MathCastle

NATIONAL COMPETITION

Target Round

Problems 1–8

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 8 problems, handed out in pairs. You have 6 minutes for each pair.
2. When time is called for a pair, it is collected and the next pair is handed out. You cannot go back to a pair once it is collected.
3. Calculators are allowed in this round.
4. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
5. If you finish a pair early, use the time to check your answers.
6. Taken in four pairs: problems 1–2, 3–4, 5–6, 7–8.

Total Correct	Scorer's Initials

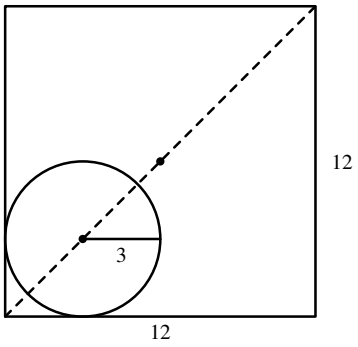
DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A perfect square is the square of an integer. How many perfect squares less than 10,000 leave a remainder of 1 when they are divided by 24? 1. _____

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2. A thin metal plate of even thickness is a square 12 cm on each side. A circular hole of radius 3 cm is drilled through it; the centre of the hole is 3 cm from each of two neighbouring edges of the square, so the hole just touches those two edges. By symmetry, the balance point (centre of mass) of the drilled plate lies on the diagonal of the square that passes through the centre of the hole. How many centimetres is the balance point of the drilled plate from the centre of the square? Express your answer as a decimal to the nearest hundredth.



2. _____

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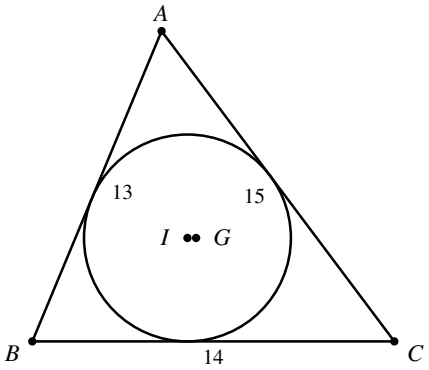
3. A standard set of dominoes has 28 tiles, one for each unordered pair of numbers from 0 to 6, where a number may be paired with itself (the doubles 0-0, 1-1, . . ., 6-6 are included). Two different tiles are drawn at random from the set, with every pair of tiles equally likely. What is the probability that the two tiles have at least one number in common, so that they can be laid end to end with matching halves? Express your answer as a common fraction. 3. _____

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4. Five positive integers form an increasing geometric sequence. The common ratio of the sequence is not an integer, and the sum of the five integers is 781. What is the largest of the five integers? 4. _____

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5. Triangle ABC has side lengths $AB = 13$, $BC = 14$ and $CA = 15$. Let I be the center of the circle inscribed in the triangle, and let G be the centroid of the triangle (the point where its three medians meet). What is the distance IG? Express your answer as a common fraction.



5. _____

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6. Seven friends sit around a round table. After a break they sit down again in a random order: every circular arrangement of the seven is equally likely, where two arrangements that differ only by a rotation of the whole table count as the same (so there are $6! = 720$ arrangements). What is the probability that no two friends who sat next to each other before the break sit next to each other after it? Express your answer as a common fraction.

6. _____

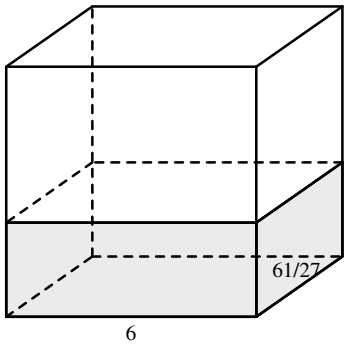
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7. What is the smallest positive integer n such that the last four digits of $73n$ are 2026, in that order?

7. _____

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8. A closed cubical box with edge length 6 contains some water. When the box rests on one of its faces, the water is $\frac{61}{27}$ deep. The box is then turned and balanced on one of its vertices, so that the long diagonal of the cube through that vertex is vertical, and the water settles. How deep is the water now, measured vertically from the lowest vertex of the box up to the water's surface? Express your answer in simplest radical form.



8. _____

Answer Key — MathCounts Nationals Target Practice Contest 1

1. 33

2. 1.04

3. $\frac{7}{18}$

4. 256

5. $\frac{1}{3}$

6. $\frac{23}{360}$

7. 7562

8. $\frac{8\sqrt{3}}{3}$

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Solutions — MathCounts Nationals Target Practice Contest 1

1. Answer: 33**. The idea.** Split $24 = 8 \cdot 3$ and ask when n^2 is 1 more than a multiple of 8 and 1 more than a multiple of 3. Both conditions turn out to depend only on whether n shares a factor with 6.

Work it out. If n is odd, $n^2 - 1 = (n-1)(n+1)$ is a product of two consecutive even numbers, one of which is a multiple of 4, so 8 divides $n^2 - 1$. If n is even, n^2 is even and cannot leave remainder 1 on division by 8. Likewise, if $3 \nmid n$ then $n \equiv \pm 1 \pmod 3$ and $n^2 \equiv 1 \pmod 3$, while if $3 \mid n$ then $n^2 \equiv 0$. So $n^2 \equiv 1 \pmod{24}$ exactly when n is odd and not a multiple of 3.

The squares below 10,000 are n^2 for $0 \le n \le 99$ (and $n = 0$ does not qualify). Among $1, \ldots, 99$: 49 are even, 33 are multiples of 3, and 16 are multiples of 6. By inclusion–exclusion, $99 - 49 - 33 + 16 = 33$ numbers are coprime to 6, giving 33 squares.

Check. In each block of 6 consecutive integers exactly two ($6k + 1$ and $6k + 5$) are coprime to 6: $1^2 = 1$, $5^2 = 25 = 24 + 1$, $7^2 = 49 = 48 + 1$, $11^2 = 121 = 120 + 1$. The numbers 1 to 96 give 16 blocks, 32 values, and 97 adds one more: 33.

2. Answer: 1.04**. The idea.** Put the drilled-out disk back. The whole square balances at its centre O, and the whole square is the drilled plate together with the disk. For an evenly thick plate, each piece pulls in proportion to its area, so the two pieces' balance points must sit on opposite sides of O at distances that balance like a seesaw: (area) × (distance) is the same for both.

Work it out. Measure along the diagonal through the hole. The hole's centre is at $(3, 3)$ if the square is $[0, 12] \times [0, 12]$, and $O = (6, 6)$, so the disk's balance point is $\sqrt{3^2 + 3^2} = 3\sqrt{2}$ cm from O. The disk has area 9π and the drilled plate has area $144 - 9\pi$. If the plate's balance point is d cm from O on the other side, then

$$(144 - 9\pi)\, d = 9\pi \cdot 3\sqrt{2}, \text{ so } d = \frac{27\sqrt{2}\,\pi}{144 - 9\pi} = \frac{3\sqrt{2}\,\pi}{16 - \pi}.$$

Numerically, $3\sqrt{2}\,\pi \approx 13.3286$ and $16 - \pi \approx 12.8584$, so $d \approx 1.0366$, which is 1.04 to the nearest hundredth.

Check. Work with one coordinate instead. The square's area-weighted x-coordinate is $144 \cdot 6 = 864$, and the disk accounts for $9\pi \cdot 3 \approx 84.82$ of it, so the plate's balance point has $x \approx \frac{864 - 84.82}{144 - 28.27} = \frac{779.18}{115.73} \approx 6.7330$, and the same for y. That point is $0.7330\sqrt{2} \approx 1.0366$ cm from $(6, 6)$, moving away from the hole, as it should, since the plate has lost weight on the hole's side.

3. Answer: $\frac{7}{18}$**. The idea.** Sort the good pairs by the number they share. The key point is that two different tiles can never share two numbers, so no good pair gets counted twice.

Work it out. There are $\binom{28}{2} = 378$ equally likely pairs of tiles. Fix a number k. The tiles showing k are k-0, k-1, …, k-6, which is 7 tiles (one of them the double k-k). Any two of them share k, giving $\binom{7}{2} = 21$ pairs, and over the 7 choices of k that is $7 \cdot 21 = 147$ pairs.

Could a pair be counted under two different numbers $j \neq k$? Then both tiles would show both j and k, so both would be the tile j-k, and they would not be two different tiles. So there is no double counting, and the probability is $\frac{147}{378} = \frac{7}{18}$.

Check. Count the pairs with no common number instead. A double k-k misses the $28 - 7 = 21$ tiles without a k. A non-double j-k misses the tiles made from the other five numbers, of which

there are $\binom{5}{2} + 5 = 15$. Adding over all 7 doubles and 21 non-doubles counts each pair twice, so there are $\frac{7\cdot 21+21\cdot 15}{2} = \frac{462}{2} = 231$ such pairs, and $378 - 231 = 147$, as before.

4. Answer: 256. **The idea.** Write the ratio as a fraction $\frac{p}{q}$ in lowest terms. For the fifth term $a\left(\frac{p}{q}\right)^4$ to be an integer, q^4 must divide the first term. That turns the sum into k times a fixed two-variable polynomial, which can only equal a divisor of 781 for very few (p, q).

Work it out. Let the ratio be $\frac{p}{q}$ with $\gcd(p, q) = 1$, $p > q$, and $q \geq 2$ (the ratio is not an integer). Since $q^4 \mid ap^4$ and $\gcd(p, q) = 1$, $q^4 \mid a$; write $a = kq^4$. The terms are $kq^4, kq^3p, kq^2p^2, kqp^3, kp^4$, all integers, and the sum is

$k\left(p^4 + p^3q + p^2q^2 + pq^3 + q^4\right) = 781 = 11 \cdot 71.$

Call the bracket $S(p, q)$. It grows quickly: $S(3, 2) = 81 + 54 + 36 + 24 + 16 = 211$, $S(4, 3) = 256 + 192 + 144 + 108 + 81 = 781$, and every other admissible pair is larger than 781, since $p \geq 5$ already gives $S(5, 2) = 1031$ and $S(5, 3)$, $S(5, 4)$ are bigger still. Now 211 does not divide 781, so the only option is $(p, q) = (4, 3)$ with $k = 1$. The sequence is $81, 108, 144, 192, 256$, and the largest term is 256.

Check. $81 + 108 + 144 + 192 + 256 = 781$, and each term is $\frac{4}{3}$ of the one before. Note that $781 = 1 + 5 + 25 + 125 + 625$ is also a sum of five terms with ratio 5; that is exactly why the statement rules out integer ratios.

5. Answer: $\frac{1}{3}$. **The idea.** Measure both points' heights above side $\overline{BC}$. The incenter is at height r (the inradius), and the centroid is at one third of the height of A. For this triangle both heights are 4, so $\overline{IG}$ is parallel to $\overline{BC}$ and the distance is just a difference of horizontal positions.

Work it out. The semiperimeter is $s = 21$, and Heron's formula gives area $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. So the altitude from A is $\frac{2\cdot 84}{14} = 12$, and $r = \frac{84}{21} = 4$.

Put $B = (0, 0)$ and $C = (14, 0)$. The foot of the altitude from A is at distance $\sqrt{13^2 - 12^2} = 5$ from B, so $A = (5, 12)$. The centroid is the average of the vertices: $G = \left(\frac{0+14+5}{3}, \frac{12}{3}\right) = \left(\frac{19}{3}, 4\right)$.

The incircle touches $\overline{BC}$ at distance $s - CA = 21 - 15 = 6$ from B, and I is directly above that point at height r: $I = (6, 4)$. The two points have the same height, so $IG = \frac{19}{3} - 6 = \frac{1}{3}$.

Check. The incenter is also the weighted average $\frac{aA+bB+cC}{a+b+c}$ with $a = 14$, $b = 15$, $c = 13$: its x-coordinate is $\frac{14\cdot 5+13\cdot 14}{42} = \frac{252}{42} = 6$ and its y-coordinate is $\frac{14\cdot 12}{42} = 4$, matching.

6. Answer: $\frac{23}{360}$. **The idea.** Label the friends $0, 1, \ldots, 6$ in their old order, so the seven forbidden pairs are $\{i, i + 1\}$, the edges of a 7-cycle. Use inclusion–exclusion over sets S of forbidden pairs that are forced to be adjacent. If S has k of the edges and they form j separate runs, glue each run into a block: there are $7 - k$ units to arrange around the table in $(7 - k - 1)!$ ways, and each block can face 2 ways, for $(6 - k)! \cdot 2^j$ arrangements.

Work it out. The number of k-edge subsets of a 7-cycle forming j runs is $\frac{7}{j}\binom{k-1}{j-1}\binom{6-k}{j-1}$. Adding up, with sign $(-1)^k$:

$k = 0$: 720.

$k = 1$: 7 sets, $5! \cdot 2 = 240$ each: -1680.

$k = 2$: 7 single runs $\times 4! \cdot 2$ plus 14 pairs of runs $\times 4! \cdot 4$: $336 + 1344 = +1680$.

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$k = 3$: $7 \times 12 + 21 \times 24 + 7 \times 48 = 84 + 504 + 336$: -924.

$k = 4$: $7 \times 4 + 21 \times 8 + 7 \times 16 = 28 + 168 + 112$: $+308$.

$k = 5$: $7 \times 2 + 14 \times 4 = 14 + 56$: -70.

$k = 6$: 7 sets, each forcing the old order in one of 2 directions: $+14$.

$k = 7$: all seven edges, again 2 arrangements: -2.

Total: $720 - 1680 + 1680 - 924 + 308 - 70 + 14 - 2 = 46$ good arrangements, so the probability is $\frac{46}{720} = \frac{23}{360}$.

Check. The same computation for 5 friends gives $24 - 60 + 60 - 30 + 10 - 2 = 2$, and indeed with five people the only way to avoid all old neighbours is to seat everyone next to the two people they were not beside, the "star" order $0, 2, 4, 1, 3$, in either direction: exactly 2 arrangements.

7. Answer: 7562**. The idea.** We need $73n \equiv 2026 \pmod{10{,}000}$. Since 73 shares no factor with 10,000, this has exactly one solution n from 1 to 9999, and we can find it by multiplying by a number that undoes 73. The useful fact is $73 \cdot 137 = 10{,}001$, which is 1 more than 10,000.

Work it out. Multiply both sides of $73n \equiv 2026$ by 137. The left side becomes $10{,}001\,n \equiv n \pmod{10{,}000}$, and the right side becomes $2026 \cdot 137 = 277{,}562$. So $n \equiv 277{,}562 \equiv 7562 \pmod{10{,}000}$. Every solution is 7562 plus a multiple of 10,000, so the smallest positive one is $n = 7562$.

Check. $73 \cdot 7562 = 552{,}026$, which does end in 2026. Without the trick you would test $10{,}000k + 2026$ for divisibility by 73 with $k = 0, 1, 2, \ldots$; the first success is $k = 55$, since $552{,}026 = 73 \cdot 7562$, and any smaller n would give a smaller k.

8. Answer: $\frac{8\sqrt{3}}{3}$**. The idea.** Put the lowest vertex at the origin with the three edges from it along the axes, so the box is $0 \le x, y, z \le 6$ and "up" is the direction $(1, 1, 1)$. A horizontal water surface is a plane $x + y + z = t$, at height $\frac{t}{\sqrt{3}}$ above the origin, and the water is the part of the cube with $x + y + z \le t$. The trap is to treat the water as a tetrahedron; at this volume the plane has already passed the three vertices next to the bottom one, so three corners of the tetrahedron stick out of the box and must be removed.

Work it out. The water's volume is $36 \cdot \frac{61}{27} = \frac{244}{3}$. The tetrahedron $x + y + z \le t$ in the positive octant has volume $\frac{t^3}{6}$. If the water were such a tetrahedron, $t^3 = 488$ and $t \approx 7.87 > 6$, which is impossible: for $t > 6$ the tetrahedron pokes through the faces $x = 6$, $y = 6$, $z = 6$. The part beyond $x = 6$ is itself a tetrahedron $x' + y + z \le t - 6$ (with $x' = x - 6$) of volume $\frac{(t-6)^3}{6}$, and likewise for y and z; these three pieces do not overlap while $t \le 12$. So for $6 \le t \le 12$ the water's volume is

$\frac{t^3 - 3(t-6)^3}{6} = \frac{244}{3}$, that is $t^3 - 3(t-6)^3 = 488$.

Expanding, $-2t^3 + 54t^2 - 324t + 648 = 488$, or $t^3 - 27t^2 + 162t - 80 = 0$. Trying $t = 8$: $512 - 1728 + 1296 - 80 = 0$. Factoring, $(t - 8)(t^2 - 19t + 10) = 0$, and the quadratic's roots $\frac{19 \pm \sqrt{321}}{2} \approx 0.54$ and 18.46 lie outside $[6, 12]$. So $t = 8$, and the depth is $\frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3} \approx 4.62$.

Check. Plug $t = 8$ back in: the big tetrahedron has volume $\frac{512}{6}$ and each of the three overhanging corners has volume $\frac{2^3}{6} = \frac{8}{6}$, leaving $\frac{512-24}{6} = \frac{488}{6} = \frac{244}{3}$. As a sanity check of the formula, $t = 9$ (the plane through the center of the cube) gives $\frac{729 - 3 \cdot 27}{6} = 108$, exactly half of 216, as symmetry

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demands. Our box is less than half full, and indeed $t = 8 < 9$, so the depth $\frac{8\sqrt{3}}{3} \approx 4.62$ is below the center's height $3\sqrt{3} \approx 5.20$.