MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. What is the value of $\dfrac{2^{2026}+2^{2024}}{2^{2025}-2^{2023}}$? Express your answer as a common fraction.

1. _____

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2. How many four-digit positive integers have digits that alternate between odd and even as you read from left to right? That is, the digits are odd, even, odd, even or else even, odd, even, odd. (Zero counts as even.)

2. _____

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3. What is the smallest positive perfect square that is divisible by each of the three numbers 12, 15 and 18? (A perfect square is the square of an integer, such as $36 = 6^2$.)

3. _____

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4. Priya writes down the units digit of each of the 2026 perfect squares $1^2, 2^2, 3^2, \ldots, 2026^2$. What is the sum of the 2026 digits she writes?

4. _____

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5. Tom already has \$12 saved toward a bicycle. If he adds \$6 each week, he will have exactly the price of the bicycle 4 weeks later than if he adds \$8 each week. What is the price of the bicycle, in dollars?

5. _____

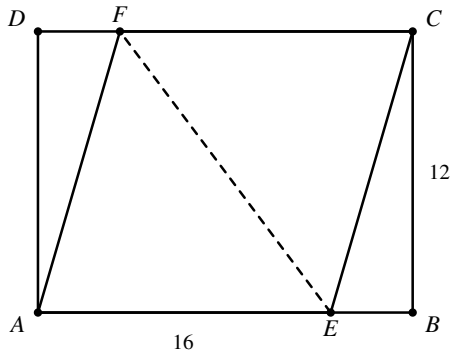
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6. Two integers a and b are chosen independently and uniformly at random from 1 to 100 inclusive (they may be equal). What is the probability that a^2+b^2 is a multiple of 5? Express your answer as a common fraction.

6. _____

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7. Rectangle $ABCD$ has $AB = 16$ and $BC = 12$, with its vertices labelled in order around it. Point E lies on side $\overline{AB}$ and point F lies on side $\overline{CD}$, placed so that $AECF$ is a rhombus (all four sides equal). What is the length of the diagonal $\overline{EF}$ of the rhombus?



7. _____

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8. Every different four-digit number that can be formed by arranging the four digits 1, 1, 2 and 3 (all four used each time) is written down, each one once. What is the sum of all the numbers written down?

8. _____

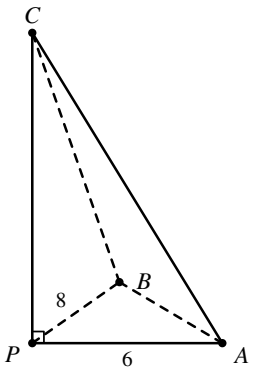
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9. A sequence begins $a_1 = 2$ and $a_2 = 3$, and every term after the second is the product of the two terms just before it. So $a_3 = 6$, $a_4 = 18$, and so on. The tenth term can be written as $a_{10} = 2^x \cdot 3^y$ for nonnegative integers x and y. What is $x + y$?

9. _____

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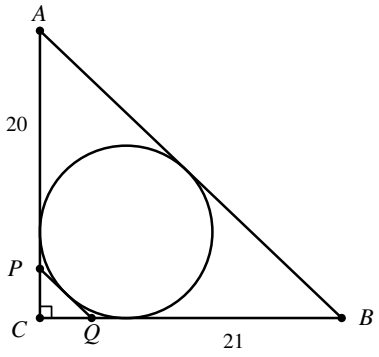
10. Tetrahedron $PABC$ has right angles $\angle APB$, $\angle BPC$ and $\angle CPA$ at vertex P, with $PA = 6$ and $PB = 8$. The sphere that passes through all four vertices P, A, B and C has radius 7. What is the volume of tetrahedron $PABC$? Express your answer in simplest radical form.



10. _____

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11. Right triangle ABC has its right angle at C, with $CA = 20$ and $CB = 21$. A line parallel to the hypotenuse $\overline{AB}$ is tangent to the inscribed circle of triangle ABC, on the side of the circle nearer to C. This line meets $\overline{CA}$ at P and $\overline{CB}$ at Q. What is the area of triangle CPQ? Express your answer as a common fraction.



11. _____

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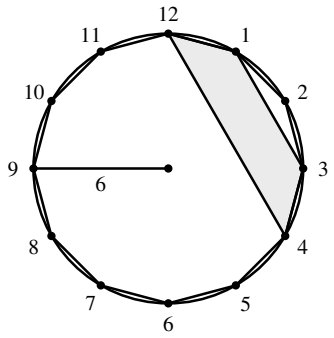
12. Real numbers x and y satisfy $xy = 5$. What is the least possible value of $(x^2 + 4)(y^2 + 9)$? 12. _____

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13. Ana rolls a fair six-sided die over and over until she rolls a 6, and then she stops. Ben, with his own fair six-sided die, rolls over and over until he rolls a 5 or a 6, and then he stops. What is the probability that Ana makes at least as many rolls as Ben? Express your answer as a common fraction. 13. _____

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14. A regular 12-sided polygon is inscribed in a circle of radius 6, and its vertices are numbered 1 through 12 in order around the circle, like the hours on a clock face. What is the area of the quadrilateral whose vertices are the vertices numbered 12, 1, 3 and 4?



14. _____

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15. For how many integers n with $1 \le n \le 2026$ is the fraction $\dfrac{2n+3}{5n+2026}$ not in lowest terms? (That is, the numerator and the denominator have a common factor greater than 1.) 15. _____

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16. What is the remainder when $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 99 \cdot 99!$ is divided by 101? 16. _____

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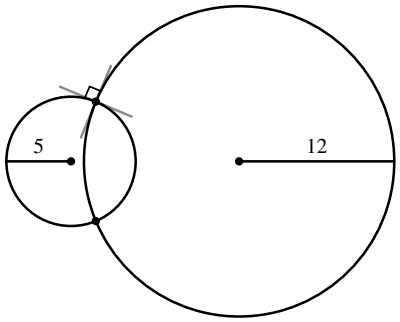
17. For a real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. How many positive integers n satisfy $\lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor = 30$? 17. _____

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18. A sequence begins with $a_1 = 1$, and each later term is given by $a_{n+1} = \dfrac{a_n}{1 + n\,a_n}$ for $n = 1, 2, 3, \ldots$. So $a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{4}$. What is the value of a_{20}? Express your answer as a common fraction. 18. _____

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19. Two circles, with radii 5 and 12, meet at two points. At each of these two points the circles cross at right angles, meaning that the two circles' tangent lines there are perpendicular. What is the distance between the two points where the circles meet? Express your answer as a common fraction.



19. _____

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20. Call a positive integer *odd-weighted* if its base-2 representation contains an odd number of digits 1. For example, $7 = 111_2$ is odd-weighted and $5 = 101_2$ is not. How many of the integers $1, 2, 3, \ldots, 2026$ are odd-weighted?

20. _____

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21. Triangle ABC has $AB = 12$ and $AC = 16$, and the circle passing through A, B and C has radius 10. What is the length of the altitude from A to line BC? Express your answer as a common fraction.

21. _____

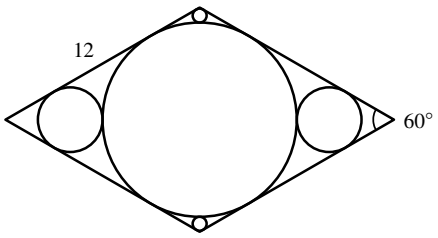
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22. An ordered sum of 14 writes 14 as a sum of positive integers in which order matters, so $3 + 1 + 3 + 3 + 1 + 3$ and $1 + 3 + 3 + 3 + 3 + 1$ are different ordered sums. How many ordered sums of 14 use only parts equal to 1, 2 or 3 and read the same forwards and backwards? (One example is $3 + 1 + 3 + 3 + 1 + 3$.)

22. _____

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23. A rhombus has side length 12 and one angle of 60°. A circle is inscribed in the rhombus. Then, in each of the rhombus's four corners, a smaller circle is drawn that is tangent to the two sides meeting at that corner and externally tangent to the inscribed circle. What is the sum of the radii of the four smaller circles? Express your answer in simplest radical form.



23. _____

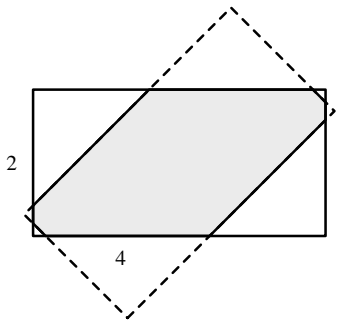
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24. How many ordered lists (a, b, c, d) of integers satisfy $0 \le a \le b \le c \le d \le 9$ and $a + d = b + c$?

24. _____

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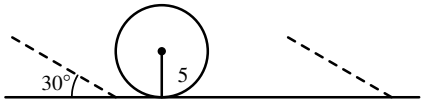
25. A 4 by 2 rectangle is rotated 45° about its centre. What is the area of the region that lies inside both the original rectangle and the rotated rectangle? Express your answer in simplest radical form.



25. _____

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26. A ball of radius 5 rests on flat, level ground. The sun's rays are parallel to one another and make an angle of 30° with the ground. What is the area of the ball's shadow on the ground? Express your answer in terms of π.



26. _____

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27. Point P is chosen uniformly at random inside an equilateral triangle ABC with side length 2. What is the probability that triangle PBC has an obtuse angle? Express your answer as a common fraction in terms of π, in simplest radical form.

27. _____

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28. What is the smallest positive integer that is a multiple of 99 and whose decimal digits add up to 99? (For example, 99 itself is a multiple of 99, but its digits add up to only 18.)

28. _____

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29. Each square of a 4×4 grid is coloured black or white so that every row contains exactly two black squares and every column contains exactly two black squares. How many such colourings are there? (Colourings that differ by a rotation or reflection of the grid count as different.)

29. _____

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30. A walker starts at the origin of the coordinate plane and takes 10 steps, each one unit up, down, left or right. How many of the 4^{10} possible sequences of steps end at the point $(3, 1)$?

30. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — MathCounts Nationals Sprint Practice Contest 6

1. $\frac{10}{3}$

2. 1125

3. 900

4. 9121

5. 108

6. $\frac{9}{25}$

7. 15

8. 23331

9. 55

10. $32\sqrt{6}$

11. $\frac{216}{35}$

12. 121

13. $\frac{3}{4}$

14. 18

15. 189

16. 99

17. 17

18. $\frac{1}{191}$

19. $\frac{120}{13}$

20. 1013

21. $\frac{48}{5}$

22. 68

23. $-72 + 44\sqrt{3}$

24. 125

25. $-20 + 18\sqrt{2}$

26. 50π

27. $\frac{\sqrt{3}\pi}{18} + \frac{1}{2}$

28. 1098999999999

29. 90

30. 25200

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Solutions — MathCounts Nationals Sprint Practice Contest 6

1. Answer: $\frac{10}{3}$**. The idea.** Nobody can compute 2^{2026}, but every power in sight is a multiple of the smallest one, 2^{2023}. Factor it out of the top and the bottom and it cancels.

Work it out. The numerator is $2^{2026} + 2^{2024} = 2^{2023}(2^3 + 2^1) = 2^{2023} \cdot 10$. The denominator is $2^{2025} - 2^{2023} = 2^{2023}(2^2 - 1) = 2^{2023} \cdot 3$. The quotient is $\frac{10}{3}$.

Check. The same pattern with small exponents, $\frac{2^3+2^1}{2^2-2^0} = \frac{10}{3}$, gives the same value, as it must: shifting every exponent by the same amount multiplies the top and bottom by the same power of 2.

2. Answer: 1125**. The idea.** Once the parity pattern is fixed, each place has its own independent list of allowed digits, so multiply. The only catch is the leading digit, which cannot be 0.

Work it out. Odd, even, odd, even: 5 odd choices for each odd place and 5 even choices $(0, 2, 4, 6, 8)$ for each even place, so $5 \cdot 5 \cdot 5 \cdot 5 = 625$. Even, odd, even, odd: the first digit is even but not 0, so it has only 4 choices, giving $4 \cdot 5 \cdot 5 \cdot 5 = 500$. The two patterns do not overlap, so the total is $625 + 500 = 1125$.

Check. Ignoring the leading-zero rule would give $2 \cdot 625 = 1250$ strings; the bad ones start with 0 and then go odd, even, odd: $5 \cdot 5 \cdot 5 = 125$ of them. $1250 - 125 = 1125$.

3. Answer: 900**. The idea.** A number divisible by 12, 15 and 18 is a multiple of their least common multiple. A perfect square needs every prime to an even power, so round each exponent of the least common multiple up to the next even number.

Work it out. $12 = 2^2 \cdot 3$, $15 = 3 \cdot 5$ and $18 = 2 \cdot 3^2$, so the least common multiple is $2^2 \cdot 3^2 \cdot 5 = 180$. A square multiple of 180 needs 2^2, 3^2 and an even power of 5, at least 5^2. The smallest is $2^2 \cdot 3^2 \cdot 5^2 = 900$.

Check. $900 = 30^2$, and $900 = 12 \cdot 75 = 15 \cdot 60 = 18 \cdot 50$. The squares of 6, 12, 18, 24 (the only candidates below 900 that are multiples of 36) are not multiples of 5.

4. Answer: 9121**. The idea.** The units digit of n^2 depends only on the units digit of n, so the list of digits repeats every ten squares. Sum one block of ten and count the blocks.

Work it out. For n ending in $1, 2, \ldots, 9, 0$ the square ends in $1, 4, 9, 6, 5, 6, 9, 4, 1, 0$, which add to 45. The squares 1^2 through 2020^2 make 202 full blocks, contributing $202 \cdot 45 = 9090$. The leftover squares $2021^2, \ldots, 2026^2$ end in $1, 4, 9, 6, 5, 6$, adding 31. The total is $9090 + 31 = 9121$.

Check. The block sum 45 can be seen by symmetry: n and $10 - n$ have squares with the same units digit, so the block is $2(1 + 4 + 9 + 6) + 5 + 0 = 45$.

5. Answer: 108**. The idea.** Only the amount still needed matters. Call it N dollars; the two saving plans take $\frac{N}{6}$ and $\frac{N}{8}$ weeks, and these differ by 4.

Work it out. $\frac{N}{6} - \frac{N}{8} = \frac{4N-3N}{24} = \frac{N}{24} = 4$, so $N = 96$. The price is $12 + 96 = 108$ dollars.

Check. At \$6 a week Tom needs $96 \div 6 = 16$ weeks; at \$8 a week he needs $96 \div 8 = 12$ weeks. The difference is 4 weeks.

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6. Answer: $\frac{9}{25}$**. The idea.** Only the remainders of a and b on division by 5 matter, and since 100 is a multiple of 5, each remainder 0, 1, 2, 3, 4 is equally likely. Squares mod 5 take only three values, which makes the count short.

Work it out. The squares of 0, 1, 2, 3, 4 leave remainders 0, 1, 4, 4, 1 on division by 5. So a^2 is $\equiv 0$ with probability $\frac{1}{5}$, $\equiv 1$ with probability $\frac{2}{5}$ and $\equiv 4$ with probability $\frac{2}{5}$, and the same for b^2. The sum is a multiple of 5 when both are 0 (probability $\frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$), or one is 1 and the other is 4 (probability $2 \cdot \frac{2}{5} \cdot \frac{2}{5} = \frac{8}{25}$). Note $1 + 1$ and $4 + 4$ do not work, and neither does 0 with a nonzero remainder. The total is $\frac{9}{25}$.

Check. Among the 25 equally likely remainder pairs, the good ones are $(0, 0)$ and the pairs with one entry in $\{1, 4\}$ and the other in $\{2, 3\}$: $1 + 2 \cdot 2 \cdot 2 = 9$ pairs out of 25.

7. Answer: 15**. The idea.** First find the rhombus's side from the right triangle EBC. Then compute the rhombus's area two ways, as base times height and as half the product of its diagonals; the other diagonal $\overline{AC}$ is the rectangle's diagonal.

Work it out. Let $AE = EC = x$. Then $EB = 16-x$, and in right triangle EBC, $x^2 = (16-x)^2+12^2$, so $32x = 400$ and $x = \frac{25}{2}$. The rhombus has base $AE = \frac{25}{2}$ and height $BC = 12$, so its area is 150. Its diagonals are $AC = \sqrt{16^2 + 12^2} = 20$ and EF, and the area is also $\frac{1}{2} \cdot 20 \cdot EF$. So $EF = 15$.

Check. In coordinates, $E = \left(\frac{25}{2}, 0\right)$ and $F = \left(\frac{7}{2}, 12\right)$, so $EF = \sqrt{9^2 + 12^2} = 15$. The diagonals of a rhombus are perpendicular: $\overline{AC}$ has direction $(16, 12)$ and $\overline{EF}$ has direction $(-9, 12)$, and $16 \cdot (-9) + 12 \cdot 12 = 0$.

8. Answer: 23331**. The idea.** Do not list them. By symmetry every place (thousands, hundreds, tens, ones) sees the same digits the same number of times, so find one place's column sum and multiply by 1111.

Work it out. The number of different arrangements is $\frac{4!}{2!} = 12$. Fix a place. It holds a 2 when the other three places hold 1, 1, 3, in 3 ways; likewise it holds a 3 in 3 ways; and it holds a 1 in the remaining $12 - 6 = 6$ ways. So each column adds up to $6 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 = 21$, and the total is $21 \cdot 1111 = 23331$.

Check. The average of the 12 numbers is $\frac{23331}{12} = 1944.25$, and each place's average digit is $\frac{21}{12} = 1.75$, the average of the digits 1, 1, 2, 3, so the average number is $1.75 \cdot 1111 = 1944.25$. Consistent.

9. Answer: 55**. The idea.** Multiplying powers adds exponents. So the exponent of 2 in each term is the sum of the exponents of 2 in the two terms before it, and the same for 3: both exponent sequences follow the Fibonacci rule.

Work it out. Exponents of 2: 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, so $x = 21$. Exponents of 3: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, so $y = 34$. Therefore $x + y = 55$.

Check. The sums $x+y$ themselves obey the same rule: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, the Fibonacci numbers, and $a_4 = 18 = 2^1 \cdot 3^2$ has exponent sum 3, matching the fourth entry.

10. Answer: $32\sqrt{6}$**. The idea.** Three perpendicular edges from P are three edges of a rectangular box with corner P. Any sphere through P, A, B, C is the sphere through all eight corners of that

box: a box has a sphere through all its corners (centred at the box's centre), and a sphere is fixed by four points not in a plane. So the sphere's diameter is the box's space diagonal.

Work it out. Let $PC = c$. The box is $6 \times 8 \times c$, so its space diagonal is $\sqrt{36 + 64 + c^2}$, and this equals the diameter 14. Then $c^2 = 196 - 100 = 96$ and $c = 4\sqrt{6}$. The tetrahedron has a right-angled corner at P, so taking triangle PAB (area $\frac{6 \cdot 8}{2} = 24$) as base and PC as height, its volume is $\frac{1}{3} \cdot 24 \cdot 4\sqrt{6} = 32\sqrt{6}$.

Check. In coordinates $P = (0, 0, 0)$, $A = (6, 0, 0)$, $B = (0, 8, 0)$, $C = (0, 0, 4\sqrt{6})$, the point $O = (3, 4, 2\sqrt{6})$ is at distance $\sqrt{9 + 16 + 24} = 7$ from P, and from A: $\sqrt{9 + 16 + 24} = 7$, and likewise from B and C. The volume is also $\frac{1}{6} \cdot PA \cdot PB \cdot PC$, the same $32\sqrt{6}$.

11. Answer: $\frac{216}{35}$. The idea. Triangle CPQ is similar to CAB, so its area follows from the ratio of perimeters. And its perimeter is easy: the incircle of ABC is an excircle of CPQ (it touches $\overline{PQ}$ and the extensions of CP and CQ), and tangent segments from C show that the perimeter of CPQ is twice the tangent length from C.

Work it out. The hypotenuse is $\sqrt{20^2 + 21^2} = 29$, so the inradius is $r = \frac{20+21-29}{2} = 6$, and the tangent length from the right-angle vertex C is $r = 6$. Let the circle touch CA at X, CB at Y and PQ at T. Then $PT = PX$ and $QT = QY$, so the perimeter of CPQ is $CP + PX + QY + CQ = CX + CY = 12$. The perimeter of ABC is $20 + 21 + 29 = 70$, so the scale factor is $\frac{12}{70} = \frac{6}{35}$, and the area is $210 \cdot \left(\frac{6}{35}\right)^2 = \frac{210 \cdot 36}{1225} = \frac{216}{35}$.

Check. By heights: the altitude from C to AB is $\frac{420}{29}$. The line PQ is on the far side of the circle from AB, at distance $2r = 12$ from it, so the altitude of CPQ is $\frac{420}{29} - 12 = \frac{72}{29}$, and the ratio $\frac{72}{420} = \frac{6}{35}$ agrees.

12. Answer: 121**. The idea.** Multiply out. The term x^2y^2 is fixed by $xy = 5$, so only $9x^2 + 4y^2$ can vary, and the product of those two terms is fixed too. That is exactly the situation where AM–GM gives the minimum.

Work it out. $(x^2+4)(y^2+9) = x^2y^2 + 9x^2 + 4y^2 + 36 = 25 + 36 + 9x^2 + 4y^2 = 61 + 9x^2 + 4y^2$. By AM–GM, $9x^2 + 4y^2 \geq 2\sqrt{9x^2 \cdot 4y^2} = 12|xy| = 60$, with equality when $9x^2 = 4y^2$. Together with $xy = 5$ that happens at $y = \frac{3}{2}x$, $\frac{3}{2}x^2 = 5$, so $x^2 = \frac{10}{3}$ and $y^2 = \frac{15}{2}$. The least value is $61 + 60 = 121$.

Check. At that point $x^2 + 4 = \frac{22}{3}$ and $y^2 + 9 = \frac{33}{2}$, and $\frac{22}{3} \cdot \frac{33}{2} = \frac{726}{6} = 121$. Another route: $(x^2+4)(y^2+9) = (xy-6)^2 + (3x+2y)^2 = 1 + (3x+2y)^2$, and $(3x+2y)^2 = (3x-2y)^2 + 24xy \geq 120$, giving 121 again.

13. Answer: $\frac{3}{4}$. The idea. Let them roll at the same moment, one roll each per round. The question is settled in the first round in which somebody stops: if Ben stops in that round (whether or not Ana also stops), Ana makes at least as many rolls as Ben; if only Ana stops, she makes fewer. Rounds where nobody stops change nothing.

Work it out. In any one round, Ben stops with probability $\frac{1}{3}$. Ana stops while Ben does not with probability $\frac{1}{6} \cdot \frac{2}{3} = \frac{1}{9}$. Nobody stops with probability $\frac{5}{6} \cdot \frac{2}{3} = \frac{5}{9}$, and then the next round is a fresh start. So the probability is $\dfrac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{9}} = \dfrac{3}{4}$.

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Check. Directly: Ana needs exactly a rolls with probability $\left(\frac{5}{6}\right)^{a-1}\frac{1}{6}$, and Ben needs at most a rolls with probability $1-\left(\frac{2}{3}\right)^{a}$. Summing over a gives $1-\frac{1}{6}\cdot\frac{2}{3}\left(1+\frac{5}{9}+\left(\frac{5}{9}\right)^{2}+\cdots\right)=1-\frac{1}{9}\cdot\frac{9}{4}=\frac{3}{4}$.

14. Answer: 18**. The idea.** Vertices 1 and 3 are mirror images in the radius through vertex 2, and so are vertices 12 and 4. So both chords $\overline{1\,3}$ and $\overline{12\,4}$ are perpendicular to that radius, hence parallel: the quadrilateral is an isosceles trapezoid, and each chord's length and distance from the centre come from its central angle.

Work it out. Neighbouring vertices are 30° apart as seen from the centre O. Chord $\overline{1\,3}$ subtends 60°, so it forms an equilateral triangle with O: length 6, at distance $6\cos 30^\circ = 3\sqrt{3}$ from O. Chord $\overline{12\,4}$ subtends 120°: length $2\cdot 6\sin 60^\circ = 6\sqrt{3}$, at distance $6\cos 60^\circ = 3$ from O. Both lie on the same side of O (the side of vertex 2), so the trapezoid's height is $3\sqrt{3}-3$. Its area is $\frac{1}{2}(6+6\sqrt{3})(3\sqrt{3}-3) = 3(\sqrt{3}+1)\cdot 3(\sqrt{3}-1) = 9\cdot 2 = 18$.

Check. Use triangles from the centre, each of area $\frac{1}{2}\cdot 6^2\sin(\text{central angle}) = 18\sin(\text{angle})$. The centre lies outside the quadrilateral (the arc from 4 back to 12 is 240°), so the area is $[O\,12\,1] + [O\,1\,3] + [O\,3\,4] - [O\,12\,4] = 18\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+\frac{1}{2}-\frac{\sqrt{3}}{2}\right) = 18$.

15. Answer: 189**. The idea.** Any common factor of $2n+3$ and $5n+2026$ also divides the combination that wipes out n: $5(2n+3)-2(5n+2026) = 15-4052 = -4037 = -11\cdot 367$, with 367 prime. So the only possible common prime factors are 11 and 367. Conversely, if an odd prime p divides both $2n+3$ and 4037, then it divides $5(2n+3)+4037 = 2(5n+2026)$, hence divides $5n+2026$. So the fraction reduces exactly when 11 or 367 divides $2n+3$.

Work it out. $11 \mid 2n+3$ exactly when $n \equiv 4 \pmod{11}$ (as $2\cdot 4+3 = 11$): $n = 4, 15, \ldots, 2017$, which is $\frac{2017-4}{11}+1 = 184$ values. $367 \mid 2n+3$: the number $2n+3$ is odd and between 5 and 4055, so it is one of the odd multiples $367, 1101, 1835, 2569, 3303, 4037$, giving $n = 182, 549, 916, 1283, 1650, 2017$: 6 values. Both happen only when $4037 \mid 2n+3$, that is $n = 2017$. By inclusion–exclusion, $184+6-1 = 189$.

Check. $n = 4$: $\frac{11}{2046}$, and $2046 = 11\cdot 186$. $n = 182$: $\frac{367}{2936}$, and $2936 = 367\cdot 8$. $n = 2017$: $\frac{4037}{12111}$, and $12111 = 3\cdot 4037$. A value such as $n = 5$ gives $\frac{13}{2051}$, and $2051 = 7\cdot 293$ shares nothing with 13.

16. Answer: 99**. The idea.** Each term telescopes: $k\cdot k! = (k+1)\cdot k! - k! = (k+1)! - k!$. So the whole sum collapses to $100! - 1$, and then we need 100! modulo the prime 101.

Work it out. Adding $(2!-1!)+(3!-2!)+\cdots+(100!-99!)$, everything cancels except $100!-1! = 100!-1$.

Now work modulo 101, which is prime. Every number a from 1 to 100 has a partner b in that range with $ab \equiv 1 \pmod{101}$. The only numbers that are their own partners satisfy $a^2 \equiv 1$, i.e. $101 \mid (a-1)(a+1)$, so $a = 1$ or $a = 100$. The numbers $2, 3, \ldots, 99$ therefore split into pairs whose products are each $\equiv 1$, so $2\cdot 3\cdots 99 \equiv 1$ and $100! \equiv 1\cdot 1\cdot 100 \equiv -1 \pmod{101}$. (This is Wilson's theorem.)

Hence the sum is $100! - 1 \equiv -2 \equiv 99 \pmod{101}$.

Check. The same argument with the prime 7: $1\cdot 1!+\cdots+5\cdot 5! = 6!-1 = 719$, and $719 = 7\cdot 102+5$, which is $7-2$, just as predicted.

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17. Answer: 17. **The idea.** As n grows, neither floor ever decreases, so $f(n) = \lfloor\sqrt{n}\rfloor + \lfloor\sqrt[3]{n}\rfloor$ never decreases. The solutions form one block of consecutive integers; find where f first reaches 30 and where it first passes 30. Organise by the cube-root floor, which changes rarely.

Work it out. If $\lfloor\sqrt[3]{n}\rfloor = 7$, then $343 \le n \le 511$ and $\lfloor\sqrt{n}\rfloor \le 22$ (since $23^2 = 529$), so $f(n) \le 29$. If $\lfloor\sqrt[3]{n}\rfloor = 9$ or more, then $n \ge 729$ and $\lfloor\sqrt{n}\rfloor \ge 27$, so $f(n) \ge 36$. So we need $\lfloor\sqrt[3]{n}\rfloor = 8$, i.e. $512 \le n \le 728$, together with $\lfloor\sqrt{n}\rfloor = 22$, i.e. $484 \le n \le 528$. Both hold exactly for $512 \le n \le 528$: that is 17 integers.

Check. $f(511) = 22 + 7 = 29$, $f(512) = 22 + 8 = 30$, $f(528) = 22 + 8 = 30$ and $f(529) = 23 + 8 = 31$. Note that the jump at 512 comes from the cube root, the jump at 529 from the square root.

18. Answer: $\frac{1}{191}$. **The idea.** The rule looks awkward, but its reciprocal is simple: flipping both sides turns the division into an addition.

Work it out. $\frac{1}{a_{n+1}} = \frac{1+n\,a_n}{a_n} = \frac{1}{a_n} + n$. So each step adds n to the reciprocal, and

$\frac{1}{a_{20}} = \frac{1}{a_1} + (1 + 2 + \cdots + 19) = 1 + 190 = 191$.

Hence $a_{20} = \frac{1}{191}$.

Check. The reciprocals run $1, 2, 4, 7, 11, \ldots$, growing by $1, 2, 3, 4, \ldots$; this matches $a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{4}$, and in general $\frac{1}{a_n} = 1 + \frac{n(n-1)}{2}$, which gives $1 + \frac{20\cdot 19}{2} = 191$ for $n = 20$.

19. Answer: $\frac{120}{13}$. **The idea.** A tangent line is perpendicular to the radius at its point of tangency. So if the two tangent lines at a meeting point are perpendicular, the two radii drawn to that point are perpendicular too, and the centres and the meeting point form a right triangle.

Work it out. Let the centres be O_1 (radius 5) and O_2 (radius 12), and let the circles meet at P and Q. At P the tangent to the first circle is perpendicular to O_1P and the tangent to the second is perpendicular to O_2P; turning both tangents through 90° gives the radii, so $O_1P \perp O_2P$. Triangle O_1PO_2 is right-angled at P with legs 5 and 12, so $O_1O_2 = 13$.

The whole figure is symmetric about the line O_1O_2, so $\overline{PQ}$ is perpendicular to O_1O_2 and is bisected by it: half of PQ is the altitude from P in the right triangle. Twice the area of the triangle is $5 \cdot 12 = 13 \cdot h$, so $h = \frac{60}{13}$ and $PQ = \frac{120}{13}$.

Check. The foot of the altitude is $\frac{5^2}{13} = \frac{25}{13}$ from O_1, and $\left(\frac{25}{13}\right)^2 + \left(\frac{60}{13}\right)^2 = \frac{625+3600}{169} = 25 = 5^2$, so P really is on the first circle; similarly $\left(\frac{144}{13}\right)^2 + \left(\frac{60}{13}\right)^2 = \frac{24336}{169} = 144$.

20. Answer: 1013. **The idea.** The numbers $2m$ and $2m + 1$ have the same binary digits except the last, which is 0 for $2m$ and 1 for $2m+1$. So in every such pair exactly one number is odd-weighted. Pair up a convenient range and adjust the ends.

Work it out. The integers $0, 1, \ldots, 2027$ form the 1014 pairs $\{0, 1\}, \{2, 3\}, \ldots, \{2026, 2027\}$, so exactly 1014 of them are odd-weighted. Now remove the two numbers outside our range. 0 has no 1s, so it is not odd-weighted. $2027 = 1024+512+256+128+64+32+8+2+1 = 11111101011_2$ has nine 1s, so it is odd-weighted. Hence $1, \ldots, 2026$ contain $1014 - 1 = 1013$ odd-weighted numbers.

Check. For $1, \ldots, 6$ the same method gives 4 odd-weighted numbers in $0, \ldots, 7$, minus $7 = 111_2$: 3. Directly: 1, 2, 4 are odd-weighted and 3, 5, 6 are not.

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21. Answer: $\frac{48}{5}$. The idea. Draw the diameter $\overline{AA'}$ of the circle. The altitude $\overline{AH}$ and the diameter make two similar right triangles, which ties the altitude to the two given sides and the diameter.

Work it out. $\angle AHB = 90°$, and $\angle ACA' = 90°$ because $\overline{AA'}$ is a diameter. Also $\angle ABH = \angle ABC = \angle AA'C$, since both are inscribed angles on the arc AC (if B and A' lie on opposite sides of AC the angles are supplementary instead, and the sines, which are all we use, still agree). So right triangles ABH and $AA'C$ are similar, giving $\dfrac{AH}{AB} = \dfrac{AC}{AA'}$, i.e. $AH = \dfrac{AB \cdot AC}{2R} = \dfrac{12 \cdot 16}{20} = \dfrac{48}{5}$.

Check. Place the circle's centre at the origin and $A = (0, 10)$. The points of the circle at distance 12 and 16 from A have $y = 10 - \frac{144}{20} = 2.8$ and $y = 10 - \frac{256}{20} = -2.8$, with $x = \pm 9.6$. Taking $B = (-9.6, 2.8)$ and $C = (-9.6, -2.8)$, line BC is $x = -9.6$ and the altitude is $9.6 = \frac{48}{5}$; taking $C = (9.6, -2.8)$ instead, line BC passes through the origin and the distance from A is again $\frac{10 \cdot 9.6}{10} = 9.6$. Both possible triangles agree.

22. Answer: 68**. The idea.** A palindrome is decided by its first half. Split by whether the number of parts is even (the middle falls between two parts) or odd (there is a middle part).

Work it out. Let $c(n)$ be the number of ordered sums of n with parts $1, 2, 3$. Looking at the first part, $c(n) = c(n-1) + c(n-2) + c(n-3)$, and $c(0), c(1), \ldots, c(7) = 1, 1, 2, 4, 7, 13, 24, 44$.

Even number of parts. The first half is any ordered sum of 7 and the second half is its mirror image: $c(7) = 44$ ways.

Odd number of parts. If the middle part is m, each half is an ordered sum of $\frac{14-m}{2}$, so m must be even; the only allowed even part is $m = 2$, and each half is an ordered sum of 6: $c(6) = 24$ ways.

Total: $44 + 24 = 68$.

Check. Try 4 instead of 14: the rule gives $c(2) + c(1) = 2 + 1 = 3$, and indeed the palindromes are $1 + 1 + 1 + 1$, $2 + 2$ and $1 + 2 + 1$.

23. Answer: $-72 + 44\sqrt{3}$**. The idea.** Each small circle sits in an angle with its centre on the angle's bisector, which passes through the centre O of the rhombus. A circle of radius t tangent to both sides of an angle θ has its centre at distance $t/\sin(\theta/2)$ from the vertex. Tangency to the incircle says: (distance from vertex to O) = (distance to small centre) $+t + R$.

Work it out. The rhombus has angles 60° and 120°. The diagonal joining the two 120° corners cuts off two equilateral triangles, so it has length 12; the other diagonal has length $12\sqrt{3}$. So O is $6\sqrt{3}$ from each 60° corner and 6 from each 120° corner. The incircle radius is half the rhombus's height: $R = \frac{1}{2} \cdot 12 \sin 60° = 3\sqrt{3}$.

60° *corner.* The centre is $t/\sin 30° = 2t$ from the vertex: $2t + t = 6\sqrt{3} - 3\sqrt{3}$, so $t = \sqrt{3}$.

120° *corner.* The centre is $t/\sin 60° = \frac{2t}{\sqrt{3}}$ from the vertex: $\frac{2t}{\sqrt{3}} + t = 6 - 3\sqrt{3}$, so $t = \frac{(6-3\sqrt{3})\sqrt{3}}{2+\sqrt{3}} = (6\sqrt{3} - 9)(2 - \sqrt{3}) = 21\sqrt{3} - 36$.

Sum of the four radii: $2\sqrt{3} + 2(21\sqrt{3} - 36) = 44\sqrt{3} - 72$.

Check. $21\sqrt{3} - 36 \approx 0.373$ is positive and much smaller than $6 - 3\sqrt{3} \approx 0.804$, the gap between the 120° corner and the incircle, as it must be; the total is about 4.21.

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24. Answer: 125**. The idea.** Describe a non-decreasing list by its starting value and its three gaps. The condition $a + d = b + c$ says $d - c = b - a$: the first and last gaps are equal. Then only a starting value and two gaps are free, subject to one inequality.

Work it out. Write $p = b - a$, $q = c - b$ and $r = d - c$, all at least 0. The condition $a + d = b + c$ is the same as $d - c = b - a$, i.e. $r = p$. So a good list is fixed by three nonnegative integers a, p, q, and it is $(a,\ a + p,\ a + p + q,\ a + 2p + q)$; the only remaining requirement is $d = a + 2p + q \le 9$.

Count by p, which can be $0, 1, 2, 3, 4$. For a fixed p we need $a + q \le m$ with $m = 9 - 2p$, and the number of pairs of nonnegative integers with $a + q \le m$ is $1 + 2 + \cdots + (m + 1) = \frac{(m+1)(m+2)}{2}$. For $m = 9, 7, 5, 3, 1$ this gives $55, 36, 21, 10, 3$, and $55 + 36 + 21 + 10 + 3 = 125$.

Check. Try the smaller bound $d \le 1$: only $p = 0$ and $a + q \le 1$ are possible, giving 3 lists, $(0, 0, 0, 0)$, $(1, 1, 1, 1)$ and $(0, 0, 1, 1)$. Listing all five non-decreasing lists of 0s and 1s directly, $(0, 0, 0, 1)$ and $(0, 1, 1, 1)$ fail and the other three work.

25. Answer: $-20+18\sqrt{2}$**. The idea.** The rotated rectangle is the region between two pairs of parallel lines at 45°. Each of those four lines slices one corner off the original rectangle, and every slice is an isosceles right triangle. Subtract the four corners.

Work it out. Put the centre at the origin, with the original rectangle $-2 \le x \le 2$, $-1 \le y \le 1$. After rotating, the long sides of the new rectangle point along the line $y = x$. Its long sides are at distance 1 from the centre and its short sides at distance 2, so it is the region between the lines $y - x = \pm\sqrt{2}$ and between the lines $x + y = \pm 2\sqrt{2}$ (a line $y - x = k$ is at distance $\frac{|k|}{\sqrt{2}}$ from the origin).

The line $y = x + \sqrt{2}$ meets the top side $y = 1$ at $x = 1 - \sqrt{2}$ and the left side $x = -2$ at $y = \sqrt{2} - 2$, so it cuts off the top-left corner: an isosceles right triangle with legs $1 - \sqrt{2} - (-2) = 3 - \sqrt{2}$. The line $x + y = 2\sqrt{2}$ meets the top side at $x = 2\sqrt{2} - 1$ and the right side at $y = 2\sqrt{2} - 2$, cutting off the top-right corner with legs $2 - (2\sqrt{2} - 1) = 3 - 2\sqrt{2}$. By symmetry through the centre, the other two lines cut off matching triangles at the bottom-right and bottom-left corners. These four triangles do not overlap (along the top side, one ends at $x \approx -0.41$ and the other starts at $x \approx 1.83$).

So the overlap has area $8 - 2 \cdot \frac{(3-\sqrt{2})^2}{2} - 2 \cdot \frac{(3-2\sqrt{2})^2}{2} = 8 - (11 - 6\sqrt{2}) - (17 - 12\sqrt{2}) = 18\sqrt{2} - 20$.

Check. $18\sqrt{2} - 20 \approx 5.46$, which is less than the rectangle's area 8. As a second check, the overlap is an octagon whose top edge runs from $x = 1 - \sqrt{2}$ to $x = 2\sqrt{2} - 1$, a length of $3\sqrt{2} - 2 \approx 2.24$. That edge is part of the top side of the original rectangle, which has length 4, and the two corner cuts on that side have legs $3 - \sqrt{2}$ and $3 - 2\sqrt{2}$; indeed $(3 - \sqrt{2}) + (3\sqrt{2} - 2) + (3 - 2\sqrt{2}) = 4$.

26. Answer: 50π**. The idea.** The rays that touch the ball form a cylinder of radius 5 around the ray through the ball's centre. The shadow is where the ground cuts this cylinder: an ellipse. Across the direction of the sun it is as wide as the ball; along that direction it is stretched, because the ground meets the cylinder at a slant.

Work it out. Across the sun's direction, the shadow's half-width is the cylinder's radius, 5. Now look at the vertical plane through the centre that contains the sun's rays: the two extreme rays are parallel lines 10 apart (the ball's diameter), and they cross the ground at an angle of 30°. So the length they cut off on the ground is $\frac{10}{\sin 30°} = 20$, a half-length of 10. An ellipse with semi-axes 5 and 10 has area $\pi \cdot 5 \cdot 10 = 50\pi$.

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Check. If the sun were straight overhead (90°), the same reasoning gives a disk of area 25π, the ball's own cross-section. At 30° the shadow is stretched by the factor $\frac{1}{\sin 30°} = 2$ in one direction only, doubling the area to 50π.

27. Answer: $\frac{\sqrt{3}\pi}{18} + \frac{1}{2}$. **The idea.** The angles of triangle PBC at B and C are smaller than the 60° angles of triangle ABC, so only the angle at P can be obtuse. By Thales' theorem, $\angle BPC$ is obtuse exactly when P lies inside the circle with diameter $\overline{BC}$. So we need the area of the part of that disc inside the triangle.

Work it out. Let M be the midpoint of $\overline{BC}$; the circle has centre M and radius 1. Let F and E be the midpoints of $\overline{AB}$ and $\overline{AC}$. Since MF joins midpoints, $MF = \frac{1}{2}AC = 1$, so F is on the circle, and likewise E is. The circle meets line AB only at B and F, and line AC only at C and E. So inside the triangle, the disc is bounded by the segments $\overline{BF}$ and $\overline{CE}$ and by the arc from F to E.

Split this region at M. Triangles MBF and MCE have all sides equal to 1, so each is equilateral with area $\frac{\sqrt{3}}{4}$. That leaves the sector FME, with angle $180° - 60° - 60° = 60°$ and area $\frac{1}{6}\pi$. The region's area is $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$.

Triangle ABC has area $\frac{\sqrt{3}}{4}\cdot 4 = \sqrt{3}$, so the probability is $\dfrac{\frac{\sqrt{3}}{2} + \frac{\pi}{6}}{\sqrt{3}} = \dfrac{1}{2} + \dfrac{\pi}{6\sqrt{3}} = \dfrac{1}{2} + \dfrac{\pi\sqrt{3}}{18} = \dfrac{9 + \pi\sqrt{3}}{18}$.

Check. This is about 0.80, which is sensible: the whole semicircle has area $\frac{\pi}{2} \approx 1.57$ and the triangle's area is about 1.73, so the disc covers most of the triangle. Only the two corner slivers of the semicircle beyond $\overline{BF}$ and $\overline{CE}$ are lost, and they are small.

28. Answer: 1098999999999. **The idea.** A digit sum of 99 already makes the number a multiple of 9, so the real condition is divisibility by 11: the digits in the odd places (counting from the units digit) and the digits in the even places must have sums A and B with $A - B$ a multiple of 11. Since $A + B = 99$ is odd, $A - B$ is odd, so $\{A, B\}$ is $\{55, 44\}$ or something even more lopsided. First find the fewest digits that can carry this, then make the number as small as possible from the left.

Work it out. Each place holds at most 9, so a sum of 55 needs at least 7 places of one kind and 44 needs at least 5 of the other; the lopsided options (66 and 33, and so on) need even more. With 12 or fewer digits, no kind of place has 7 places, so we need 13 digits: 7 odd places (including the leading digit) and 6 even places. The other assignment, $A = 44$ and $B = 55$, fails, since 6 even places hold at most 54. So the odd places sum to 55 and the even places to 44. The leading digit is an odd place; the other 6 odd places hold at most 54, so the leading digit is at least 1, and 1 works if they are all 9s. The second digit is an even place and can be 0, leaving 44 for the remaining 5 even places, so the fourth digit is at least $44 - 36 = 8$. The smallest number is $1\,0\,9\,8\,9\,9\,9\,9\,9\,9\,9\,9 = 1098999999999$.

Check. The digits sum to $1 + 0 + 9 + 8 + 9 \cdot 9 = 99$. Odd places from the right: $9 \cdot 6 + 1 = 55$; even places: $9 \cdot 4 + 8 + 0 = 44$; the difference 11 gives divisibility by 11, and the digit sum gives divisibility by 9. Indeed $1098999999999 = 99 \cdot 11101010101$.

29. Answer: 90. **The idea.** Fix the top row, which can be chosen in $\binom{4}{2} = 6$ ways; by symmetry each choice leads to the same number of completions. Then look at the two columns that are still missing two black squares: in the remaining three rows, each of them is white in exactly one row.

Work it out. Say the top row has black squares in columns 1 and 2. In rows 2 to 4, columns 1 and 2 need one more black square each and columns 3 and 4 need two each, so column 3 is white

in exactly one of these rows and so is column 4. If both are white in the same row (3 choices), that row is black in columns 1, 2 and the other two rows are black in columns 3, 4: 3 colourings. If column 3 is white in row r and column 4 in a different row s ($3 \cdot 2 = 6$ choices), the third row is black in columns 3, 4; row r is black in column 4 and one of columns 1, 2, and row s in column 3 and the other one: 2 ways, so 12 colourings. That gives $3 + 12 = 15$ completions, and $6 \cdot 15 = 90$ colourings in all.

Check. The case count shows that the four rows always form two complementary pairs (a row together with the row of opposite colours). Counting that way: using one complementary pair twice gives $3 \cdot \frac{4!}{2!\,2!} = 18$ grids, and using two different pairs out of the three, once each, gives $3 \cdot 4! = 72$ grids. Again $18 + 72 = 90$.

30. Answer: 25200**. The idea.** Counting directly means juggling four step counts at once. Instead, look at the walk through the two diagonal coordinates $u = x + y$ and $v = x - y$. Every step changes each of u and v by exactly ± 1, and the four steps give the four sign combinations. So a 10-step walk is the same as two independent 10-step walks on a number line.

Work it out. A step right changes (u, v) by $(+1, +1)$, left by $(-1, -1)$, up by $(+1, -1)$ and down by $(-1, +1)$. These are all four sign pairs, each exactly once, so choosing a sequence of 10 steps is the same as choosing a sequence of 10 signs for u and, independently, a sequence of 10 signs for v.

Ending at $(3, 1)$ means ending with $u = 4$ and $v = 2$. For u, the signs must include 7 pluses and 3 minuses: $\binom{10}{3} = 120$ ways. For v, 6 pluses and 4 minuses: $\binom{10}{4} = 210$ ways. In total there are $120 \cdot 210 = 25200$ sequences.

Check. Count directly with R, L, U, D steps: we need $R - L = 3$, $U - D = 1$ and $R + L + U + D = 10$, so $L + D = 3$. The cases $(L, D) = (0, 3), (1, 2), (2, 1), (3, 0)$ give $\frac{10!}{3!\,0!\,4!\,3!} + \frac{10!}{4!\,1!\,3!\,2!} + \frac{10!}{5!\,2!\,2!\,1!} + \frac{10!}{6!\,3!\,1!\,0!} = 4200 + 12600 + 7560 + 840 = 25200$.