4! = 24$ and $5! = 120 = 36 \cdot 3 + 12$ are not multiples of 36, so the first five terms all count: $1 + 2 + 6 + 24 + 120 = 153$.

Since $153 = 4 \cdot 36 + 9$, the remainder is 9.

Check. A common slip is to stop at 4! as one would for 12 or 24; here 5! has only one factor of 3, so it is not divisible by 9, and it must be kept. Mod 4: $S \equiv 1+2+2+0+0 = 5 \equiv 1$, and $9 \equiv 1$. Mod 9: $S \equiv 1+2+6+6+3 = 18 \equiv 0$, and $9 \equiv 0$. Both agree.

11. Answer: 900**. The idea.** Measure the mix in "parts". Each ingredient she is short of puts a cap on how big a part can be, and the smallest cap wins; that ingredient is the one that runs out.

Work it out. Flour is 5 parts, and she has 450 g, so a part can be at most $450 \div 5 = 90$ g. Sugar is 2 parts, and she has 200 g, so a part can be at most $200 \div 2 = 100$ g. Butter is unlimited. So a part is at most 90 g, and flour is the limiting ingredient. The whole mix is $5 + 2 + 3 = 10$ parts, weighing at most $10 \cdot 90 = 900$ g.

Check. With 90 g parts she uses 450 g of flour (all of it), 180 g of sugar (20 g left over) and 270 g of butter: $450 + 180 + 270 = 900$. Any bigger batch would need more than 450 g of flour.

12. Answer: 49**. The idea.** The last two digits of a product depend only on the last two digits of the factors. Look for a power of 7 that ends in 01: after that, the pattern of endings repeats.

Work it out. $7^2 = 49$ and $7^4 = 49^2 = 2401$, which ends in 01. So multiplying by 7^4 does not change the last two digits. Since $2026 = 4 \cdot 506 + 2$, we have $7^{2026} = (7^4)^{506} \cdot 7^2$, and it ends in the same two digits as $7^2 = 49$.

Check. The endings of $7^1, 7^2, 7^3, 7^4, 7^5$ are $07, 49, 43, 01, 07$. They repeat every 4, and exponent 2026 is 2 more than a multiple of 4, so the ending is 49.

13. Answer: 70**. The idea.** Three points form a triangle unless they lie on one line. Any three of the marked points that are all on ℓ, or all on m, are collinear; any other choice uses both lines, and then the three points are not collinear (a line through two points of ℓ is ℓ itself, which misses m). So every triangle takes two vertices from one line and one from the other.

Work it out. Two vertices on ℓ and one on m: $\binom{5}{2} \cdot 4 = 10 \cdot 4 = 40$. Two vertices on m and one on ℓ: $\binom{4}{2} \cdot 5 = 6 \cdot 5 = 30$. Total $40 + 30 = 70$.

Check. Count all choices of three points and remove the bad ones: $\binom{9}{3} = 84$, minus $\binom{5}{3} = 10$ collinear triples on ℓ and $\binom{4}{3} = 4$ on m, gives $84 - 14 = 70$.

14. Answer: 210**. The idea.** Watch the gap between the runners rather than the runners themselves. Pat pulls ahead at a steady rate, and catching up from behind means the lead has grown to exactly

one whole lap. (The tempting answer, the least common multiple 420 of 60 and 84, is when both are next back at the start together; Pat has caught Quinn earlier than that, somewhere else on the track.)

Work it out. Each second Pat runs $\frac{1}{60}$ of a lap and Quinn runs $\frac{1}{84}$ of a lap, so Pat gains $\frac{1}{60} - \frac{1}{84} = \frac{7}{420} - \frac{5}{420} = \frac{2}{420} = \frac{1}{210}$ of a lap per second. Gaining one full lap takes 210 seconds.

Check. In 210 seconds Pat runs $210 \div 60 = 3.5$ laps and Quinn runs $210 \div 84 = 2.5$ laps: exactly one lap apart, and both are halfway round the track, side by side. Before that the difference is less than one lap.

15. Answer: $8\pi - 16$**. The idea.** Add up the four semicircle areas. Together the semicircles cover the whole square, and the only parts counted twice are the petals. So the extra over the square's area is exactly the petal area.

Work it out. Each semicircle has radius 2 and area $\frac{1}{2}\pi \cdot 2^2 = 2\pi$, so the four have total area 8π. The square has area 16. Every point of the square lies in at least one semicircle (from any point P of the square, the four sides are seen under four angles that add up to 360°, so some side is seen under an angle of at least 90°, which puts P inside that side's semicircle), and the petals are exactly the doubly covered parts. So the petals' total area is $8\pi - 16$.

Check. One petal is the lens of two circles of radius 2 whose centers (midpoints of adjacent sides) are $2\sqrt{2}$ apart, so they meet at right angles. The lens is two circular segments of 90°: each is a quarter circle π minus a right triangle of area 2, so a lens has area $2(\pi - 2) = 2\pi - 4$, and four lenses give $8\pi - 16 \approx 9.13$.

16. Answer: 12**. The idea.** Turn every average into a total. The first four and the last four together cover all seven numbers, but the fourth number belongs to both groups, so it gets counted twice. Comparing with the total of all seven isolates it.

Work it out. The first four numbers add to $4 \cdot 6 = 24$ and the last four add to $4 \cdot 11 = 44$. Together that is 68, which counts each of the seven numbers once and the fourth number one extra time. All seven add to $7 \cdot 8 = 56$. So the fourth number is $68 - 56 = 12$.

Check. The row 4, 4, 4, 12, 10, 11, 11 works: the first four add to 24 (average 6), the last four add to $12 + 10 + 11 + 11 = 44$ (average 11), and all seven add to 56 (average 8). The other six numbers can change, but the fourth is always 12.

17. Answer: 13**. The idea.** Feeding a linear function into itself gives another linear function whose slope and constant we can write in terms of a and b. Two linear expressions are equal for every x only when their slopes match and their constants match, so we get two equations.

Work it out. $f(f(x)) = a(ax + b) + b = a^2x + (ab + b)$. Matching this with $4x + 9$: $a^2 = 4$, and since $a > 0$, $a = 2$. Then $ab + b = 3b = 9$, so $b = 3$. Hence $f(x) = 2x + 3$ and $f(5) = 13$.

Check. $f(f(5)) = f(13) = 29$, and $4 \cdot 5 + 9 = 29$. The condition $a > 0$ matters: $a = -2$ would give $-b = 9$, the different function $f(x) = -2x - 9$, and then $f(5) = -19$.

18. Answer: 69843**. The idea.** The biggest digits belong in the highest places, so 9 and 7 lead the two numbers, and the next biggest digit, 5, belongs in one of the two tens places. That leaves only six candidates, and we compare them carefully, because the answer is not the one most people guess.

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Work it out. If 9 leads the three-digit number and 7 leads the two-digit one: $953 \times 71 = 67663$, $951 \times 73 = 69423$, $931 \times 75 = 69825$. If 7 leads the three-digit number and 9 leads the two-digit one: $753 \times 91 = 68523$, $751 \times 93 = 69843$, $731 \times 95 = 69445$. The largest is $751 \times 93 = 69843$, just 18 more than 931×75.

Check. Why does 751×93 beat 951×73? Both are built from the same pieces: $751 \times 93 = (700+51)(90+3)$ and $951 \times 73 = (900+51)(70+3)$. The products $700 \cdot 90$ and $900 \cdot 70$ are equal, and the cross terms decide it: $700 \cdot 3 + 51 \cdot 90 = 6690$ beats $900 \cdot 3 + 51 \cdot 70 = 6270$. Putting the 9 in the short number lets it multiply the 51 as well. A computer search of all 120 arrangements also gives 69843 as the largest.

19. Answer: $12\pi - 9\sqrt{3}$. **The idea.** In an equilateral triangle the center of the circle through the vertices is the same point as the centroid, where the medians (which are also the altitudes) meet. The centroid sits $\frac{2}{3}$ of the way down each median, which gives the radius.

Work it out. The altitude of an equilateral triangle with side 6 is $\frac{\sqrt{3}}{2} \cdot 6 = 3\sqrt{3}$, so the radius is $R = \frac{2}{3} \cdot 3\sqrt{3} = 2\sqrt{3}$ and the circle has area $\pi R^2 = 12\pi$. The triangle has area $\frac{1}{2} \cdot 6 \cdot 3\sqrt{3} = 9\sqrt{3}$. The region between them has area $12\pi - 9\sqrt{3}$.

Check. The region is three equal pieces, one cut off by each side. Each piece is a 120° sector of area $\frac{1}{3} \cdot 12\pi = 4\pi$ minus the triangle formed by two radii and a side, which has area $\frac{1}{3} \cdot 9\sqrt{3} = 3\sqrt{3}$. Three pieces give $12\pi - 9\sqrt{3}$ again, which is about $37.70 - 15.59 = 22.1$, a sensible size.

20. Answer: 6. **The idea.** Call the number who went n, so the planned group had $n + 2$ people. The two shares differ by \$5, and that single fact gives an equation that turns into a product of two numbers two apart.

Work it out. Each person who went paid $\frac{120}{n}$ instead of $\frac{120}{n+2}$, so $\frac{120}{n} - \frac{120}{n+2} = 5$. Combining the fractions, $\frac{120(n+2)-120n}{n(n+2)} = \frac{240}{n(n+2)} = 5$, so $n(n+2) = 48$. Two positive integers that differ by 2 and multiply to 48 are 6 and 8, so $n = 6$.

Check. The plan was 8 friends at $\frac{120}{8} = \$15$ each. With 6 friends, each paid $\frac{120}{6} = \$20$, which is indeed \$5 more. Be careful: 8 is the number who planned to go, not the number who went.

21. Answer: $\frac{6}{11}$. **The idea.** Every roll that is neither a 7 nor an 8 changes nothing: the game simply starts over. So only the deciding roll matters, and on that roll we just compare how likely a 7 is with how likely an 8 is.

Work it out. Of the 36 equally likely rolls, a sum of 7 comes from $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$, which is 6 rolls, and a sum of 8 from $(2,6), (3,5), (4,4), (5,3), (6,2)$, which is 5 rolls. Given that a roll is a 7 or an 8, it is a 7 with probability $\frac{6}{6+5} = \frac{6}{11}$.

Check. Let x be the answer. On the first roll we win with probability $\frac{6}{36}$, lose with probability $\frac{5}{36}$, and start over with probability $\frac{25}{36}$, so $x = \frac{6}{36} + \frac{25}{36}x$. Then $\frac{11}{36}x = \frac{6}{36}$, and $x = \frac{6}{11}$.

22. Answer: 10. **The idea.** After the fold, the flipped edge $\overline{CD}$ crosses side $\overline{AB}$ at some point E, and the double layer is triangle AEC. The fold copies angle DCA onto angle ECA, and angle DCA equals angle CAB because $\overline{AB} \parallel \overline{CD}$. So triangle AEC has two equal angles: it is isosceles with $EA = EC$.

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Work it out. Let $EA = EC = x$. Then $EB = 8 - x$, and triangle EBC has a right angle at B, so $x^2 = (8-x)^2 + 4^2 = 64 - 16x + x^2 + 16$. This gives $16x = 80$, so $x = 5$. Triangle AEC has base $AE = 5$ on line AB and height $BC = 4$ (the distance from C to that line), so its area is $\frac{1}{2} \cdot 5 \cdot 4 = 10$.

Check. $EB = 3$, so triangle EBC is a 3-4-5 right triangle with area 6, and its mirror image on the flap is another single-layer piece of area 6. The whole sheet is the double layer counted twice plus the two single pieces: $2 \cdot 10 + 6 + 6 = 32 = 8 \cdot 4$.

23. Answer: 28**. The idea.** Group the six terms in consecutive pairs. Each pair is the previous pair multiplied by r^2, so the three pair sums are 7, $7r^2$ and $7r^4$. This turns the problem into a quadratic in $x = r^2$, and since x is a square it cannot be negative.

Work it out. Terms 1 and 2 add to 7, terms 3 and 4 add to $7x$, and terms 5 and 6 add to $7x^2$, where $x = r^2$. So $7(1 + x + x^2) = 91$, which gives $x^2 + x - 12 = 0$, that is $(x+4)(x-3) = 0$. Since $x = r^2 \geq 0$, $x = 3$. The sum of the first four terms is $7 + 7x = 7 + 21 = 28$.

Check. Take $r = \sqrt{3}$: then $a = \frac{7}{1+\sqrt{3}}$, and the six terms sum to $a(1+r)(1+r^2+r^4) = 7 \cdot 13 = 91$. Taking $r = -\sqrt{3}$ instead also works (with $a = \frac{7}{1-\sqrt{3}}$), and gives the same four-term sum $7(1+3) = 28$, so the answer does not depend on which ratio is used.

24. Answer: 82**. The idea.** A number is divisible by 11 exactly when its alternating digit sum is. For $\overline{abcba}$ that is $a - b + c - b + a = 2a - 2b + c$. Once a and b are chosen, this condition fixes c modulo 11, so there is at most one good c; the only question is when the required residue is the non-digit 10.

Work it out. There are $9 \cdot 10 = 90$ choices of (a, b). For each, we need $c \equiv 2b - 2a \pmod{11}$, and exactly one c in $\{0, 1, \ldots, 10\}$ does this. It fails to be a digit only when $2(b-a) \equiv 10 \pmod{11}$, that is, $b - a \equiv 5 \pmod{11}$, so $b - a = 5$ or $b - a = -6$. The first happens for $(a, b) = (1, 6), (2, 7), (3, 8), (4, 9)$ and the second for $(a, b) = (6, 0), (7, 1), (8, 2), (9, 3)$: 8 bad pairs. So $90 - 8 = 82$ palindromes work.

Check. Try $(a, b) = (3, 7)$: $c \equiv 14 - 6 = 8$, and $37873 = 11 \cdot 3443$. Try the bad pair $(1, 6)$: c would have to be 10, then $2a - 2b + c = c - 10$, which leaves remainder $c + 1 \neq 0$ on division by 11 for every digit c; for instance $16061 = 11 \cdot 1460 + 1$. So that pair is correctly dropped. A computer count of all 900 palindromes also gives 82.

25. Answer: 32**. The idea.** Look for what a move cannot change. Turning over two coins changes the number of tails by +2 (two heads become tails), −2 (two tails become heads) or 0 (one of each). So the number of tails always stays even. Then show that every pattern with an even number of tails really can be made.

Work it out. To turn over just two coins in positions $i < j$, flip the pairs $(i, i+1), (i+1, i+2), \ldots, (j-1, j)$: every coin strictly between i and j is turned twice and ends where it started, so only coins i and j change. Any pattern with an even number of tails can be built by pairing up its tails and doing this once for each pair. So the reachable patterns are exactly those with 0, 2, 4 or 6 tails: $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 1 + 15 + 15 + 1 = 32$.

Check. Count another way. There are 5 possible moves, the order of moves does not matter, and doing a move twice undoes it, so a final pattern depends only on which of the 5 moves were done an odd number of times: at most $2^5 = 32$ patterns. Different sets of moves give different patterns

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(reading left to right, coin 1 shows tails exactly when move $(1, 2)$ was used, and so on), so there are exactly 32.

26. Answer: 14**. The idea.** If two numbers leave the same remainder when divided by d, the remainders cancel when the numbers are subtracted, so their difference is a multiple of d. So d must divide every difference of the three numbers, and the differences are much easier to factor than the numbers themselves.

Work it out. The differences are $2146 - 2023 = 123 = 3 \cdot 41$ and $2310 - 2146 = 164 = 4 \cdot 41$. Their greatest common divisor is 41, which is prime, so the only possible d greater than 1 is $d = 41$. Now $41 \cdot 49 = 2009$, so $2023 = 41 \cdot 49 + 14$ and $r = 14$.

Check. $41 \cdot 52 = 2132$ and $2146 - 2132 = 14$; $41 \cdot 56 = 2296$ and $2310 - 2296 = 14$. All three remainders are 14. The third difference, $2310 - 2023 = 287 = 7 \cdot 41$, is also a multiple of 41, as it must be.

27. Answer: $(2 + \sqrt{2})\pi$**. The idea.** During one roll the square turns 90° about a fixed pivot corner, so every point of the square moves along a quarter circle centred at that pivot. Its path has length $\frac{\pi}{2}$ times its distance from the pivot. So we only need to track where P is relative to the pivot at the start of each roll.

Work it out. Roll 1: P is the bottom-left corner and the pivot is the bottom-right corner, at distance 2; P ends up as the top-left corner. Roll 2: P is the top-left corner, diagonally opposite the pivot, at distance $2\sqrt{2}$; P ends up as the top-right corner. Roll 3: P is directly above the pivot, at distance 2; P ends up as the bottom-right corner. Roll 4: P is the pivot itself, so it does not move. The total is $\frac{\pi}{2}(2 + 2\sqrt{2} + 2 + 0) = (2 + \sqrt{2})\pi$.

Check. In each roll the four corners are at distances 0, 2, 2 and $2\sqrt{2}$ from the pivot, and over the four rolls every corner takes each role once, so all four corners travel the same distance, $(2 + \sqrt{2})\pi \approx 10.7$. Compare the centre of the square, which is always $\sqrt{2}$ from the pivot and travels $4 \cdot \frac{\pi}{2}\sqrt{2} = 2\sqrt{2}\pi \approx 8.9$: a corner swings wider on some rolls and not at all on one, and ends up a little ahead.

28. Answer: 175**. The idea.** The number is a multiple of 5, so it ends in 0 or 5. It cannot end in 0: then the product of its digits would be 0, and so would the number. So the last digit is 5, and the number is $5 \cdot 5 \cdot$ (product of the first two digits), a multiple of 25. That pins down the tens digit.

Work it out. Write the number as $\overline{ab5}$ with digits $a \neq 0$ and b. It equals $5 \cdot a \cdot b \cdot 5 = 25ab$. A multiple of 25 ends in 00, 25, 50 or 75; ending in 5 means it ends in 25 or 75, so $b = 2$ or $b = 7$. The number is odd, so $25ab$ is odd, so b is odd: $b = 7$. Now $100a + 75 = 25 \cdot 7a = 175a$, so $75a = 75$ and $a = 1$. The number is 175.

Check. The digits of 175 multiply to $1 \cdot 7 \cdot 5 = 35$, and $5 \cdot 35 = 175$. A computer search of all three-digit numbers finds no other.

29. Answer: 88π**. The idea.** Away from the shed the rope is a straight radius, so the goat sweeps most of a circle. Behind the shed the rope has to wrap around a corner, and from there only the leftover rope swings, which gives smaller quarter circles.

Work it out. Put the stake at the corner O with the 6 m wall running east to corner E and the 4 m wall running north to corner N. The shed blocks exactly one quarter of the full circle, so the

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goat freely sweeps $\frac{3}{4}$ of a circle of radius 10: area $\frac{3}{4} \cdot 100\pi = 75\pi$. Going east along the 6 m wall, 4 m of rope is left at E; it swings through a quarter circle of radius 4 on the far side of that wall: $\frac{1}{4} \cdot 16\pi = 4\pi$. Going north along the 4 m wall, 6 m is left at N; it swings through a quarter circle of radius 6 on the far side: $\frac{1}{4} \cdot 36\pi = 9\pi$. The first quarter circle reaches along the back wall only up to the far corner of the shed (it has radius 4, the length of that wall), and so does the second (radius 6, the length of the other back wall). So the two just touch at the far corner and do not overlap, and no rope is left to go around that corner. The total is $75\pi + 4\pi + 9\pi = 88\pi$.

Check. The rope is exactly half the shed's perimeter, $6 + 4 = 10$, which is why both wrapped pieces end exactly at the far corner. A computer check that finds the shortest rope path to every point on a fine grid gives an area of about $276.5 \approx 88\pi$.

30. Answer: $\frac{55}{216}$. The idea. If the smallest roll is m and the largest is $m+3$, all four dice land in the window $\{m, m+1, m+2, m+3\}$ and both ends of the window are actually used. Counting rolls inside a window is easy (4^4); inclusion-exclusion then removes the rolls that miss an end.

Work it out. Fix the window $\{m, \ldots, m+3\}$. There are $4^4 = 256$ ordered rolls using only these four values. Those that never show m use only 3 values: $3^4 = 81$; likewise 81 never show $m+3$; and $2^4 = 16$ show neither. So $256 - 81 - 81 + 16 = 110$ rolls have smallest m and largest $m+3$. The window must fit on a die, so $m = 1, 2$ or 3, giving $3 \cdot 110 = 330$ rolls out of $6^4 = 1296$. The probability is $\frac{330}{1296} = \frac{55}{216}$.

Check. Count the 110 for the window $\{1, 2, 3, 4\}$ another way, by how many dice show 1 and how many show 4 (at least one each), with the rest showing 2 or 3. One 1 and one 4: $4 \cdot 3 \cdot 2^2 = 48$. One 1 and two 4s, or two 1s and one 4: $2 \cdot 4 \cdot 3 \cdot 2 = 48$. One 1 and three 4s, or three 1s and one 4: $2 \cdot 4 = 8$. Two of each: $\binom{4}{2} = 6$. Total $48 + 48 + 8 + 6 = 110$. A computer check of all 1296 rolls also gives 330.

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