MathCastle

NATIONAL COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

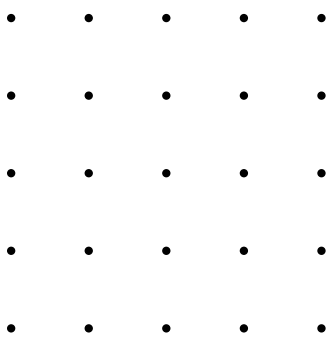
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The fish in a pond are counted at the start of a two-year study. During the first spring the population grows by 20%, and that autumn 300 fish are harvested. During the second spring the population grows by 25%, and that autumn 450 fish are harvested. After the second harvest, the pond holds exactly as many fish as it did at the start of the study. How many fish were in the pond at the start?

1. _____

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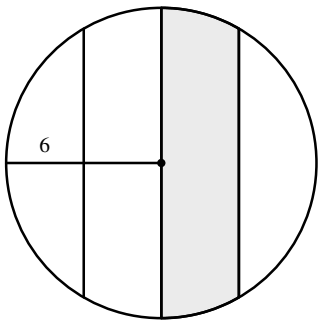
2. The 25 points (x, y) with x and y each in $\{0, 1, 2, 3, 4\}$ form a 5×5 square array. Four of these 25 points are chosen at random, with every set of four points equally likely. What is the probability that the four chosen points are the four vertices of a square? (The square's sides do not have to be horizontal and vertical.) Express your answer as a common fraction.



2. _____

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3. A round pizza has radius 6 inches. Three parallel straight cuts divide it into four strips, and the four strips all have the same width (measured perpendicular to the cuts). What is the area, in square inches, of one of the two middle strips? Express your answer in terms of π and in simplest radical form.



3. _____

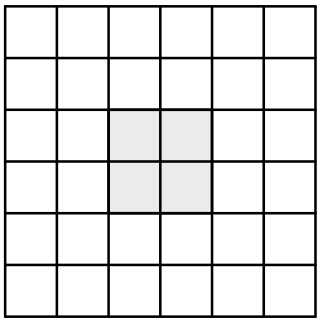
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4. Some of the positive divisors of a positive integer may be perfect squares. For example, the divisors of 48 that are perfect squares are 1, 4 and 16, exactly three of them, while 36 has four such divisors: 1, 4, 9 and 36. For how many positive integers $n \leq 1000$ are exactly three of the positive divisors of n perfect squares?

4. _____

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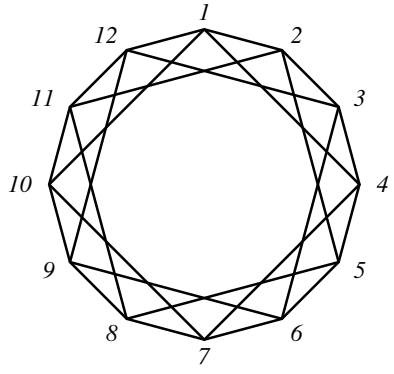
5. A 6×6 board is divided into 36 unit squares. The four unit squares at the very middle of the board form a 2×2 block. Consider every rectangle whose sides lie along the grid lines of the board (squares count as rectangles, and rectangles of every size and position count). How many of these rectangles contain at least one of the four middle unit squares, but not all four of them?



5. _____

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6. The vertices of a regular dodecagon (a regular 12-sided polygon) are labeled 1 through 12 in order around it. Three squares are drawn: one with vertices at the points labeled $1, 4, 7, 10$; one with vertices at $2, 5, 8, 11$; and one with vertices at $3, 6, 9, 12$. What fraction of the dodecagon's area lies inside all three squares? Express your answer in simplest radical form.



6. _____

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7. Consider the list $1, 2, 3, 5, 8, 13, 21, \ldots$, in which each number after the second is the sum of the two numbers before it. Every positive integer can be written in exactly one way as a sum of different numbers from this list, no two of which are next to each other in the list. For example, $20 = 13 + 5 + 2$ and $7 = 5 + 2$. How many positive integers less than 1000 use exactly three numbers from the list when written this way?

7. _____

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8. A positive integer n is called abundant if the sum of its positive divisors other than n itself is greater than n. For example, 12 is abundant, because $1+2+3+4+6 = 16 > 12$. Every abundant number less than 100 is even. What is the sum of all odd abundant numbers less than 2026?

8. _____

Answer Key — MathCounts Nationals Target Practice Contest 2

1. 1650

2. $\frac{1}{253}$

3. $6\pi + 9\sqrt{3}$

4. 48

5. 144

6. $4 - 2\sqrt{3}$

7. 226

8. 2520

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Solutions — MathCounts Nationals Target Practice Contest 2

1. **Answer:** 1650**. The idea.** Follow an unknown starting population x through the four events in order. A percentage growth multiplies, a harvest subtracts, and the order matters: the second spring's 25% growth also acts on what was left after the first harvest. Setting the final count equal to x gives one linear equation.

   Work it out. After the first spring there are $1.2x$ fish, and after the first harvest $1.2x - 300$. The second spring multiplies this by 1.25: $1.25(1.2x - 300) = 1.5x - 375$. The second harvest leaves $1.5x - 375 - 450 = 1.5x - 825$. Setting this equal to x gives $0.5x = 825$, so $x = 1650$.

   Check. Start with 1650: the first spring gives 1980, the harvest leaves 1680, the second spring gives 2100, and the harvest leaves 1650, the starting number. Every count along the way is a whole number, as it must be for fish.

2. **Answer:** $\frac{1}{253}$**. The idea.** Tilted squares are easy to miss and hard to list directly. Instead, file every square under the smallest horizontal-and-vertical square that contains it. Each tilted square with lattice-point vertices touches each side of that box at exactly one vertex, so it is determined by the box and by how far along the bottom side its bottom vertex sits.

   Work it out. Take any square whose vertices are lattice points and draw its bounding box, a square of side s with horizontal and vertical sides. If the square is not itself horizontal and vertical, each of its four vertices lies on a different side of the box, and going around the box, each side is cut into two pieces of lengths q and $s - q$ in the same pattern (the four corner triangles are congruent right triangles with legs q and $s - q$). If the square is horizontal and vertical it is the box itself; call that $q = 0$. So for each box of side s there are exactly s squares filed under it, one for each $q = 0, 1, \ldots, s - 1$, and all their vertices are lattice points in the box.

   In the 5×5 array, a box of side s can be placed in $(5 - s)^2$ positions, for $s = 1, 2, 3, 4$. The number of squares is

   $$1 \cdot 16 + 2 \cdot 9 + 3 \cdot 4 + 4 \cdot 1 = 16 + 18 + 12 + 4 = 50.$$

   The number of ways to choose four points is $\binom{25}{4} = 12650$, so the probability is $\frac{50}{12650} = \frac{1}{253}$.

   Check. The squares with horizontal and vertical sides number $16+9+4+1 = 30$. The tilted ones: side vector $(1, 1)$ in a 2×2 box, 9 places; side vectors $(1, 2)$ and $(2, 1)$ in a 3×3 box, $2 \cdot 4 = 8$; side vectors $(1, 3)$, $(2, 2)$, $(3, 1)$ in the whole 4×4 array, 3. That is 20 tilted squares and 50 in all. Counting only the 30 upright squares gives the trap $\frac{3}{1265}$.

3. **Answer:** $6\pi + 9\sqrt{3}$**. The idea.** The diameter perpendicular to the cuts has length 12, so each strip is 3 wide and the middle cut passes through the centre. A middle strip is therefore half of the pizza minus the thin outer strip, and the outer strip is a circular segment whose chord is 3 from the centre. That chord cuts off exactly a third of the circle's angle.

   Work it out. The half-pizza on one side of the central cut has area $\frac{1}{2}\pi \cdot 6^2 = 18\pi$.

   For the outer strip: the chord lies 3 from the centre O. The radius to an end of the chord, the perpendicular from O to the chord, and half the chord form a right triangle with hypotenuse 6 and one leg 3, a 30-60-90 triangle. So half the chord is $3\sqrt{3}$ and each half subtends 60° at O: the chord subtends 120°. The segment is the 120° sector minus the triangle formed by O and the chord:

$\frac{120}{360} \cdot 36\pi - \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 12\pi - 9\sqrt{3}$.

The middle strip is $18\pi - (12\pi - 9\sqrt{3}) = 6\pi + 9\sqrt{3} \approx 34.4$ square inches.

Check. The two outer strips and two middle strips should fill the pizza: $2(12\pi - 9\sqrt{3}) + 2(6\pi + 9\sqrt{3}) = 36\pi$, the whole area. The middle strips are larger than the outer ones (34.4 against 22.1), as they should be, since they are the wider part of the circle.

4. Answer: 48. **The idea.** A square divisor of n is k^2 with $k^2 \mid n$. If $n = p_1^{a_1} p_2^{a_2} \cdots$, then $k^2 \mid n$ exactly when each prime p_i appears in k at most $\left\lfloor \frac{a_i}{2} \right\rfloor$ times. So the number of square divisors is the product of the factors $\left\lfloor \frac{a_i}{2} \right\rfloor + 1$. Since 3 is prime, this product is 3 only when one factor is 3 and every other factor is 1.

Work it out. A factor $\left\lfloor \frac{a}{2} \right\rfloor + 1 = 3$ means $a = 4$ or $a = 5$, and a factor 1 means the exponent is 0 or 1. So $n = p^4 m$ or $n = p^5 m$, where p is prime and m is squarefree (no repeated prime) and not a multiple of p. Count each case with $n \le 1000$.

$p = 2$, $n = 16m$: $m \le 62$, odd and squarefree. There are 31 odd numbers up to 62; the odd ones with a repeated prime are $9, 27, 45, 25, 49$. That leaves 26.

$p = 2$, $n = 32m$: $m \le 31$, odd and squarefree: 16 odd numbers minus $9, 25, 27$, leaving 13.

$p = 3$, $n = 81m$: $m \le 12$, squarefree, not a multiple of 3: $m = 1, 2, 5, 7, 10, 11$, so 6.

$p = 3$, $n = 243m$: $m \le 4$: $m = 1, 2$ ($m = 4$ has a repeated prime), so 2.

$p = 5$: $625m \le 1000$ forces $m = 1$, giving 625; and $5^5 > 1000$. For $p \ge 7$, $p^4 \ge 2401$ is too big.

The total is $26 + 13 + 6 + 2 + 1 = 48$.

Check. $n = 80 = 2^4 \cdot 5$: the square divisors are $1, 4, 16$, three of them, and 80 was counted as $16 \cdot 5$. $n = 144 = 2^4 \cdot 3^2$ has $3 \cdot 2 = 6$ square divisors $(1, 4, 16, 9, 36, 144)$ and was rightly left out, because the 3^2 makes $m = 9$ not squarefree. $n = 64 = 2^6$ has $1, 4, 16, 64$, four square divisors, which is why the exponent must be 4 or 5 and not 6.

5. Answer: 144. **The idea.** A rectangle on the board is a choice of a block of consecutive rows together with a block of consecutive columns, and the two choices are independent. A rectangle contains a middle square exactly when its rows reach one of the two middle rows and its columns reach one of the two middle columns. Count "at least one", then subtract "all four".

Work it out. Number the rows 1 to 6; the middle rows are 3 and 4. There are $\binom{7}{2} = 21$ blocks of consecutive rows. The blocks that miss both rows 3 and 4 lie inside rows 1 to 2 or inside rows 5 to 6: $3 + 3 = 6$ of them. So $21 - 6 = 15$ row blocks reach a middle row, and likewise 15 column blocks. A rectangle contains at least one middle square exactly when both of its blocks do (then the rectangle contains the square where a reached middle row meets a reached middle column): $15^2 = 225$ rectangles.

A rectangle contains all four middle squares when its row block contains both rows 3 and 4: its top row is 1, 2 or 3 and its bottom row is 4, 5 or 6, so 9 row blocks, and 9 column blocks. That gives 81 rectangles.

The answer is $225 - 81 = 144$.

Check. Count the 15 row blocks directly: a block from row i to row j reaches row 3 or 4 when $i \le 4$ and $j \ge 3$. Starting at rows 1, 2 or 3, the end can be any of rows 3 to 6: $3 \cdot 4 = 12$; starting at row 4, the end is 4, 5 or 6: 3 more. Total 15. The board has $21^2 = 441$ rectangles, of which $441 - 225 = 216$ miss the middle block entirely.

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6. **Answer:** $4 - 2\sqrt{3}$. **The idea.** All three squares are inscribed in the dodecagon's circumscribed circle, so all 12 of their sides are at the same distance from the centre, and each square is the next one turned by 30°. The region common to all three is therefore itself a regular dodecagon, with its sides at that common distance from the centre. Two regular dodecagons are similar, so the area ratio is the square of the ratio of their apothems (centre-to-side distances).

   Work it out. Let the circumradius be R and the centre O. Each square is inscribed in the circle of radius R, so its side is $R\sqrt{2}$ and each side is $\frac{R\sqrt{2}}{2} = \frac{R}{\sqrt{2}}$ from O. The sides of the first square face directions $45°, 135°, 225°, 315°$ from O (measuring from vertex 1); the other squares add 30° and 60°. The twelve directions are 30° apart, so the common region is a regular dodecagon with apothem $\frac{R}{\sqrt{2}}$, and apothem squared $\frac{R^2}{2}$.

   For the original dodecagon, find its side s. Take the triangle formed by O and two neighbouring vertices A and B: $OA = OB = R$ and $\angle AOB = 30°$. The perpendicular from B to line OA has length $\frac{R}{2}$ (it is opposite the 30° angle in a 30-60-90 triangle) and its foot is $\frac{\sqrt{3}}{2}R$ from O, hence $R - \frac{\sqrt{3}}{2}R$ from A. So

   $$s^2 = \frac{R^2}{4} + R^2\left(1 - \frac{\sqrt{3}}{2}\right)^2 = R^2(2 - \sqrt{3}).$$

   Its apothem squared is $R^2 - \frac{s^2}{4} = R^2 \cdot \frac{2+\sqrt{3}}{4}$. The area ratio is

   $$\frac{R^2/2}{R^2(2+\sqrt{3})/4} = \frac{2}{2+\sqrt{3}} = 2(2 - \sqrt{3}) = 4 - 2\sqrt{3} \approx 0.536.$$

   Check. Use areas directly with $R = 1$. The original dodecagon is 12 triangles with two sides 1 and a 30° angle between them, each of area $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$: total 3. A regular dodecagon with apothem h has side $2h(2 - \sqrt{3})$ (scale the original: its apothem $\frac{\sqrt{2+\sqrt{3}}}{2}$ goes with side $\sqrt{2 - \sqrt{3}}$, ratio $\frac{2\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} = 2(2 - \sqrt{3})$), so its area is $12 \cdot \frac{1}{2} \cdot h \cdot 2h(2 - \sqrt{3}) = 12h^2(2 - \sqrt{3})$. With $h^2 = \frac{1}{2}$ this is $12 - 6\sqrt{3}$, and $\frac{12-6\sqrt{3}}{3} = 4 - 2\sqrt{3}$.

7. **Answer:** 226. **The idea.** The list begins $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987$. Choosing non-neighbouring entries from the first 14 numbers (up to 610) gives sums that are all different, by the uniqueness in the statement, and the biggest possible such sum is $610 + 233 + 89 + 34 + 13 + 5 + 2 = 986$. So these choices produce exactly the integers 1 to 986, each once, and counting three-term numbers there is the same as counting ways to pick 3 non-neighbouring places out of 14. The few numbers from 987 to 999 are handled separately.

   Work it out. Picking places $i < j < k$ out of $1, \ldots, 14$ with no two next to each other is the same as picking $i < j - 1 < k - 2$ out of $1, \ldots, 12$ with no restriction (shift each chosen place left by the number of chosen places before it). So there are $\binom{12}{3} = 220$ three-term numbers from 1 to 986.

   For $987 \le n \le 999$: without 987 the largest sum available is 986, so 987 is used, and $n = 987 + r$ with $0 \le r \le 12$, where r is written with numbers at most 8 (so 610, the neighbour of 987, is never involved). Then n has three terms exactly when r has two. Writing out $r = 1, \ldots, 12$: 1, 2, 3, 3 + 1, 5, 5 + 1, 5 + 2, 8, 8 + 1, 8 + 2, 8 + 3, 8 + 3 + 1. The two-term ones are $r = 4, 6, 7, 9, 10, 11$: six numbers.

   The answer is $220 + 6 = 226$.

   Check. On a smaller scale: the numbers below 13 use places among 1, 2, 3, 5, 8, and the formula gives $\binom{3}{3} = 1$ three-term number; the list above shows it is $12 = 8 + 3 + 1$, the only one. Another

check on the correspondence: the number of ways to choose non-neighbouring places from 14 (including choosing none) is 987, one for each integer from 0 to 986, as it should be.

8. Answer: 2520**. The idea.** Let $\sigma(n)$ be the sum of all positive divisors of n, including n. Then n is abundant exactly when $\frac{\sigma(n)}{n} > 2$. This ratio multiplies over prime powers: for $n = p_1^{a_1} p_2^{a_2} \cdots$ it is the product of the factors $1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{a_i}}$, and each such factor is less than $\frac{p_i}{p_i - 1}$ (the full infinite sum). Odd primes give small factors ($\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{6}$, ...), so an odd number needs several small primes to push the product past 2. That narrows the search to a short list.

Work it out. Let $n < 2026$ be odd.

At most two different primes: the ratio is less than $\frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} < 2$. Not abundant.

Four or more different primes: the smallest candidates are $3 \cdot 5 \cdot 7 \cdot 11 = 1155$, $3 \cdot 5 \cdot 7 \cdot 13 = 1365$, $3 \cdot 5 \cdot 7 \cdot 17 = 1785$ and $3 \cdot 5 \cdot 7 \cdot 19 = 1995$; any other choice of primes, a fifth prime, or a squared prime pushes n past 2026. For 1155, $\sigma = 4 \cdot 6 \cdot 8 \cdot 12 = 2304 < 2310$, so it just misses. The other three swap 11 for a larger prime, which only lowers the ratio, so they miss too.

Exactly three different primes $p < q < s$: if $p \geq 5$, the ratio is below $\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} < 1.61$; if $p = 3$ and $q \geq 7$, it is below $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} = 1.925$. So $p = 3$ and $q = 5$, and then $\frac{15}{8} \cdot \frac{s}{s-1} > 2$ forces $s < 16$, so $s = 7, 11$ or 13. If no prime is repeated, the ratio is $\frac{4}{3} \cdot \frac{6}{5} \cdot \frac{s+1}{s} \leq \frac{8}{5} \cdot \frac{8}{7} < 2$, so some prime is repeated. The odd numbers below 2026 of this shape are

$s = 7$: $315, 525, 735, 945, 1575$; $s = 11$: $495, 825, 1485, 1815$; $s = 13$: $585, 975, 1755$.

Compare $\sigma(n)$ with $2n$: $\sigma(315) = 13 \cdot 6 \cdot 8 = 624 < 630$; $\sigma(525) = 4 \cdot 31 \cdot 8 = 992 < 1050$; $\sigma(735) = 4 \cdot 6 \cdot 57 = 1368 < 1470$; $\sigma(945) = 40 \cdot 6 \cdot 8 = 1920 > 1890$; $\sigma(1575) = 13 \cdot 31 \cdot 8 = 3224 > 3150$; $\sigma(495) = 13 \cdot 6 \cdot 12 = 936 < 990$; $\sigma(825) = 4 \cdot 31 \cdot 12 = 1488 < 1650$; $\sigma(1485) = 40 \cdot 6 \cdot 12 = 2880 < 2970$; $\sigma(1815) = 4 \cdot 6 \cdot 133 = 3192 < 3630$; $\sigma(585) = 13 \cdot 6 \cdot 14 = 1092 < 1170$; $\sigma(975) = 4 \cdot 31 \cdot 14 = 1736 < 1950$; $\sigma(1755) = 40 \cdot 6 \cdot 14 = 3360 < 3510$.

Only $945 = 3^3 \cdot 5 \cdot 7$ and $1575 = 3^2 \cdot 5^2 \cdot 7$ are abundant, and the answer is $945 + 1575 = 2520$.

Check. For 945 the proper divisors add up to $1920 - 945 = 975 > 945$; for 1575 they add up to $3224 - 1575 = 1649 > 1575$. The near misses show how tight the bound is: 1155 falls short by only 6 ($2304 - 1155 = 1149$), and 1485 by 45.

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