MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. In a certain sequence of numbers, each term from the third term on is the sum of the two terms just before it. The fifth term of the sequence is 19 and the eighth term is 71. What is the first term?

1. _____

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2. For some positive integers n, the decimal expansion of $\frac{1}{n}$ terminates with exactly five digits after the decimal point, the fifth being nonzero (for example, $\frac{1}{32} = 0.03125$). How many positive integers n have this property?

2. _____

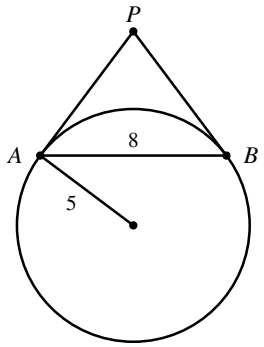
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3. Four fair coins are flipped. Every coin that lands tails is then flipped one more time, and the result of that second flip is final (coins that landed heads the first time are left alone). What is the probability that at least three of the four coins end up showing heads? Express your answer as a common fraction.

3. _____

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4. Points A and B lie on a circle of radius 5, and chord $\overline{AB}$ has length 8. The line tangent to the circle at A and the line tangent to the circle at B meet at a point P. What is the length PA? Express your answer as a common fraction.



4. _____

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5. One lighthouse flashes every 24 seconds. A second lighthouse flashes every n seconds, where n is a positive integer. Both flash together at time 0, and the next time they flash together is exactly 360 seconds later. What is the sum of all possible values of n?

5. _____

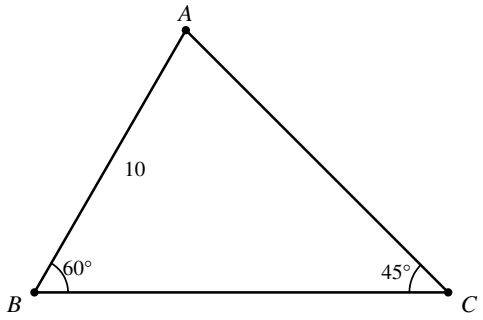
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6. Twelve cards are numbered 1 through 12. How many ways are there to choose a set of 4 of the cards (order does not matter) so that no two of the chosen cards have numbers that add up to 13?

6. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

7. Triangle ABC has $AB = 10$, $\angle ABC = 60°$ and $\angle ACB = 45°$. What is the area of triangle ABC? Express your answer in simplest radical form.



7. _____

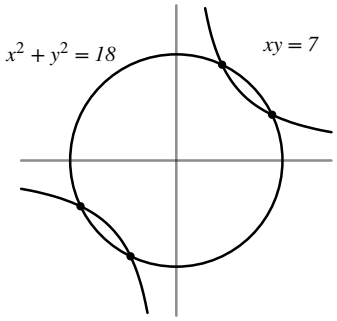
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8. Real numbers a and b satisfy $a - b = 3$ and $a^3 - b^3 = 117$. What is the value of $a^4 + b^4$? (You may find a and b, but the question can be answered without doing so.)

8. _____

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9. In the coordinate plane, the circle $x^2 + y^2 = 18$ and the curve $xy = 7$ meet at exactly four points. What is the area of the quadrilateral whose vertices are these four points? Express your answer in simplest radical form.



9. _____

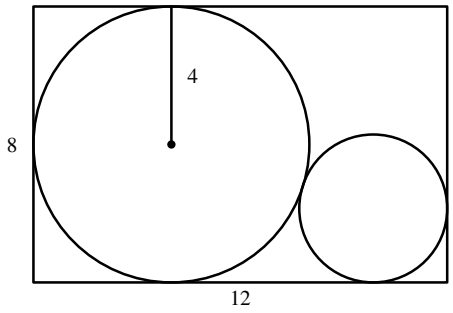
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10. Eight books, numbered 1 through 8, are stacked in a single pile. From top to bottom the pile reads 3, 7, 1, 8, 2, 4, 5, 6. In one move you may take the book currently on top of the pile and slide it back into the pile at any position you like (including the very bottom, or back on top). What is the least number of moves needed to make the pile read 1, 2, 3, 4, 5, 6, 7, 8 from top to bottom?

10. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

11. A rectangle is 12 units wide and 8 units tall. A circle of radius 4 inside the rectangle is tangent to its left side, its top side and its bottom side. A smaller circle inside the rectangle is tangent to the rectangle's bottom side and right side, and is externally tangent to the first circle. What is the radius of the smaller circle? Express your answer in simplest radical form.



11. _____

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12. Three different unit squares are chosen at random from a 4×4 board of 16 unit squares, with all sets of three squares equally likely. What is the probability that the three chosen squares form an L-shaped piece, that is, that they are three of the four squares of some 2×2 block of the board? Express your answer as a common fraction.

12. _____

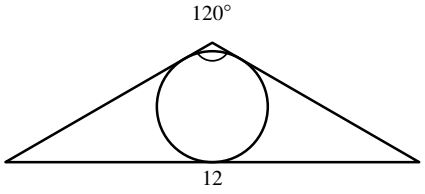
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13. For a certain constant k, the parabola $y = x^2 - 4x + k$ and the line $y = 2x - 1$ meet at two points, and the distance between those two points is $4\sqrt{5}$. What is k?

13. _____

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14. An isosceles triangle has an angle of 120° between its two equal sides, and its longest side has length 12. What is the radius of the circle inscribed in the triangle? Express your answer in simplest radical form.



14. _____

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15. For how many integers k with $1 \le k \le 100$ does the number k^k have units digit 6?

15. _____

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16. A string of six digits is formed, each digit being 1, 2 or 3 (for example 313223). Call the string *good* if every 1 in it has at least one 2 somewhere to its right (not necessarily right next to it). A string with no 1s at all is good. How many of the $3^6 = 729$ strings are good?

16. _____

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17. How many positive integers less than 1000 are divisible by their own units digit? (For example, 36 counts because 6 divides 36, while 34 does not count. Numbers whose units digit is 0 do not count.)

17. _____

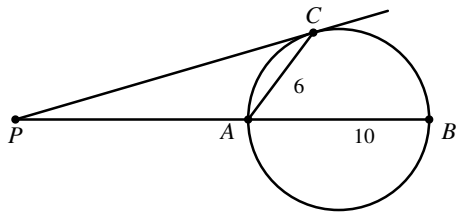
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18. How many subsets of $\{1, 2, 3, \ldots, 10\}$, including the empty set, have elements whose sum is at most 10? (The empty set has sum 0.)

18. _____

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19. Segment $\overline{AB}$ is a diameter of a circle, and $AB = 10$. Point C lies on the circle with $AC = 6$. The line tangent to the circle at C meets line AB at point P. What is the length PC? Express your answer as a common fraction.



19. _____

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20. Let $S(n)$ be the sum of the digits of the positive integer n. Find the sum of all positive integers n for which $n + S(n)^2 = 2000$.

20. _____

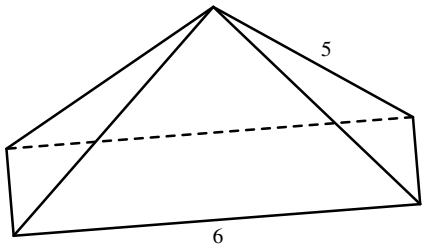
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21. Real numbers a, b and c satisfy $ab + c = bc + a = ca + b = 5$. What is the largest possible value of the product abc? Express your answer in simplest radical form.

21. _____

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22. A pyramid has a square base with side length 6, and its four slanted edges (from the apex to the base vertices) each have length 5. A sphere passes through all five vertices of the pyramid. What is the radius of the sphere? Express your answer as a common fraction in simplest radical form.



22. _____

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23. What is the largest positive integer whose digits are all different and in which every two adjacent digits, read left to right, form a two-digit prime number? (For example, 3137 does not qualify: 31, 13 and 37 are all prime, but the digit 3 is used twice.)

23. _____

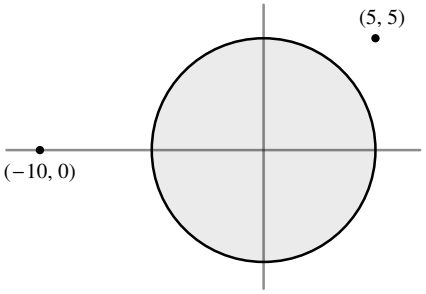
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24. Real numbers x and y satisfy $x^2 + xy + y^2 = 3$. As x and y vary, what is the largest possible value of $x^2 - xy + y^2$? (For example, $x = y = 1$ satisfies the condition and gives the value 1.)

24. _____

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25. A circular pond of radius 5 is centered at the origin of a coordinate plane. A path from the point $(-10, 0)$ to the point $(5, 5)$ may touch the edge of the pond but may not go inside it. What is the length of the shortest such path? Express your answer in simplest radical form in terms of π.



25. _____

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26. A grid has 3 rows and 4 columns of unit squares. Some of its 12 squares are shaded (possibly none, possibly several) so that no two shaded squares share a side. How many different shadings are possible?

26. _____

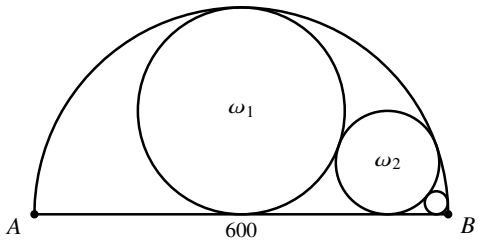
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27. A path from $(0, 0)$ to $(7, 7)$ in the coordinate plane is made of unit steps, each either one unit right (increasing x by 1) or one unit up (increasing y by 1). How many such paths have every lattice point (x, y) on them satisfying $-1 \leq x - y \leq 2$?

27. _____

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28. A semicircle has diameter $\overline{AB}$ of length 600. Circle ω_1, of radius 150, lies inside the semicircle, tangent to $\overline{AB}$ at its midpoint and tangent to the arc. Circle ω_2 lies inside the semicircle, on the B side of ω_1, tangent to $\overline{AB}$, to the arc, and to ω_1. Then ω_3 is the circle other than ω_1 that lies inside the semicircle and is tangent to $\overline{AB}$, to the arc, and to ω_2; and ω_4 is the circle other than ω_2 that lies inside the semicircle and is tangent to $\overline{AB}$, to the arc, and to ω_3. What is the radius of ω_4?



28. _____

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29. For how many positive integers $n \leq 100$ does $6n$ have exactly 12 more positive divisors than n has?

29. _____

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30. Each of the nine squares of a 3×3 grid is painted red, white or blue, each color chosen at random with probability $\frac{1}{3}$, independently of the other squares. The grid has four possible mirror lines: the horizontal line and the vertical line through its center, and its two diagonals. What is the probability that the painted grid looks exactly the same after reflection in at least one of these four lines? Express your answer as a common fraction.

30. _____

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Answer Key — MathCounts Nationals Sprint Practice Contest 5

1. 17

2. 11

3. $\frac{189}{256}$

4. $\frac{20}{3}$

5. 675

6. 240

7. $\frac{75+25\sqrt{3}}{2}$

8. 641

9. $16\sqrt{2}$

10. 4

11. $16 - 8\sqrt{3}$

12. $\frac{9}{140}$

13. 4

14. $12 - 6\sqrt{3}$

15. 30

16. 365

17. 470

18. 43

19. $\frac{120}{7}$

20. 7114

21. $3\sqrt{21} - 8$

22. $\frac{25\sqrt{7}}{14}$

23. 89731

24. 9

25. $5 + 5\sqrt{3} + \frac{5\pi}{6}$

26. 227

27. 610

28. 3

29. 16

30. $\frac{101}{729}$

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Solutions — MathCounts Nationals Sprint Practice Contest 5

1. Answer: 17. **The idea.** Two known terms that are not next to each other do not let you step backward right away. Give the sixth term a name, x; then every later term is written in terms of x, the eighth term pins x down, and after that you can walk backward using "each term is the one after it minus the one after that".

Work it out. With $a_5 = 19$ and $a_6 = x$: $a_7 = 19 + x$ and $a_8 = a_6 + a_7 = 19 + 2x$. Setting $19 + 2x = 71$ gives $x = 26$. Now go backward with $a_n = a_{n+2} - a_{n+1}$: $a_4 = a_6 - a_5 = 26 - 19 = 7$, $a_3 = a_5 - a_4 = 19 - 7 = 12$, $a_2 = a_4 - a_3 = 7 - 12 = -5$, $a_1 = a_3 - a_2 = 12 - (-5) = 17$.

Check. Run forward from $17, -5$: $17, -5, 12, 7, 19, 26, 45, 71$. The fifth term is 19 and the eighth is 71, as required. (A term can be negative; nothing in the rule forbids it.) The answer is 17.

2. Answer: 11. **The idea.** $\frac{1}{n}$ terminates with at most k digits after the point exactly when $10^k \cdot \frac{1}{n}$ is an integer, that is, when n divides 10^k. So we want the divisors of 10^5 that do not divide 10^4.

Work it out. Every divisor of $10^5 = 2^5 \cdot 5^5$ has the form $2^a 5^b$ with $0 \le a, b \le 5$: that is $6 \cdot 6 = 36$ divisors. Those that also divide $10^4 = 2^4 5^4$ have $0 \le a, b \le 4$: $5 \cdot 5 = 25$ of them. The ones left over, with $\max(a, b) = 5$, number $36 - 25 = 11$.

Check. List them: $2^5 5^b$ for $b = 0, \ldots, 5$ gives $32, 160, 800, 4000, 20000, 100000$, and $2^a 5^5$ for $a = 0, \ldots, 4$ gives $3125, 6250, 12500, 25000, 50000$. That is $6 + 5 = 11$ values; for instance $\frac{1}{160} = 0.00625$ and $\frac{1}{3125} = 0.00032$ both stop at the fifth place. The answer is 11.

3. Answer: $\frac{189}{256}$. **The idea.** Forget the order in which things happen and look at one coin at a time. A coin ends up tails only if it lands tails on both of its flips, which has probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. So each coin independently ends up heads with probability $\frac{3}{4}$, and the question becomes an ordinary count with a biased coin.

Work it out. All four coins end heads with probability $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$. Exactly three end heads: choose the one tails coin in 4 ways, giving $4 \cdot \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} = \frac{108}{256}$. The total is $\frac{81+108}{256} = \frac{189}{256}$.

Check. Give each coin two flips up front, used in order (the second one only matters if the first is tails): there are $4^4 = 256$ equally likely combinations, and a coin ends tails in exactly 1 of its 4 flip pairs. The combinations with at most one coin ending tails number $3^4 + 4 \cdot 3^3 = 81 + 108 = 189$. The answer is $\frac{189}{256}$.

4. Answer: $\frac{20}{3}$. **The idea.** A tangent is perpendicular to the radius at its point of contact, so $\triangle OAP$ (with O the center) has a right angle at A. The segment OP is the perpendicular bisector of $\overline{AB}$, so the midpoint M of $\overline{AB}$ is the foot of the altitude from A in that right triangle. Similar triangles finish it.

Work it out. $AM = 4$ and $OA = 5$, so $OM = \sqrt{25 - 16} = 3$. Right triangles OMA and OAP share the angle at O, so they are similar, with $\frac{PA}{OA} = \frac{MA}{OM}$. Hence $PA = 5 \cdot \frac{4}{3} = \frac{20}{3}$.

Check. Also $\frac{OP}{OA} = \frac{OA}{OM}$ gives $OP = \frac{25}{3}$, and then $PA = \sqrt{\frac{625}{9} - 25} = \sqrt{\frac{400}{9}} = \frac{20}{3}$. The answer is $\frac{20}{3}$.

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5. Answer: 675**. The idea.** The lighthouses flash together at the common multiples of 24 and n, so the condition says $\operatorname{lcm}(n, 24) = 360$. Compare prime factorizations one prime at a time: $24 = 2^3 \cdot 3$ and $360 = 2^3 \cdot 3^2 \cdot 5$.

Work it out. For each prime, the lcm takes the larger exponent. For 2: 24 already supplies 2^3, so n may carry 2^0, 2^1, 2^2 or 2^3 (not more). For 3: 24 has only 3^1, so n must supply 3^2 exactly. For 5: n must supply 5^1 exactly. No other prime may divide n. So $n = 45 \cdot 2^a$ with $0 \le a \le 3$: $n = 45, 90, 180, 360$, whose sum is $45(1 + 2 + 4 + 8) = 675$.

Check. $\operatorname{lcm}(45, 24) = 360$ and $\operatorname{lcm}(360, 24) = 360$; a value such as $n = 120$ fails because $\operatorname{lcm}(120, 24) = 120$. The answer is 675.

6. Answer: 240**. The idea.** The twelve numbers fall into six "partner" pairs with sum 13: $\{1, 12\}, \{2, 11\}, \{3, 10\}, \{4$ A good set may use at most one card from each pair, so it uses four different pairs and one card from each.

Work it out. Choose which 4 of the 6 pairs are used: $\binom{6}{4} = 15$ ways. From each chosen pair pick one of its two cards: $2^4 = 16$ ways. In total $15 \cdot 16 = 240$ sets.

Check. Count the complement: of the $\binom{12}{4} = 495$ sets, those containing at least one full pair number $6\binom{10}{2} - \binom{6}{2} = 270 - 15 = 255$ by inclusion–exclusion (a set of 4 can contain at most two full pairs). Then $495 - 255 = 240$. The answer is 240.

7. Answer: $\frac{75+25\sqrt{3}}{2}$**. The idea.** Drop the altitude from A to $\overline{BC}$. It splits the triangle into a 30-60-90 triangle (at B) and a 45-45-90 triangle (at C), and both are easy to solve from $AB = 10$.

Work it out. Let D be the foot of the altitude from A. In right triangle ABD the angle at B is $60°$, so $BD = 5$ and $AD = 5\sqrt{3}$. In right triangle ADC the angle at C is $45°$, so $DC = AD = 5\sqrt{3}$. Both angles at B and C are acute, so D lies between them and $BC = 5 + 5\sqrt{3}$. The area is $\frac{1}{2} \cdot (5 + 5\sqrt{3}) \cdot 5\sqrt{3} = \frac{25\sqrt{3}+75}{2}$.

Check. By the law of sines, $BC = \frac{10 \sin 75°}{\sin 45°} = 10 \cdot \frac{\sqrt{6}+\sqrt{2}}{4} \cdot \sqrt{2} = 5(\sqrt{3} + 1)$, the same length. The area is about $\frac{75+43.3}{2} \approx 59.2$.

8. Answer: 641**. The idea.** Factor the difference of cubes to get $a^2 + ab + b^2$, compare it with $(a - b)^2 = a^2 - 2ab + b^2$ to find ab, and then build up to fourth powers through $a^2 + b^2$.

Work it out. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, so $a^2 + ab + b^2 = \frac{117}{3} = 39$. Also $a^2 - 2ab + b^2 = 9$. Subtracting, $3ab = 30$, so $ab = 10$ and $a^2 + b^2 = 39 - 10 = 29$. Then $a^4 + b^4 = (a^2 + b^2)^2 - 2(ab)^2 = 841 - 200 = 641$.

Check. The numbers are $a = 5$, $b = 2$ (or $a = -2$, $b = -5$): $5 - 2 = 3$, $125 - 8 = 117$, and $625 + 16 = 641$.

9. Answer: $16\sqrt{2}$**. The idea.** Don't solve for the points. Both curves are unchanged when x and y are swapped and when both signs are flipped, so if (p, q) is one intersection point, the others are (q, p), $(-p, -q)$ and $(-q, -p)$. These form a rectangle whose side lengths depend only on $p + q$ and $p - q$, and those come straight from the two equations.

Work it out. From $p^2 + q^2 = 18$ and $pq = 7$: $(p + q)^2 = 18 + 14 = 32$ and $(p - q)^2 = 18 - 14 = 4$. The distance from (p, q) to (q, p) is $\sqrt{2(p - q)^2} = \sqrt{8} = 2\sqrt{2}$, and from (p, q) to $(-q, -p)$ it is

$\sqrt{2(p+q)^2} = \sqrt{64} = 8$. The quadrilateral is a rectangle (its two diagonals are diameters of the circle, so they are equal and bisect each other), and its area is $2\sqrt{2} \cdot 8 = 16\sqrt{2}$.

Check. Solving, $p + q = 4\sqrt{2}$ and $p - q = 2$ give $p = 2\sqrt{2} + 1$ and $q = 2\sqrt{2} - 1$: indeed $pq = 8 - 1 = 7$ and $p^2 + q^2 = 2(8 + 1) = 18$. The four vertices are (p, q), (q, p), $(-p, -q)$, $(-q, -p)$, and the shoelace formula gives $2(p^2 - q^2) = 2(p + q)(p - q) = 2 \cdot 4\sqrt{2} \cdot 2 = 16\sqrt{2}$. The answer is $16\sqrt{2}$.

10. Answer: 4**. The idea.** Each move touches only the top book. For a book that starts at depth d to reach the top, each of the $d - 1$ books originally above it must first be lifted off at least once, so moving it costs at least d moves in total. Hence after m moves the bottom $8 - m$ books of the original pile have never been touched, and they are still in their original order. Since they end up in the sorted pile, they must already be in increasing order.

Work it out. Lower bound: the untouched books form a bottom block of the original pile, and that block must already be in increasing order from top to bottom. The longest such bottom block is $2, 4, 5, 6$ (adding 8 above it breaks the order). So at most 4 books can stay untouched, and at least $8 - 4 = 4$ moves are needed. (Beware: $1, 2, 4, 5, 6$ appear in increasing order in the pile and form a longer list, but they are not a bottom block, so they cannot all stay put.) Upper bound: move 3, then 7, then 1, then 8, each time inserting the book into its correct place among the sorted books below: after the four moves the pile is $1, 2, 3, 4, 5, 6, 7, 8$. So the minimum is 4.

Check. After moving 3 the pile is $7, 1, 8, 2, 3, 4, 5, 6$; after 7: $1, 8, 2, 3, 4, 5, 6, 7$; after 1 (slid in just above 2): $8, 1, 2, 3, 4, 5, 6, 7$; after 8: $1, 2, 3, 4, 5, 6, 7, 8$. A computer search over all 8! orders confirms that no 3 moves suffice. The answer is 4.

11. Answer: $16 - 8\sqrt{3}$**. The idea.** Put the rectangle in coordinates. Each tangency to a side fixes a coordinate of a center, and external tangency of the two circles says the distance between the centers is the sum of the radii. That gives one equation in the unknown radius r.

Work it out. Place the rectangle with corners $(0, 0)$ and $(12, 8)$. The big circle touches the left, top and bottom sides, so its center is $(4, 4)$. The small circle touches the bottom and right sides, so its center is $(12 - r, r)$. The centers are $4 + r$ apart: $(8 - r)^2 + (4 - r)^2 = (4 + r)^2$. Expanding, $64 - 16r + r^2 + 16 - 8r + r^2 = 16 + 8r + r^2$, that is, $r^2 - 32r + 64 = 0$. So $r = 16 \pm \sqrt{256 - 64} = 16 \pm 8\sqrt{3}$. The plus sign gives a circle far bigger than the rectangle, so $r = 16 - 8\sqrt{3}$.

Check. $r \approx 16 - 13.856 = 2.144$. The small center is then about $(9.856, 2.144)$, and its distance to $(4, 4)$ is $\sqrt{5.856^2 + 1.856^2} \approx \sqrt{34.29 + 3.44} \approx 6.144 = 4 + r$. The small circle stays below the top side ($2r < 8$). The answer is $16 - 8\sqrt{3}$.

12. Answer: $\frac{9}{140}$**. The idea.** Count L-shapes by the 2×2 block they sit in. The key point to check is that no L-shape is counted twice: an L spans exactly two rows and two columns, so it determines its block.

Work it out. A 2×2 block is fixed by its top-left square, which can be in any of the top 3 rows and left 3 columns: 9 blocks. Each block has 4 ways to leave out one square, giving 4 L-shapes, so there are $9 \cdot 4 = 36$ L-shapes, all different. The number of ways to choose 3 squares is $\binom{16}{3} = 560$. The probability is $\frac{36}{560} = \frac{9}{140}$.

Check. Count L-shapes by their corner square instead (the one sharing a side with both others). A corner square of the board has 1 orientation available, an edge square 2, an interior square 4: $4 \cdot 1 + 8 \cdot 2 + 4 \cdot 4 = 36$. The answer is $\frac{9}{140}$.

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13. Answer: 4. **The idea.** Both points lie on a line of slope 2. Moving d units in x along such a line moves $2d$ units in y, a distance of $d\sqrt{1+4} = d\sqrt{5}$. So the two x-coordinates must differ by exactly 4, and the difference of the roots of a quadratic is controlled by its discriminant.

Work it out. Setting $x^2 - 4x + k = 2x - 1$ gives $x^2 - 6x + (k+1) = 0$. If its roots are r and s, then $r + s = 6$ and $rs = k + 1$, so $(r-s)^2 = (r+s)^2 - 4rs = 36 - 4(k+1)$. We need $(r-s)^2 = 16$: $36 - 4(k+1) = 16$, so $k + 1 = 5$ and $k = 4$.

Check. With $k = 4$ the equation is $x^2 - 6x + 5 = 0$, with roots 1 and 5. The points are $(1, 1)$ and $(5, 9)$, at distance $\sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$. The answer is 4.

14. Answer: $12-6\sqrt{3}$. **The idea.** The altitude to the longest side splits the triangle into two 30-60-90 triangles, which gives every length. Then use area = inradius × semiperimeter.

Work it out. Half of the long side is 6, opposite the 60° angle of each half, so the altitude is $\frac{6}{\sqrt{3}} = 2\sqrt{3}$ and each equal side is $4\sqrt{3}$. The area is $\frac{1}{2} \cdot 12 \cdot 2\sqrt{3} = 12\sqrt{3}$ and the semiperimeter is $\frac{12+8\sqrt{3}}{2} = 6 + 4\sqrt{3}$. So

$r = \frac{12\sqrt{3}}{6+4\sqrt{3}} = \frac{6\sqrt{3}}{3+2\sqrt{3}} = \frac{6\sqrt{3}\,(2\sqrt{3}-3)}{12-9} = 2\sqrt{3}\,(2\sqrt{3}-3) = 12 - 6\sqrt{3}$.

Check. $12 - 6\sqrt{3} \approx 1.61$, less than the altitude $2\sqrt{3} \approx 3.46$, as it must be. The incenter sits on the altitude, and its distance 1.61 to the long side equals its distance to an equal side: the apex is $3.46 - 1.61 = 1.85$ above it, and $1.85 \cdot \sin 60° \approx 1.61$.

15. Answer: 30. **The idea.** The units digit of k^k depends on the units digit of k and on k's position in the short repeating cycle of powers of that digit. Only even k can end in 6 (odd numbers have odd powers), and 0 never gives 6, so check the units digits 2, 4, 6, 8 one at a time.

Work it out. Units digit 6: every power of a number ending in 6 ends in 6, giving $k = 6, 16, \ldots, 96$: 10 values. Units digit 4: powers of 4 end in $4, 6, 4, 6, \ldots$, so an even exponent gives 6; here the exponent k is itself even, so all 10 values $4, 14, \ldots, 94$ work. Units digit 2: powers of 2 end in $2, 4, 8, 6$ in a cycle of 4, so we need $4 \mid k$; with k ending in 2 this means $k = 12, 32, 52, 72, 92$: 5 values. Units digit 8: powers of 8 end in $8, 4, 2, 6$, so again $4 \mid k$: $k = 8, 28, 48, 68, 88$: 5 values. The total is $10 + 10 + 5 + 5 = 30$.

Check. Small cases: $4^4 = 256$ ends in 6; $2^2 = 4$ does not, but 12^{12} ends like $2^{12} = 4096$, in 6; 8^8 ends like 2^{24}, in 6; 18^{18} ends like 8^{18}, and 18 leaves remainder 2 on division by 4, so it ends in 4. The answer is 30.

16. Answer: 365. **The idea.** Only one 1 really matters: the last 1 in the string. If the last 1 has a 2 to its right, then so does every earlier 1 (the same 2 works). So a string is bad exactly when it contains a 1 and everything after its last 1 is a 3. Count the bad strings instead.

Work it out. Suppose the last 1 of a bad string is in position k ($1 \le k \le 6$). The $6 - k$ digits after it must all be 3s (no 1 because this is the last 1, no 2 because the string is bad): one choice. The $k - 1$ digits before it are completely free: 3^{k-1} choices. So the number of bad strings is $1 + 3 + 9 + 27 + 81 + 243 = 364$, and the number of good strings is $729 - 364 = 365$.

Check. For strings of length 2 the same count gives $9 - (1 + 3) = 5$ good strings, and indeed they are 22, 23, 32, 33 (no 1) and 12. In general the answer is $3^n - \frac{3^n-1}{2} = \frac{3^n+1}{2}$, which is $\frac{730}{2} = 365$ for $n = 6$.

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17. Answer: 470**. The idea.** Write the number as $n = 10k + d$ with units digit $d \neq 0$ and $0 \leq k \leq 99$ (so $n < 1000$). For each digit d, whether $d \mid 10k + d$ is the same as whether $d \mid 10k$, a simple condition on k. So count the good values of k among $0, 1, \ldots, 99$ digit by digit.

Work it out. $d = 1, 2, 5$: d divides $10k$ always: 100 each, 300 in all. $d = 3$: need $3 \mid 10k$, i.e. $3 \mid k$: $k = 0, 3, \ldots, 99$, 34 values. $d = 4$: need $4 \mid 10k$, i.e. $2 \mid k$: 50. $d = 6$: need $6 \mid 10k$, i.e. $3 \mid k$: 34. $d = 7$: need $7 \mid k$: $k = 0, 7, \ldots, 98$, 15 values. $d = 8$: need $8 \mid 10k$, i.e. $4 \mid k$: 25. $d = 9$: need $9 \mid k$: $k = 0, 9, \ldots, 99$, 12 values. Total: $300 + 34 + 50 + 34 + 15 + 25 + 12 = 470$.

Check. Below 20 the rule predicts $k = 0$ gives all of 1 through 9, and $k = 1$ gives $d = 1, 2, 5$ only: 11, 12, 15. Directly, 13, 14, 16, 17, 18, 19 fail and 11, 12, 15 work: 12 numbers below 20, as predicted.

18. Answer: 43**. The idea.** A subset with small sum cannot have many elements: four different positive integers already add up to at least $1 + 2 + 3 + 4 = 10$, and five add up to at least 15. So sort the subsets by their number of elements and count each size directly.

Work it out. Size 0: the empty set, 1. Size 1: every one of $\{1\}, \ldots, \{10\}$, 10. Size 2: $\{a, b\}$ with $a < b$ and $a + b \leq 10$: for $a = 1, 2, 3, 4$ the number of choices of b is 8, 6, 4, 2 (b from $a + 1$ to $10 - a$), total 20. Size 3: $\{a, b, c\}$ with $a < b < c$ and sum at most 10. With $\{1, 2\}$: $c = 3, \ldots, 7$, 5 sets; with $\{1, 3\}$: $c = 4, 5, 6$, 3; with $\{1, 4\}$: $c = 5$, 1; with $\{2, 3\}$: $c = 4, 5$, 2; anything else already sums to more than 10 (for example $2 + 4 + 5 = 11$). Total 11. Size 4: only $\{1, 2, 3, 4\}$, 1. Altogether $1 + 10 + 20 + 11 + 1 = 43$.

Check. Count by the sum instead. A subset with sum s is a way to write s as a sum of *different* positive integers, and the numbers of such ways for $s = 0, 1, \ldots, 10$ are 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10 (for example $6 = 6 = 5 + 1 = 4 + 2 = 3 + 2 + 1$). Their total is again 43.

19. Answer: $\frac{120}{7}$**. The idea.** Angle ACB is a right angle, so $BC = 8$. The angle between the tangent and the chord $\overline{CA}$ equals the inscribed angle $\angle CBA$ on the other side, so triangles PCA and PBC are similar, and their ratio is $\frac{CA}{BC} = \frac{3}{4}$.

Work it out. Since C is nearer to A than to B, the tangent meets line AB beyond A. Triangles PCA and PBC share the angle at P, and $\angle PCA = \angle PBC$, so they are similar with $\frac{PA}{PC} = \frac{PC}{PB} = \frac{CA}{BC} = \frac{3}{4}$. Write $PC = t$; then $PA = \frac{3t}{4}$ and $PB = \frac{4t}{3}$. Since $PB - PA = AB = 10$, $t\left(\frac{4}{3} - \frac{3}{4}\right) = \frac{7t}{12} = 10$, so $t = \frac{120}{7}$.

Check. Then $PA = \frac{90}{7}$ and $PB = \frac{160}{7}$, and $PA \cdot PB = \frac{14400}{49} = PC^2$, the tangent–secant relation. In coordinates the tangent at $C = \left(-\frac{7}{5}, \frac{24}{5}\right)$ on $x^2 + y^2 = 25$ is $-\frac{7}{5}x + \frac{24}{5}y = 25$, which meets $y = 0$ at $x = -\frac{125}{7}$, and the distance from there to C is $\frac{120}{7}$.

20. Answer: 7114**. The idea.** Every number is congruent to its digit sum modulo 9. So with $s = S(n)$, the equation forces $s + s^2 \equiv 2000 \pmod 9$, which leaves only a few possible digit sums; for each one, $n = 2000 - s^2$ is determined and can be tested.

Work it out. Since $n < 2000$, n has at most four digits and $s \leq 1 + 9 + 9 + 9 = 28$. Now $2000 \equiv 2 \pmod 9$, and $s^2 + s$ for $s \equiv 0, 1, \ldots, 8 \pmod 9$ is $\equiv 0, 2, 6, 3, 2, 3, 6, 2, 0$. So $s \equiv 1, 4$ or $7 \pmod 9$, i.e. $s \in \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28\}$. Test $n = 2000 - s^2$: $s = 1$: 1999 has digit sum 28, no. $s = 4$: 1984 has 22, no. $s = 7$: 1951 has 16, no. $s = 10$: 1900 has 10, yes. $s = 13$: 1831 has 13, yes. $s = 16$: 1744 has 16, yes. $s = 19$: 1639 has 19, yes. $s = 22$: 1516 has 13, no.

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$s = 25$: 1375 has 16, no. $s = 28$: 1216 has 10, no. The solutions are $1900, 1831, 1744, 1639$, with sum 7114.

Check. $1639 + 19^2 = 1639 + 361 = 2000$ and $1744 + 256 = 2000$; the mod-9 filter is only a filter, so each surviving candidate was tested directly.

21. Answer: $3\sqrt{21} - 8$**. The idea.** Subtract the equations in pairs: each difference factors, and each factorization says that either two of the numbers are equal or one of them is 1. That splits the problem into two small cases, and the answer comes from comparing them.

Work it out. From $ab + c = bc + a$: $b(a - c) - (a - c) = (a - c)(b - 1) = 0$. In the same way $(b - a)(c - 1) = 0$ and $(c - b)(a - 1) = 0$.

Case 1: one of the numbers is 1, say $b = 1$ (the equations are symmetric under cycling $a \to b \to c \to a$, so the other choices give the same products). Then the equations become $a + c = 5$ and $ac + 1 = 5$, so a and c are 1 and 4, and $abc = 4$.

Case 2: none is 1. Then $a = c$, $b = a$ and $c = b$, so $a = b = c$ with $a^2 + a = 5$, giving $a = \frac{-1\pm\sqrt{21}}{2}$. Using $a^2 = 5 - a$: $abc = a^3 = a(5 - a) = 5a - a^2 = 5a - (5 - a) = 6a - 5$. The larger root gives $6 \cdot \frac{-1+\sqrt{21}}{2} - 5 = 3\sqrt{21} - 8$, and the other gives a negative product.

Since $\sqrt{21} > 4$, $3\sqrt{21} - 8 > 4$, so the largest value is $3\sqrt{21} - 8$.

Check. With $a = b = c = \frac{-1+\sqrt{21}}{2} \approx 1.791$: $a^2 + a \approx 3.208 + 1.791 \approx 5.000$, and $a^3 \approx 5.748 = 3\sqrt{21} - 8$. In Case 1, $(1, 1, 4)$ gives $1 + 4 = 4 + 1 = 4 + 1 = 5$. The tempting guess "all equal" happens to win here, but only a comparison with Case 1 shows it.

22. Answer: $\frac{25\sqrt{7}}{14}$**. The idea.** By symmetry the sphere's center lies on the pyramid's axis (the line through the apex perpendicular to the base). The pyramid is so low that the center turns out to lie *below* the base, so set up the equation with a signed height rather than assuming the center is inside.

Work it out. The base center is $\frac{6\sqrt{2}}{2} = 3\sqrt{2}$ from each base vertex, so the height is $h = \sqrt{25 - 18} = \sqrt{7}$. Put the center at signed height z above the base center. Equal distances to a base vertex and to the apex: $18 + z^2 = (\sqrt{7} - z)^2 = 7 - 2\sqrt{7}\,z + z^2$, so $z = -\frac{11}{2\sqrt{7}}$ (below the base). The radius is the distance to the apex: $R = \sqrt{7} + \frac{11}{2\sqrt{7}} = \frac{14+11}{2\sqrt{7}} = \frac{25}{2\sqrt{7}} = \frac{25\sqrt{7}}{14}$.

Check. Cut through the apex and two opposite base vertices: the triangle has sides $5, 5, 6\sqrt{2}$ and height $\sqrt{7}$, and the sphere's great circle is its circumcircle. For an isosceles triangle with legs ℓ and height h, $R = \frac{\ell^2}{2h} = \frac{25}{2\sqrt{7}}$, the same value.

23. Answer: 89731**. The idea.** A two-digit prime ends in 1, 3, 7 or 9. So every digit except the first must be one of those four, and with all digits different the number has at most $1 + 4 = 5$ digits. Then look for the largest chain.

Work it out. Starting with 9: the only two-digit prime $9x$ with $x \in \{1, 3, 7\}$ is 97. From 7: 71 or 73. From 973: 31 (then 1 needs 13, 17, 19, all digits used) giving 9731; from 971: 13 then 3 needs 31 or 37, used, giving 9713. So a number starting with 9 has at most 4 digits. A five-digit number beats any four-digit one, so try first digit 8: the prime must be 83 or 89, and 89 is larger. Then 97, then 73 (larger than 71), then 31: 89731, which uses $1, 3, 7, 9$, so it cannot be longer. The pairs $89, 97, 73, 31$ are all prime.

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Check. A five-digit example cannot start with 9 (shown above), so the best possible start is 8, and 89 . . . beats 83 . . .. After 89 the only prime is 97, after 97 the larger choice 73 still leads on to 31, and the only other five-digit continuation, 89713, is smaller than 89731. A computer search over all chains confirms 89731.

24. Answer: 9**. The idea.** Both expressions use only $s = x^2 + y^2$ and $p = xy$. The condition gives $s + p = 3$ and the target is $s - p = 3 - 2p$, so we want p as small as possible. What stops p from being very negative is the fact that $(x + y)^2 \geq 0$, that is, $s \geq -2p$.

Work it out. From $s = 3 - p$ and $s + 2p = (x + y)^2 \geq 0$ we get $3 + p \geq 0$, so $p \geq -3$. Therefore $x^2 - xy + y^2 = 3 - 2p \leq 3 + 6 = 9$. Equality needs $x + y = 0$ and $xy = -3$: take $x = \sqrt{3}$, $y = -\sqrt{3}$.

Check. With $x = \sqrt{3}$, $y = -\sqrt{3}$: $x^2 + xy + y^2 = 3 - 3 + 3 = 3$ and $x^2 - xy + y^2 = 3 + 3 + 3 = 9$. The smallest value, by the other bound $(x - y)^2 \geq 0$, is 1, which the example in the statement reaches.

25. Answer: $5 + 5\sqrt{3} + \frac{5\pi}{6}$**. The idea.** The straight segment passes through the pond, so a shortest path runs along a tangent line to the pond, follows the shore along an arc, and leaves along another tangent line. Going around on the two different sides gives two candidates; take the shorter.

Work it out. Let O be the origin, $A = (-10, 0)$, $B = (5, 5)$. The segment AB is too close to O: line AB is $x - 3y + 10 = 0$, at distance $\frac{10}{\sqrt{10}} = \sqrt{10} < 5$ from O, and the nearest point $(-1, 3)$ lies between A and B. From A ($OA = 10$) a tangent has length $\sqrt{100 - 25} = 5\sqrt{3}$, and the radius to its point of tangency makes an angle of 60° with OA (since $\cos = \frac{5}{10}$). From B ($OB = 5\sqrt{2}$) a tangent has length $\sqrt{50 - 25} = 5$ and the angle is 45°. The angle $\angle AOB$ is 135° (from direction 180° to direction 45°). Going around the upper side, the arc between the two tangency points is $135° - 60° - 45° = 30°$, of length $5 \cdot \frac{\pi}{6} = \frac{5\pi}{6}$. The lower side would need an arc of $225° - 105° = 120°$, much longer. So the shortest path has length $5\sqrt{3} + 5 + \frac{5\pi}{6}$.

Check. The tangency points on the upper route are $(-\frac{5}{2}, \frac{5\sqrt{3}}{2})$ (at 120°) and $(0, 5)$ (at 90°); indeed $(0, 5)$ to $(5, 5)$ is a horizontal tangent of length 5. Numerically the answer is about 8.66+5+2.62 = 16.28, a bit more than $|AB| = \sqrt{250} \approx 15.81$, as expected.

26. Answer: 227**. The idea.** Build the grid one column at a time. A single column of 3 squares can be shaded in only 5 allowed patterns, and a column's pattern only has to be compatible with its neighbours. Keep a running count for each pattern of the latest column.

Work it out. Write a column top to bottom: the allowed patterns are 000, 100, 010, 001, 101. Two adjacent columns are compatible when no row is shaded in both. So 000 fits next to all five; 100 and 001 each fit next to 000, 010 and the other of the two (001 or 100); 010 fits next to 000, 100, 001, 101; 101 fits next to 000, 010. For one column, each pattern has count 1. For two columns, the counts ending in 000, 100, 010, 001, 101 are 5, 3, 4, 3, 2 (total 17). For three: 000: 17; 100: 5 + 4 + 3 = 12; 010: 5 + 3 + 3 + 2 = 13; 001: 12; 101: 5 + 4 = 9 (total 63). For four: 000: 63; 100: 17 + 13 + 12 = 42; 010: 17 + 12 + 12 + 9 = 50; 001: 42; 101: 17 + 13 = 30. Total: 63 + 42 + 50 + 42 + 30 = 227.

Check. The total for k columns equals the count of 000-ending shadings with $k + 1$ columns (adding an empty column changes nothing): the 3-column total 63 reappears as the 000 count for 4 columns. A 3×1 grid gives 5 and a 3×2 grid gives 17, both easy to list by hand.

27. Answer: 610**. The idea.** Watch the value $d = x - y$. Each step changes it by ± 1, so after an even

number of steps d is even, and it must be 0 or 2 (the only even numbers from -1 to 2). Group the steps in pairs and count how a pair of steps moves between these two states.

Work it out. From $d = 0$: two steps return to 0 by going to 1 and back or to -1 and back (2 ways), or reach 2 (1 way, right-right). From $d = 2$: two steps can go to 1 and back to 2 (1 way; 3 is not allowed), or down to 0 (1 way). If p and q count the ways to be at $d = 0$ and $d = 2$ after some number of pairs, one more pair gives $p' = 2p+q$ and $q' = p+q$. Starting from $(p, q) = (1, 0)$, the successive values are $(2, 1), (5, 3), (13, 8), (34, 21), (89, 55), (233, 144), (610, 377)$. The point $(7, 7)$ is reached after 14 steps, i.e. 7 pairs, with $d = 0$: there are 610 paths.

Check. The numbers are every other Fibonacci number, and they obey $p_{k+1} = 3p_k - p_{k-1}$: $3 \cdot 233 - 89 = 610$. For $(1, 1)$ the rule gives 2 (both paths stay within the corridor) and for $(2, 2)$ it gives 5: of the 6 paths only up-up-right-right, which reaches $x - y = -2$, is excluded.

28. Answer: 3**. The idea.** Describe each circle by the distance a from the midpoint O of AB to its point of contact with AB. Tangency to the arc ties the radius to a; tangency to the previous circle then gives a quadratic for the next a whose *other* root is the circle before it. Vieta's formulas turn each step into a subtraction.

Work it out. Let $R = 300$. A circle of radius r tangent to AB at distance a from O has center (a, r); internal tangency to the arc says the center is $R - r$ from O: $a^2 + r^2 = (R - r)^2$, so $a^2 = R^2 - 2Rr$ and $r = \frac{R^2-a^2}{2R} = \frac{90000-a^2}{600}$. Two such circles are externally tangent when the horizontal distance between their centers satisfies $(a - a')^2 + (r - r')^2 = (r + r')^2$, i.e. $(a - a')^2 = 4rr'$.

ω_1: $a_1 = 0$, $r_1 = 150$. ω_2: $a^2 = 4 \cdot 150 \cdot r = 600r$ and $a^2 = 90000 - 600r$, so $r_2 = 75$, $a_2 = 150\sqrt{2}$.

Given the circle (a_n, r_n), the next contact point satisfies $(a - a_n)^2 = 4r_n \cdot \frac{90000-a^2}{600}$. With $k = \frac{r_n}{150}$ this is $(1 + k)a^2 - 2a_n a + a_n^2 - 90000k = 0$, whose two roots are the previous circle's a_{n-1} and the new a_{n+1}, so $a_{n+1} = \frac{2a_n}{1+k} - a_{n-1}$.

ω_3: $k = \frac{1}{2}$, $a_3 = \frac{2 \cdot 150\sqrt{2}}{3/2} - 0 = 200\sqrt{2}$, so $r_3 = \frac{90000-80000}{600} = \frac{50}{3}$.

ω_4: $k = \frac{1}{9}$, $a_4 = \frac{2 \cdot 200\sqrt{2}}{10/9} - 150\sqrt{2} = 360\sqrt{2} - 150\sqrt{2} = 210\sqrt{2}$, so $r_4 = \frac{90000-88200}{600} = 3$.

Check. Directly: $(a_4 - a_3)^2 = (10\sqrt{2})^2 = 200$ and $4r_3r_4 = 4 \cdot \frac{50}{3} \cdot 3 = 200$. The radii $150, 75, \frac{50}{3}, 3$ are $\frac{R}{2}, \frac{R}{4}, \frac{R}{18}, \frac{R}{100}$, shrinking fast as the circles crowd toward B.

29. Answer: 16**. The idea.** Split off the primes 2 and 3: write $n = 2^a 3^b m$ with m not divisible by 2 or 3, and let D be the number of divisors of m. Multiplying by 6 only bumps the exponents of 2 and 3, so the gain in divisors has a clean formula, and the condition becomes a small factorisation problem.

Work it out. n has $(a + 1)(b + 1)D$ divisors and $6n$ has $(a + 2)(b + 2)D$. The difference is $\big[(a+2)(b+2) - (a+1)(b+1)\big]D = (a+b+3)D$. We need $(a+b+3)D = 12$ with $a+b+3 \geq 3$:

$a + b + 3 = 3$, $D = 4$: $a = b = 0$ and $m = n$ has 4 divisors, with no factor 2 or 3: $m = pq$ or p^3 with primes $p, q \geq 5$. Up to 100: $35, 55, 65, 85, 95, 77, 91$ ($5^3 = 125$ is too big): 7 numbers.

$a + b + 3 = 4$, $D = 3$: $n = 2m$ or $3m$ with $m = p^2$, $p \geq 5$: $50, 98, 75$: 3 numbers.

$a + b + 3 = 6$, $D = 2$: $a + b = 3$ and $m = p \geq 5$ prime: $n = 8p$ ($p = 5, 7, 11$), $12p$ ($p = 5, 7$), $18p$ ($p = 5$), $27p$ (none): 6 numbers.

$a + b + 3 = 12$, $D = 1$: $a + b = 9$, but then $n \geq 2^9 > 100$: none.

Total: $7 + 3 + 6 = 16$.

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Check. $n = 35$: 4 divisors; $6n = 210$ has 16. $n = 40 = 8 \cdot 5$: 8 divisors; 240 has 20. $n = 50$: 6 divisors; 300 has 18.

30. Answer: $\frac{101}{729}$**. The idea.** Use inclusion–exclusion over the four mirrors. A coloring symmetric in some set of mirrors is also symmetric under everything those mirrors generate, so the squares fall into groups (*orbits*) that must be one color, and the count is $3^{\text{number of orbits}}$. The key facts: two perpendicular mirrors together force 180° rotational symmetry, while two mirrors at 45° force all eight symmetries of the square.

Work it out. One mirror. The horizontal midline fixes the middle row and swaps the top and bottom rows: $3 + 3 = 6$ orbits, $3^6 = 729$ colorings. A diagonal fixes its 3 squares and swaps the other 6 in 3 pairs: again 6 orbits, 729. Four mirrors: $4 \cdot 729 = 2916$.

Two mirrors. Both midlines: the orbits are the center, the four corners, the top and bottom middle squares, and the left and right middle squares: 4 orbits, 81. Both diagonals: the center, {top-left, bottom-right}, {top-right, bottom-left} and the four edge-middle squares: 4 orbits, 81. A midline with a diagonal (4 such pairs): reflections in lines 45° apart generate a 90° rotation, so the coloring has every symmetry of the square, leaving 3 orbits (center, corners, edge-middles): 27 each. Pairs: $81 + 81 + 4 \cdot 27 = 270$.

Three or four mirrors always include a midline and a diagonal, so each of the 4 triples and the one quadruple gives 27.

By inclusion–exclusion the number of colorings with at least one mirror line is $2916 - 270 + 4 \cdot 27 - 27 = 2727$. Out of $3^9 = 19683$ colorings the probability is $\frac{2727}{19683} = \frac{101}{729}$.

Check. Count by exact symmetry type instead. Fully symmetric: 27. Symmetric in both midlines but not fully: $81 - 27 = 54$; the same for both diagonals: 54. Symmetric in exactly one mirror: each mirror has 729 colorings, minus those with more symmetry; a midline's colorings include the $54+27$ with both midlines, so $729-81 = 648$ for each midline (having one midline and a diagonal already means full symmetry), and likewise 648 for each diagonal. Total $27 + 54 + 54 + 4 \cdot 648 = 2727$.

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