MathCastle

NATIONAL COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. An arithmetic sequence has 20 terms. The sum of all 20 terms is 710, and the sum of the 10 terms in the odd-numbered positions (the 1st, 3rd, 5th, . . ., 19th terms) is 340. What is the 20th term?

1. _____

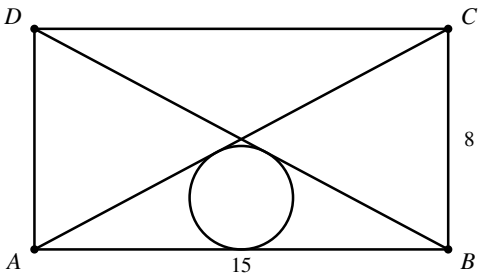
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2. Ten cards numbered 1 through 10 are shuffled and laid out in a single row, with all 10! orders equally likely. A card is called *lonely* if no card directly next to it in the row has a number that differs from its own number by exactly 1. (A card at either end of the row has only one card next to it.) What is the expected number of lonely cards? Express your answer as a common fraction.

2. _____

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3. Rectangle $ABCD$ has $AB = CD = 15$ cm and $BC = DA = 8$ cm, and both diagonals $\overline{AC}$ and $\overline{BD}$ are drawn. A circle lies inside the rectangle and is tangent to side $\overline{AB}$ and to both diagonals. What is the radius of the circle, in centimeters? Express your answer as a common fraction.



3. _____

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4. An arcade game pays out prize tokens worth 8, 13 and 18 points, and there is an unlimited supply of each kind. A player can claim a total of exactly N points if some collection of these tokens (any numbers of each kind, possibly none of a kind) is worth exactly N points. What is the largest positive integer N that cannot be claimed?

4. _____

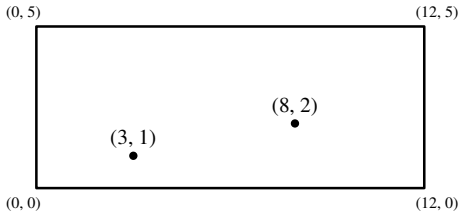
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5. Eight guests at a party wear name tags numbered 1 through 8. For a game they are split at random into four pairs, with all 105 ways of splitting the eight guests into four unordered pairs equally likely. What is the probability that in every pair the two tag numbers add up to a prime number? Express your answer as a common fraction.

5. _____

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6. A rectangular billiard table has corners at $(0, 0)$, $(12, 0)$, $(12, 5)$ and $(0, 5)$. A ball, treated as a point, sits at $(3, 1)$. It is struck in a straight line and bounces off the cushions (the sides of the rectangle) with the angle of reflection equal to the angle of incidence, never hitting a corner. The ball must bounce off each of the four cushions at least once and afterwards roll through the point $(8, 2)$. What is the shortest possible length of the ball's path from $(3, 1)$ to $(8, 2)$? Express your answer in simplest radical form.



6. _____

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7. A positive integer is called *tidy* if every one of its digits is even and no digit appears more than once (so 4068 has tidy digits but 4488 does not). How many tidy positive integers are multiples of 36?

7. _____

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8. The numbers $1, 2, 3, \ldots, 9$ are placed in the nine cells of a 3×3 grid, one number in each cell. The placement is called *friendly* if every two numbers in cells that share a side are relatively prime. (Cells that touch only at a corner are not restricted.) Rotations and reflections of the grid count as different placements. How many friendly placements are there?

8. _____

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Answer Key — MathCounts Nationals Target Practice Contest 3

1. 64

2. $\frac{296}{45}$

3. $\frac{15}{8}$

4. 59

5. $\frac{2}{35}$

6. $\sqrt{442}$

7. 18

8. 2016

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Nationals Target Practice Contest 3

1. Answer: 64**. The idea.** Split the sequence into the 10 odd-position terms and the 10 even-position terms and pair them up: the 2nd term with the 1st, the 4th with the 3rd, and so on. Each even-position term is exactly one common difference d bigger than its partner, so the two half-sums differ by exactly $10d$. That gives d at once, and the total then gives the first term.

Work it out. The even-position terms add to $710 - 340 = 370$, so $10d = 370 - 340 = 30$ and $d = 3$. With first term a, the sum of all twenty terms is $20a + (0 + 1 + \cdots + 19)d = 20a + 190d$, so $20a + 570 = 710$, $20a = 140$ and $a = 7$. The 20th term is $a + 19d = 7 + 57 = 64$.

Check. The sequence is $7, 10, 13, \ldots, 64$. Its sum is $20 \cdot \frac{7+64}{2} = 710$. The odd-position terms $7, 13, 19, \ldots, 61$ form an arithmetic sequence with difference 6, and their sum is $10 \cdot \frac{7+61}{2} = 340$. Both conditions hold. The answer is 64.

2. Answer: $\frac{296}{45}$**. The idea.** Linearity of expectation: the expected number of lonely cards is the sum, over the ten cards, of the probability that that card is lonely. A card's chance depends on only two things: whether it lands at an end of the row (one neighbour) or inside (two neighbours), and whether its number has two "dangerous" values next to it (2 through 9) or only one (1 and 10). So there are just four cases to price.

Work it out. Fix a card and its position. Its neighbours are a uniformly random set of the other 9 cards, of size 1 or 2; the card is lonely when none of them is dangerous. Inside the row with 2 dangerous values: $\binom{7}{2}/\binom{9}{2} = \frac{21}{36} = \frac{7}{12}$. Inside with 1 dangerous value: $\binom{8}{2}/\binom{9}{2} = \frac{28}{36} = \frac{7}{9}$. At an end with 2 dangerous values: $\frac{7}{9}$. At an end with 1 dangerous value: $\frac{8}{9}$. Each card lands inside with probability $\frac{8}{10}$ and at an end with probability $\frac{2}{10}$. A card numbered 2 to 9 is therefore lonely with probability $\frac{8}{10} \cdot \frac{7}{12} + \frac{2}{10} \cdot \frac{7}{9} = \frac{7}{15} + \frac{7}{45} = \frac{28}{45}$, and a card numbered 1 or 10 with probability $\frac{8}{10} \cdot \frac{7}{9} + \frac{2}{10} \cdot \frac{8}{9} = \frac{28}{45} + \frac{8}{45} = \frac{36}{45}$. Adding over the eight middle numbers and the two extreme ones, the expected number of lonely cards is $8 \cdot \frac{28}{45} + 2 \cdot \frac{36}{45} = \frac{224+72}{45} = \frac{296}{45}$.

Check. The answer is about 6.58 of the 10 cards, which is sensible: a card has on average 1.8 neighbours and each is dangerous with chance only about $\frac{2}{9}$. The method also survives a small case done by hand. With only the cards $1, 2, 3$: card 2 has both other cards as dangerous values, so it is never lonely; card 1 is lonely only at an end next to the 3, probability $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, and likewise card 3; expected total $\frac{2}{3}$. Listing the six orders $123, 132, 213, 231, 312, 321$ gives $0, 1, 1, 1, 1, 0$ lonely cards, average $\frac{4}{6} = \frac{2}{3}$, as predicted. A computer run over all 10! orders confirms $\frac{296}{45}$. The answer is $\frac{296}{45}$.

3. Answer: $\frac{15}{8}$**. The idea.** The diagonals cross at the centre P of the rectangle and cut it into four triangles. A circle inside the rectangle touching $\overline{AB}$ and both diagonals sits in the bottom triangle ABP, touching all three of its sides, so it is the incircle of triangle ABP, and an incircle's radius is area divided by semiperimeter.

Work it out. P is the midpoint of each diagonal. The diagonal has length $\sqrt{15^2 + 8^2} = 17$, so $PA = PB = \frac{17}{2}$. Triangle ABP has base $AB = 15$ and height 4 (half of BC, since P is the centre), so its area is $\frac{1}{2} \cdot 15 \cdot 4 = 30$. Its semiperimeter is $\frac{15+\frac{17}{2}+\frac{17}{2}}{2} = 16$. The radius is $\frac{30}{16} = \frac{15}{8}$.

Check. Coordinates: $A = (0, 0)$, $B = (15, 0)$, $C = (15, 8)$. By symmetry the centre is $(7.5, r)$. Line AC is $8x - 15y = 0$, and the distance from $(7.5, r)$ to it is $\frac{|60-15r|}{17}$. Setting this equal to r

gives $60 - 15r = 17r$, so $r = \frac{60}{32} = \frac{15}{8}$, the same. The circle reaches up to $2 \cdot \frac{15}{8} = 3.75 < 4$, below P, so it really lies inside the triangle and the rectangle; the only other circles tangent to line AB and both diagonal lines have centres at $(7.5, 30)$ or $(7.5, -30)$ or off to the sides, far outside the rectangle. The answer is $\frac{15}{8}$.

4. Answer: 59**. The idea.** Sort totals by their remainder when divided by 8. Adding an 8-point token never changes the remainder, so once some total in a remainder class can be made, every larger total in the same class can be made too. So for each of the 8 classes find the smallest total you can build from 13s and 18s; the answer is the largest of these eight smallest totals, minus 8.

Work it out. Modulo 8, a 13 counts as 5 and an 18 counts as 2. With a thirteens and b eighteens, the smallest totals in each class are: remainder 0: 0; remainder 5: 13; remainder 2: 18; remainder 7: 13+18 = 31; remainder 4: 18+18 = 36 (note $13 \cdot 4 = 52$ is larger); remainder 1: 13+18+18 = 49; remainder 6: $18 \cdot 3 = 54$. For remainder 3 we need $5a + 2b \equiv 3 \pmod 8$, so a must be odd. With $a = 1$ we need $2b \equiv 6$, so $b \equiv 3 \pmod 4$ and the total is at least $13 + 54 = 67$; with $a = 3$, $b \equiv 2 \pmod 4$ gives at least $39 + 36 = 75$; with $a = 5$, at least $65 + 18 = 83$; larger a is larger still. So the smallest total with remainder 3 is 67, the biggest of the eight. Every number from 60 to 67 is the smallest member of its class or is reached from one by adding 8s, so everything from 60 on can be claimed, while $67 - 8 = 59$ cannot.

Check. Directly: 59 minus 0, 13, 26, 39, 52 gives 59, 46, 33, 20, 7, and none of these is a sum of 8s and 18s (59, 33, 7 are odd; $46 - 18k \in \{46, 28, 10\}$ and $20 - 18k \in \{20, 2\}$ are not multiples of 8). And everything from 60 to 67 can be claimed: $60 = 2 \cdot 18 + 3 \cdot 8$, $61 = 13 + 6 \cdot 8$, $62 = 3 \cdot 18 + 8$, $63 = 13 + 18 + 4 \cdot 8$, $64 = 8 \cdot 8$, $65 = 5 \cdot 13$, $66 = 18 + 6 \cdot 8$, $67 = 13 + 3 \cdot 18$; adding 8s then reaches every larger total. The answer is 59.

5. Answer: $\frac{2}{35}$**. The idea.** Parity does most of the work. Two odd tags or two even tags add to an even number that is at least $1 + 3 = 4$, which is never prime. So every pair must be one odd tag with one even tag, and the problem shrinks to matching the odd tags $1, 3, 5, 7$ with the even tags $2, 4, 6, 8$.

Work it out. List which even partners give a prime sum. Tag 1: 2, 4, 6 (sums 3, 5, 7; $1 + 8 = 9$ fails). Tag 3: 2, 4, 8 (sums 5, 7, 11; $3 + 6 = 9$ fails). Tag 5: 2, 6, 8 (sums 7, 11, 13; $5 + 4 = 9$ fails). Tag 7: 4, 6 (sums 11, 13; $7 + 2 = 9$ and $7 + 8 = 15$ fail). Start with the most restricted tag, 7. If 7 takes 4: then $1 \in \{2, 6\}$, $3 \in \{2, 8\}$, $5 \in \{2, 6, 8\}$; 1 takes 2 forces 3–8, 5–6 (one way), and 1 takes 6 leaves 3 and 5 to share $\{2, 8\}$ either way (two ways): 3 matchings. If 7 takes 6: then $1 \in \{2, 4\}$, $3 \in \{2, 4, 8\}$, $5 \in \{2, 8\}$; 1 takes 2 forces 5–8, 3–4 (one way), and 1 takes 4 leaves 3 and 5 to share $\{2, 8\}$ (two ways): 3 matchings. That is 6 good splits out of 105, a probability of $\frac{6}{105} = \frac{2}{35}$.

Check. The 105 comes from $7 \cdot 5 \cdot 3 \cdot 1$: guest 1 has 7 possible partners, the lowest remaining guest then has 5, and so on. One good split to test: $\{1, 6\}, \{3, 8\}, \{5, 2\}, \{7, 4\}$ has sums 7, 11, 7, 11, all prime. The answer is $\frac{2}{35}$.

6. Answer: $\sqrt{442}$**. The idea.** Unfold the table. Instead of bouncing the ball, reflect the table across the cushion it hits; the path becomes a straight segment from $(3, 1)$ to a mirror image of $(8, 2)$ in a tiling of the plane by reflected tables. Crossing a vertical grid line $x = 12m$ is a bounce off the left cushion when m is even and off the right cushion when m is odd; likewise the lines $y = 5n$ are the bottom cushion (n even) and top cushion (n odd). The horizontal and vertical parts of the segment can then be chosen separately.

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Work it out. Horizontally, the images of $x = 8$ are $\ldots, -16, -8, 8, 16, 32, \ldots$ (reflecting in $x = 0$, $x = \pm 12$, and so on). To hit both side cushions the segment must cross two consecutive vertical lines. Going left, -16 lies beyond $x = 0$ and $x = -12$, a horizontal run of $3 - (-16) = 19$. Going right, the first image beyond $x = 12$ and $x = 24$ is 32, a run of 29. So the horizontal run is at least 19. Vertically, the images of $y = 2$ are $\ldots, -8, -2, 2, 8, 12, \ldots$; going down, -8 lies beyond $y = 0$ and $y = -5$, a vertical run of $1 - (-8) = 9$; going up, 12 lies beyond $y = 5$ and $y = 10$, a run of 11. So the vertical run is at least 9. The target image $(-16, -8)$ achieves both minima at once, so the shortest path has length $\sqrt{19^2 + 9^2} = \sqrt{361 + 81} = \sqrt{442}$.

Check. The segment from $(3, 1)$ to $(-16, -8)$ crosses $y = 0$ at $x = 3 - \frac{19}{9}$, $x = 0$ at $y = 1 - \frac{27}{19}$, $y = -5$ at $x = -\frac{29}{3}$ and $x = -12$ at $y = -\frac{116}{19}$: four bounces (bottom, left, top, right), none at a corner. $442 = 2 \cdot 13 \cdot 17$ has no square factor, and $\sqrt{442} \approx 21.02$. The answer is $\sqrt{442}$.

7. Answer: 18**. The idea.** Split $36 = 4 \cdot 9$. Divisibility by 9 is about the digit sum, and the even digits $0, 2, 4, 6, 8$ are so limited that the digit sum is pinned down completely. Divisibility by 4 is then about the last two digits only.

Work it out. A tidy number uses distinct digits from $\{0, 2, 4, 6, 8\}$, so its digit sum is even and at most 20. It must be a positive multiple of 9, hence exactly 18. The subsets of $\{0, 2, 4, 6, 8\}$ with sum 18 are $\{4, 6, 8\}$ and $\{0, 4, 6, 8\}$ (any subset containing 2 would need the others to sum to 16, but $4 + 6 + 8 = 18$, $4 + 8 = 12$, $6 + 8 = 14$, $4 + 6 = 10$ all miss). Now the last two digits must form a multiple of 4. With $\{4, 6, 8\}$: the two-digit endings 48, 64, 68, 84 work (not 46 or 86), giving 648, 864, 468, 684: 4 numbers. With $\{0, 4, 6, 8\}$: the endings from distinct digits of the set that are multiples of 4 are 04, 08, 40, 48, 60, 64, 68, 80, 84 (the others, 06, 46, 86, fail). When the ending contains 0 (04, 08, 40, 60, 80), the other two digits are nonzero and can go in front in 2 orders: $5 \cdot 2 = 10$ numbers. When it does not (48, 64, 68, 84), the two leading digits are 0 and something, and only 1 order avoids a leading zero: 4 numbers. Total $4 + 10 + 4 = 18$.

Check. The smallest is $468 = 36 \cdot 13$ and the largest is $8640 = 36 \cdot 240$. A sample from the middle: $6048 = 36 \cdot 168$, with digits 6, 0, 4, 8. The answer is 18.

8. Answer: 2016**. The idea.** Only two primes can cause trouble: 2 (shared by $2, 4, 6, 8$) and 3 (shared by $3, 6, 9$), since 5 and 7 appear once each and 1 is coprime to everything. So a placement is friendly exactly when (i) no two of $2, 4, 6, 8$ share a side, and (ii) no two of $3, 6, 9$ share a side. Condition (i) forces the even numbers onto a set of four mutually non-adjacent cells, and there are very few of those; after that only 3, 9 and 6 need care.

Work it out. Colour the grid like a checkerboard: the 4 corners and the centre are one colour, the 4 edge-middles the other; cells sharing a side always differ in colour. Four pairwise non-adjacent cells: the 4 edge-middles; or 4 of the 5 corner/centre cells (the 4 corners, or the centre with 3 corners); a mixture fails, because one edge-middle blocks two corners and the centre, leaving too few cells. So there are 6 possible sets for the evens. In each case place 6, then place 3 and 9 (on cells not beside 6 and not beside each other), then $2, 4, 8$ in $3! = 6$ ways and $1, 5, 7$ in $3! = 6$ ways.

Case A, evens on the corners. The odd cells are the edge-middles and the centre. The corner holding 6 touches two edge-middles; 3 and 9 may use the centre or the other two edge-middles, but the centre touches every edge-middle, so they take the two far edge-middles in 2 orders. Total $4 \cdot 2 \cdot 6 \cdot 6 = 288$.

Case B, evens on the edge-middles. The edge-middle holding 6 touches two corners and the centre; 3 and 9 take the two far corners (which do not touch) in 2 orders: $4 \cdot 2 \cdot 6 \cdot 6 = 288$.

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Case C, evens on the centre and three corners; say corner X is left for an odd number, 4 choices. The odd cells are the four edge-middles and X. If 6 is in the centre it touches all four edge-middles, leaving only X for both 3 and 9: impossible. So 6 is on one of the three even corners. If that corner is opposite X, the free cells are the two far edge-middles and X, and X touches both of them, so $3, 9$ take the two edge-middles: 2 ways. If the corner of 6 is beside X (same row or column), the free cells are the two far edge-middles and X, and X touches only one of them, so the non-touching pairs are the two edge-middles or X with the other edge-middle: 2 pairs, 4 ways. For each X that is $2 + 4 + 4 = 10$ ways to place $6, 3, 9$, so Case C gives $4 \cdot 10 \cdot 6 \cdot 6 = 1440$.

Altogether $288 + 288 + 1440 = 2016$.

Check. A friendly example from Case C: rows $(6, 5, 2)$, $(7, 4, 9)$, $(8, 3, 1)$: here X is the bottom-right corner (holding 1), 6 is top-left, opposite X, and $3, 9$ sit on the two far edge-middles; every side-sharing pair is coprime. As a probability, $\frac{2016}{9!} = \frac{1}{180}$ of all placements are friendly, which a computer search over all 362880 placements confirms. The answer is 2016.

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