MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A rectangle's length is increased by 20% and its width is decreased by 20%. The new rectangle's perimeter is 4 units more than the original perimeter, and its area is 8 square units less than the original area. What was the perimeter of the original rectangle?

1. _____

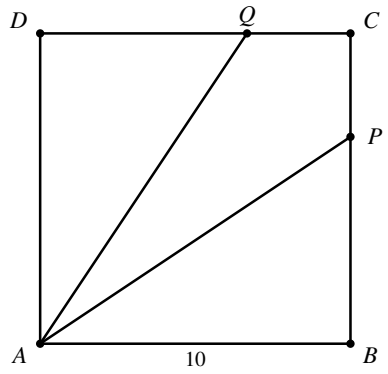
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2. Every integer from 1 to 2026, inclusive, is written out in the usual way, with no leading zeros. In total, how many times is the digit 0 written?

2. _____

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3. Square $ABCD$ has side length 10. Point P lies on side $\overline{BC}$ and point Q lies on side $\overline{CD}$, and segments $\overline{AP}$ and $\overline{AQ}$ cut the square into three regions of equal area. What is the length of $\overline{AP}$? Express your answer in simplest radical form.



3. _____

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4. Lena writes the whole numbers from 1 to 90 on the board. She then circles every number that has no common factor greater than 1 with 90. What is the sum of the circled numbers?

4. _____

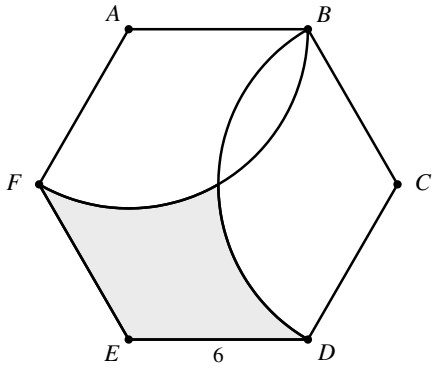
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5. Positive real numbers x, y and z satisfy $x(y+z) = 16$, $y(z+x) = 21$ and $z(x+y) = 25$. What is $x+y+z$?

5. _____

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6. Regular hexagon $ABCDEF$ has side length 6, with its vertices labeled in order around it. A circle of radius 6 is drawn centered at A, and another circle of radius 6 is drawn centered at C. What is the area of the part of the hexagon's interior that lies outside both circles? Express your answer in simplest radical form in terms of π.



6. _____

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7. Two different vertices of a regular 12-gon are chosen at random, with every pair of vertices equally likely. What is the probability that the distance between the two chosen vertices is greater than the radius of the circle passing through all twelve vertices? Express your answer as a common fraction.

7. _____

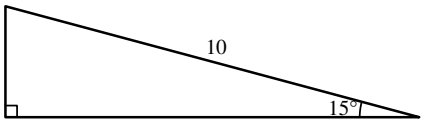
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8. A post office sells only 17-cent stamps and 29-cent stamps. Maya buys at least one stamp of each kind, and her stamps cost exactly \$10.00 in total. What is the fewest number of stamps she could have bought?

8. _____

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9. A right triangle has a hypotenuse of length 10, and one of its two acute angles measures 15°. What is the area of the triangle? Express your answer as a common fraction.



9. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

10. How many ordered pairs (x, y) of positive integers satisfy $x+2y = 100$ and have no common factor greater than 1? (For example, $(98, 1)$ counts, but $(96, 2)$ does not, since 2 divides both.)

10. _____

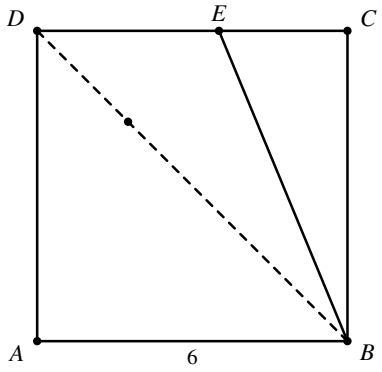
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11. Five friends each buy one gift. Each friend, independently, chooses one of the other four friends uniformly at random and gives that friend the gift. What is the probability that there is at least one pair of friends who give their gifts to each other? Express your answer as a common fraction.

11. _____

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12. Square $ABCD$ has side length 6, with vertices labeled in order around the square. Point E lies on side $\overline{CD}$. The square is folded along the line BE, and vertex C lands exactly on a point of diagonal $\overline{BD}$. What is the length CE? Express your answer in simplest radical form.



12. _____

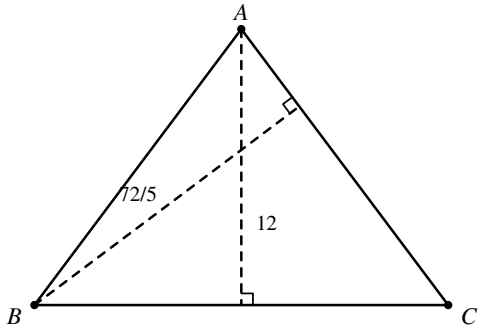
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13. At some moment between 4:00 and 5:00, the hour hand and the minute hand of an ordinary clock are mirror images of each other across the line through the 12 and the 6. How many minutes after 4:00 is this? Express your answer as a common fraction.

13. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

14. Triangle ABC is isosceles with $AB = AC$. The altitude from A to side $\overline{BC}$ has length 12, and the altitude from B to side $\overline{AC}$ has length $\frac{72}{5}$. What is the area of triangle ABC?



14. _____

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15. Six different balls are placed into three different boxes. Every ball goes into some box, and every box must end up with an even number of balls (a box may be empty, since 0 is even). In how many ways can this be done?

15. _____

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16. Two numbers a and b are each chosen from $1, 2, 3, \ldots, 30$. The pair is ordered, and a and b may be equal, so there are 900 possible pairs (a, b). For how many of these 900 pairs is the product ab divisible by 30?

16. _____

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17. For some positive integers n, the number $2^n + 65$ is a perfect square; for example, $2^4 + 65 = 81 = 9^2$. What is the sum of all positive integers n for which $2^n + 65$ is a perfect square?

17. _____

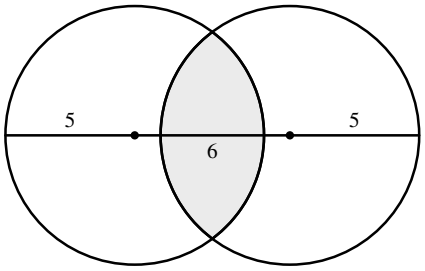
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18. For how many integers n with $1 < n < 200$ is the product of all the positive divisors of n (including 1 and n itself) equal to n^3?

18. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

19. Two circles of radius 5 have centers 6 units apart. The region inside both circles is a lens. A square is drawn inside the lens with two of its sides parallel to the line through the two centers, and it is as large as possible. What is the side length of the square? Express your answer in simplest radical form.



19. _____

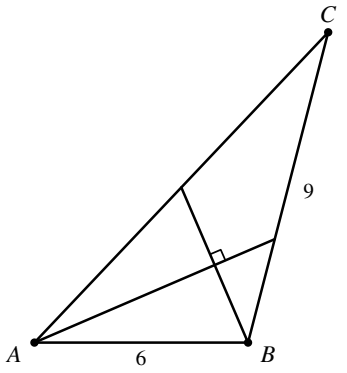
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20. Three cards are dealt one after another, without replacement, from a standard deck of 52 cards (13 ranks, 4 cards of each rank). What is the probability that the second card has a higher rank than the first and the third card has a higher rank than the second? Express your answer as a common fraction.

20. _____

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21. Triangle ABC has $AB = 6$ and $BC = 9$. The bisector of $\angle BAC$ is perpendicular to the median from B (the segment from B to the midpoint of side $\overline{AC}$). What is the area of triangle ABC? Express your answer in simplest radical form.



21. _____

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22. Three real numbers a, b and c, in that order, form a geometric sequence, so $\frac{b}{a} = \frac{c}{b}$. Their sum is $a + b + c = 26$ and the sum of their squares is $a^2 + b^2 + c^2 = 364$. What is the value of b?

22. _____

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23. Ten stepping stones are numbered 1 through 10 in a row. A frog starts on stone 1 and hops until it lands on stone 10. Each hop takes it one stone forward, two stones forward, or one stone back, and it may never land on a stone it has already visited. How many different routes can the frog take?

23. _____

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24. Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x. What is the sum of all real numbers x that satisfy $x^2 + 2\lfloor x \rfloor = 15$? Express your answer in simplest radical form.

24. _____

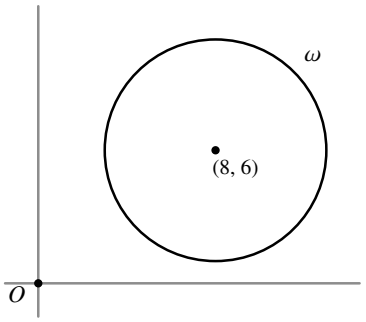
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25. Some of the 12 edges of a cube are chosen so that every one of the 8 vertices is an endpoint of exactly two chosen edges. How many different sets of chosen edges are possible?

25. _____

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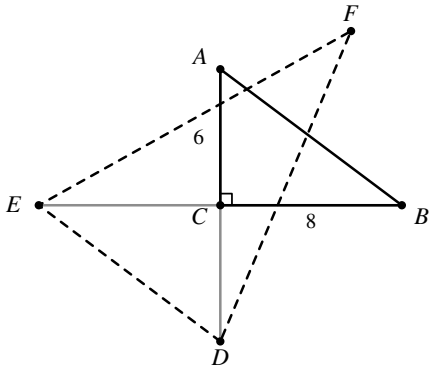
26. Let O be the origin and let ω be the circle $(x-8)^2 + (y-6)^2 = 25$. A line through O meets ω at two points P and Q, with P between O and Q, such that P is the midpoint of $\overline{OQ}$. There are exactly two such lines. What is the sum of their slopes? Express your answer as a common fraction.



26. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

27. Right triangle ABC has its right angle at C, with $CA = 6$ and $CB = 8$. Each vertex is reflected across the line containing the opposite side: A across line BC to a point D, B across line CA to a point E, and C across line AB to a point F. What is the area of triangle DEF?



27. _____

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28. A hexagon has all six interior angles equal to $120°$, and its six sides have lengths $1, 2, 3, 4, 5, 6$ in some order. What is the largest possible area of such a hexagon? Express your answer in simplest radical form.

28. _____

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29. How many subsets of $\{1, 2, 3, \ldots, 15\}$, including the empty set, contain no two numbers whose sum is a perfect square?

29. _____

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30. A sequence starts with $a_1 = 1$, and each later term is the previous term plus the greatest integer less than or equal to the square root of the previous term: $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$. So the sequence begins $1, 2, 3, 4, 6, 8, \ldots$. How many of the first 2026 terms $a_1, a_2, \ldots, a_{2026}$ are perfect squares?

30. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — MathCounts Nationals Sprint Practice Contest 4

1. 60

2. 529

3. $\frac{10\sqrt{13}}{3}$

4. 1080

5. 10

6. $36\sqrt{3} - 12\pi$

7. $\frac{7}{11}$

8. 44

9. $\frac{25}{2}$

10. 20

11. $\frac{145}{256}$

12. $6\sqrt{2} - 6$

13. $\frac{480}{13}$

14. 108

15. 183

16. 135

17. 14

18. 27

19. $-3 + \sqrt{41}$

20. $\frac{176}{1275}$

21. $\frac{27\sqrt{15}}{4}$

22. 6

23. 149

24. $-2 - 3\sqrt{3}$

25. 9

26. $\frac{256}{129}$

27. 72

28. $\frac{67\sqrt{3}}{4}$

29. 1487

30. 10

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Nationals Sprint Practice Contest 4

1. **Answer:** 60**. The idea.** Whatever the rectangle, the new area is $1.2 \times 0.8 = 0.96$ times the old one, so the area condition alone pins down LW. The perimeter condition pins down $L - W$.

   Work it out. Let the length be L and the width W. The area drops by $0.04LW = 8$, so $LW = 200$. The new perimeter is $2(1.2L+0.8W) = 2(L+W)+0.4(L-W)$, so $0.4(L-W) = 4$ and $L-W = 10$. Two numbers with product 200 and difference 10 are 20 and 10 (from $W(W+10) = 200$, $W = 10$). The original perimeter is $2(20 + 10) = 60$.

   Check. The new rectangle is 24×8: perimeter $64 = 60 + 4$ and area $192 = 200 - 8$.

2. **Answer:** 529**. The idea.** Count place by place, and remember that a 0 is never written in the leading position, so each range of lengths must be handled on its own.

   Work it out. One-digit numbers: no zeros. Two-digit numbers: a units 0 in $10, 20, \ldots, 90$: 9 zeros. Three-digit numbers abc: $c = 0$ in $9 \cdot 10 = 90$ numbers and $b = 0$ in another 90: 180 zeros. Four-digit numbers $1bcd$ from 1000 to 1999: each of the three places b, c, d is 0 in exactly 100 of them: 300 zeros. From 2000 to 2026: 2000 has 3; 2001 to 2009 have 2 each (18); 2010 and 2020 have 2 each (4); the other 15 numbers 2011–2019 and 2021–2026 have 1 each (15). That is $3 + 18 + 4 + 15 = 40$. Total: $9 + 180 + 300 + 40 = 529$.

   Check. From 1 to 999 the count is 189; a quick sanity check is that padding to three digits would give 300 zeros in 000–999, and the leading zeros removed are 3 (in 000) $+2 \cdot 9$ (in 001–009) +90 (in 010–099) = 111, and $300 - 111 = 189$. Then $189 + 300 + 40 = 529$.

3. **Answer:** $\frac{10\sqrt{13}}{3}$**. The idea.** The region next to side AB is the right triangle ABP, and its area must be one third of the square. That fixes BP with no need to look at the other two pieces.

   Work it out. The square has area 100, so triangle ABP has area $\frac{100}{3}$. Its legs are $AB = 10$ and BP, so $\frac{1}{2} \cdot 10 \cdot BP = \frac{100}{3}$ and $BP = \frac{20}{3}$. By the Pythagorean theorem, $AP = \sqrt{100 + \frac{400}{9}} = \sqrt{\frac{1300}{9}} = \frac{10\sqrt{13}}{3}$.

   Check. By symmetry $DQ = \frac{20}{3}$ too, so triangle ADQ also has area $\frac{100}{3}$, and the middle quadrilateral $APCQ$ gets $100 - \frac{200}{3} = \frac{100}{3}$, as required. Also $AP \approx 12.0$, longer than the side and shorter than the diagonal 14.1.

4. **Answer:** 1080**. The idea.** If k shares no factor with 90, neither does $90 - k$, since any common factor of $90 - k$ and 90 would also divide k. So the circled numbers pair up as k and $90 - k$, each pair summing to 90.

   Work it out. Since $90 = 2 \cdot 3^2 \cdot 5$, the circled numbers are those not divisible by 2, 3 or 5. Their count is $90 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 24$. No circled number is its own partner (45 is divisible by 3 and 5), so they form 12 pairs each summing to 90: the total is $12 \cdot 90 = 1080$.

   Check. The circled numbers below 30 are $1, 7, 11, 13, 17, 19, 23, 29$ (sum 120); adding 30 or 60 gives the rest, so the total is $3 \cdot 120 + 8 \cdot 30 + 8 \cdot 60 = 360 + 240 + 480 = 1080$.

5. **Answer:** 10**. The idea.** Expanded, the three equations are sums of two of the products xy, yz, zx. Adding all three gives twice the sum of the three products, and then each product falls out by subtraction.

Work it out. The equations read $xy + xz = 16$, $xy + yz = 21$ and $xz + yz = 25$. Adding them, $2(xy + yz + zx) = 62$, so $xy + yz + zx = 31$. Subtracting each equation from 31 gives $yz = 15$, $xz = 10$ and $xy = 6$. Multiplying these, $(xyz)^2 = 6 \cdot 10 \cdot 15 = 900$, so $xyz = 30$ (the numbers are positive). Then $x = \frac{xyz}{yz} = 2$, $y = \frac{xyz}{xz} = 3$ and $z = \frac{xyz}{xy} = 5$, and $x + y + z = 10$.

Check. $2(3 + 5) = 16$, $3(5 + 2) = 21$ and $5(2 + 3) = 25$.

6. Answer: $36\sqrt{3} - 12\pi$. **The idea.** Let O be the center of the hexagon. Since $AB = AO = AF = 6$, the circle about A passes through B, O and F, and the part of the hexagon inside it is exactly the 120° sector FAB. Likewise the circle about C cuts out the 120° sector BCD. The two sectors overlap in a lens between B and O, so use inclusion–exclusion.

Work it out. The hexagon is six equilateral triangles of side 6: area $6 \cdot \frac{\sqrt{3}}{4} \cdot 36 = 54\sqrt{3}$. Each sector has area $\frac{1}{3} \cdot 36\pi = 12\pi$ (the arc from B through O to F bulges past triangles ABO and AOF by only $6 - 3\sqrt{3} < 1$, so it stays inside the hexagon). Both circles pass through B and O, and $\angle BAO = \angle BCO = 60^\circ$ because triangles ABO and CBO are equilateral. So the overlap is two circular segments on chord $\overline{BO}$, each a 60° sector minus an equilateral triangle: $6\pi - 9\sqrt{3}$. The lens has area $12\pi - 18\sqrt{3}$, and the covered part of the hexagon is $12\pi + 12\pi - (12\pi - 18\sqrt{3}) = 12\pi + 18\sqrt{3}$. The area outside both circles is $54\sqrt{3} - 18\sqrt{3} - 12\pi = 36\sqrt{3} - 12\pi$.

Check. Numerically $62.35 - 37.70 = 24.65$, about a quarter of the hexagon's 93.53. Without the overlap correction one would get $54\sqrt{3} - 24\pi \approx 18.1$, too small, because the lens (about 6.5) would be subtracted twice.

7. Answer: $\frac{7}{11}$. **The idea.** A chord of a circle equals the radius exactly when it subtends a central angle of 60° (it is then a side of an equilateral triangle with the center). Longer chords subtend larger angles, so only very close vertices fail.

Work it out. Two vertices that are k steps apart around the 12-gon ($1 \le k \le 6$) subtend a central angle of $30k^\circ$. The chord is longer than the radius exactly when $30k > 60$, that is $k \ge 3$. Pairs with $k = 1$: 12; with $k = 2$: 12 (these chords equal the radius, which is not greater). There are $\binom{12}{2} = 66$ pairs in all, so $66 - 24 = 42$ succeed and the probability is $\frac{42}{66} = \frac{7}{11}$.

Check. Fix one vertex. Of the other 11 vertices, the two neighbours and the two vertices two steps away fail, leaving 7. So the probability is $\frac{7}{11}$ directly.

8. Answer: 44. **The idea.** We need positive integers a, b with $17a + 29b = 1000$. Reduce modulo 17 to find which b work: they form a single residue class mod 17, and only a couple of them are small enough. Then compare the totals. Since a 29-cent stamp is worth more, using more of them means fewer stamps.

Work it out. Mod 17: $29 \equiv 12$ and $1000 = 17 \cdot 58 + 14 \equiv 14$, so $12b \equiv 14 \pmod{17}$. The inverse of 12 mod 17 is 10 (since $120 = 7 \cdot 17 + 1$), so $b \equiv 140 \equiv 4 \pmod{17}$. With $29b < 1000$ we need $b \le 34$, so $b = 4$ or $b = 21$. Then $b = 4$ gives $17a = 884$, $a = 52$, total 56 stamps; $b = 21$ gives $17a = 391$, $a = 23$, total 44 stamps. The fewest is 44.

Check. $23 \cdot 17 + 21 \cdot 29 = 391 + 609 = 1000$. Going from $(52, 4)$ to $(23, 21)$ swaps 29 stamps of 17 cents for 17 stamps of 29 cents (both worth 493 cents), saving 12 stamps: $56 - 12 = 44$.

9. Answer: $\frac{25}{2}$. **The idea.** Neither leg is a nice number, so avoid them. Draw the median from the right angle to the hypotenuse instead: it has length 5 (half the hypotenuse), and it creates an isosceles triangle whose outside angle is $2 \cdot 15^\circ = 30^\circ$. A 30° angle gives the height at once.

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Work it out. Let the right angle be at C, with $\angle A = 15°$, and let M be the midpoint of hypotenuse $\overline{AB}$. Then $MA = MB = MC = 5$. Triangle AMC is isosceles, so $\angle MCA = 15°$ and the exterior angle $\angle CMB = 15° + 15° = 30°$. The height from C to $\overline{AB}$ is therefore $MC \sin 30° = 5 \cdot \frac{1}{2} = \frac{5}{2}$, and the area is $\frac{1}{2} \cdot 10 \cdot \frac{5}{2} = \frac{25}{2}$.

Check. With trigonometry the legs are $10\cos 15°$ and $10 \sin 15°$, so the area is $50 \sin 15° \cos 15° = 25 \sin 30° = \frac{25}{2}$. As a size check, $\frac{25}{2}$ is half of the largest possible area 25 (the 45° right triangle with this hypotenuse).

10. Answer: 20**. The idea.** A common factor of x and y also divides $x + 2y = 100$, and a common factor of y and 100 also divides $x = 100 - 2y$. So x and y are relatively prime exactly when y and 100 are.

Work it out. Since $x = 100 - 2y \geq 1$, the value y runs over $1, 2, \ldots, 49$, and each y gives exactly one pair. We need $\gcd(y, 100) = 1$, that is, y odd and not a multiple of 5. There are 25 odd numbers from 1 to 49, and 5 of them $(5, 15, 25, 35, 45)$ are multiples of 5. That leaves $25 - 5 = 20$ pairs.

Check. $y = 3$ gives $(94, 3)$, relatively prime; $y = 15$ gives $(70, 15)$, both multiples of 5; $y = 49$ gives $(2, 49)$, relatively prime. The count 20 is also $\frac{1}{2}\varphi(100) = \frac{40}{2}$, as it should be, since exactly half of the numbers below 100 that are prime to 100 lie below 50.

11. Answer: $\frac{145}{256}$**. The idea.** Use inclusion–exclusion over the $\binom{5}{2} = 10$ pairs of friends. The key observation is that two mutual pairs can never share a friend, since each friend gives only one gift; so only disjoint pairs can happen together, and with five friends at most two pairs can.

Work it out. A particular pair is mutual when each chooses the other: probability $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. Two disjoint pairs are both mutual with probability $\frac{1}{16^2} = \frac{1}{256}$, and there are $5 \cdot 3 = 15$ ways to choose two disjoint pairs (pick the friend left out, then split the other four into two pairs). Three disjoint pairs would need six friends. So the probability is $10 \cdot \frac{1}{16} - 15 \cdot \frac{1}{256} = \frac{160-15}{256} = \frac{145}{256}$.

Check. Count outcomes: there are $4^5 = 1024$. Outcomes with a given mutual pair: the other three friends choose freely, $4^3 = 64$; with two given disjoint mutual pairs: 4. So $10 \cdot 64 - 15 \cdot 4 = 580$ outcomes, and $\frac{580}{1024} = \frac{145}{256}$, a little over half.

12. Answer: $6\sqrt{2} - 6$**. The idea.** Folding along BE reflects C to a point C' with $BC' = BC$ and $\angle C'BE = \angle EBC$. Since C' lies on BD, line BE bisects $\angle CBD = 45°$. Instead of trigonometry, look at the small triangle near D.

Work it out. $BC' = BC = 6$ and $BD = 6\sqrt{2}$, so $C'D = 6\sqrt{2} - 6$. The reflection keeps the right angle at C: $\angle BC'E = 90°$, so $EC' \perp BD$. In triangle $DC'E$ the angle at D is 45° and the angle at C' is 90°, so it is an isosceles right triangle and $EC' = C'D = 6\sqrt{2} - 6$. Finally $CE = C'E$ because folding preserves lengths. So $CE = 6\sqrt{2} - 6$.

Check. By the angle bisector theorem in triangle BCD, $\frac{CE}{ED} = \frac{BC}{BD} = \frac{1}{\sqrt{2}}$, so $CE = \frac{6}{1+\sqrt{2}} = 6(\sqrt{2} - 1)$. Numerically $CE \approx 2.49$, less than half the side, which fits a bisector of a 45° angle.

13. Answer: $\frac{480}{13}$**. The idea.** Measure each hand's position in degrees clockwise from the 12. Reflecting across the 12–6 line sends an angle θ to $360° - \theta$, so the two hands are mirror images exactly when their angles add up to 360°.

Work it out. At m minutes after 4:00, the minute hand is at $6m$ degrees (360° per hour) and the hour hand is at $120 + \frac{m}{2}$ degrees (30° per hour, starting at the 4). Setting $6m + 120 + \frac{m}{2} = 360$ gives $\frac{13m}{2} = 240$, so $m = \frac{480}{13} \approx 36.9$. (The next possibility, a sum of 720°, would need $m > 60$.)

Check. At $m = \frac{480}{13}$ the minute hand is at $\frac{2880}{13} \approx 221.5°$ and the hour hand at $120 + \frac{240}{13} \approx 138.5°$. These are 41.5° on either side of the 6 (which is at 180°), as a mirror image across the 12–6 line should be.

14. Answer: 108**. The idea.** Compute the area in two ways, once with each altitude. That links the leg to the base, and the Pythagorean theorem in the half-triangle then fixes everything.

Work it out. Let $BC = 2x$ and $AB = AC = L$. The altitude from A meets BC at its midpoint, so $L^2 = x^2 + 144$. The area is $\frac{1}{2} \cdot 2x \cdot 12 = 12x$, and also $\frac{1}{2} \cdot L \cdot \frac{72}{5} = \frac{36L}{5}$. So $L = \frac{5x}{3}$. Then $\frac{25x^2}{9} = x^2 + 144$, so $\frac{16x^2}{9} = 144$, $x = 9$, and the area is $12 \cdot 9 = 108$.

Check. The triangle has sides 15, 15, 18 (two 9–12–15 right triangles back to back). The altitude to a side of length 15 is $\frac{2 \cdot 108}{15} = \frac{72}{5}$, as given. It is shorter than $BC = 18$, as the leg of right triangle B–foot–C must be.

15. Answer: 183**. The idea.** The only even splits of 6 into three parts are $6 + 0 + 0$, $4 + 2 + 0$ and $2 + 2 + 2$, in some order. Count each shape, choosing which box gets which size and then which balls go where.

Work it out. Shape $(6, 0, 0)$: choose the full box, 3 ways. Shape $(4, 2, 0)$: assign the sizes $4, 2, 0$ to the boxes in $3! = 6$ ways, then choose the 4 balls for the big box, $\binom{6}{4} = 15$ ways: 90. Shape $(2, 2, 2)$: $\frac{6!}{2!\,2!\,2!} = 90$ ways. Total $3 + 90 + 90 = 183$.

Check. A sign trick counts it another way. Each assignment is a word of length 6 in letters x, y, z; replacing x, y by ± 1 and averaging $(x+y+z)^6$ over the four sign choices $(\pm 1, \pm 1, 1)$ keeps exactly the words with an even number of each letter (the count of z is then even automatically). This gives $\frac{3^6+1+1+1}{4} = \frac{732}{4} = 183$.

16. Answer: 135**. The idea.** $30 = 2 \cdot 3 \cdot 5$, so $30 \mid ab$ means three separate things: $2 \mid ab$, $3 \mid ab$ and $5 \mid ab$. As a runs over $1, \ldots, 30$, its remainders mod 2, mod 3 and mod 5 run over every combination exactly once (Chinese Remainder Theorem), so the three conditions behave independently and the count is a product.

Work it out. For a prime p, $p \mid ab$ fails only when neither a nor b is a multiple of p. Fraction of pairs where $2 \mid ab$: $1 - (\frac{1}{2})^2 = \frac{3}{4}$. Where $3 \mid ab$: $1 - (\frac{2}{3})^2 = \frac{5}{9}$. Where $5 \mid ab$: $1 - (\frac{4}{5})^2 = \frac{9}{25}$. Since each number's residues mod 2, 3, 5 are independent over 1–30, so are the three events for the pair, and the count is $900 \cdot \frac{3}{4} \cdot \frac{5}{9} \cdot \frac{9}{25} = 135$.

Check. Count by $\gcd(a, 30)$ instead: b must be a multiple of $30/\gcd(a, 30)$. The numbers a with $\gcd(a, 30) = 1, 2, 3, 5, 6, 10, 15, 30$ number 8, 8, 4, 2, 4, 2, 1, 1, and they allow $1, 2, 3, 5, 6, 10, 15, 30$ values of b respectively: $8 + 16 + 12 + 10 + 24 + 20 + 15 + 30 = 135$.

17. Answer: 14**. The idea.** Split by the parity of n. For odd n a remainder argument rules out a square; for even n the power of 2 is itself a square, and a difference of two squares equal to 65 has only a few factorizations.

Work it out. If n is odd, 2^n leaves remainder 2 or 3 when divided by 5 (the remainders of $2^1, 2^3, 2^5, \ldots$ alternate $2, 3$), so $2^n + 65$ leaves remainder 2 or 3. But squares leave only 0, 1 or 4.

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So n is even: $n = 2k$ and $m^2 - (2^k)^2 = 65$, that is, $(m - 2^k)(m + 2^k) = 65$. The factor pairs are $1 \cdot 65$ and $5 \cdot 13$, giving $2 \cdot 2^k = 64$ or $2 \cdot 2^k = 8$, so $2^k = 32$ or $2^k = 4$. Then $n = 10$ or $n = 4$, and the sum is 14.

Check. $2^{10} + 65 = 1089 = 33^2$ and $2^4 + 65 = 81 = 9^2$. The pair $(m - 2^k, m + 2^k)$ must multiply to 65 with a positive gap, so no other pair exists.

18. Answer: 27. **The idea.** Pair each divisor d of n with n/d: each pair multiplies to n. If n has D divisors, multiplying all of them gives $n^{D/2}$ (for a perfect square the middle divisor $\sqrt{n}$ pairs with itself, and the formula still holds). So the condition is just $D = 6$, a question about the shape of the prime factorisation.

Work it out. Since $n > 1$, $n^{D/2} = n^3$ means $D = 6$. A number $p_1^{e_1} p_2^{e_2} \cdots$ has $(e_1+1)(e_2+1)\cdots$ divisors, and $6 = 6$ or $3 \cdot 2$, so $n = p^5$ or $n = p^2 q$ with $p \neq q$ primes. $p^5 < 200$: only 32. $p^2 q < 200$: $p = 2$: $q < 50$, $q \neq 2$, the 14 odd primes $3, 5, \ldots, 47$; $p = 3$: $q < 22.2$, $q \neq 3$: $2, 5, 7, 11, 13, 17, 19$, that is 7; $p = 5$: $q < 8$, $q \neq 5$: $2, 3, 7$, that is 3; $p = 7$: $q < 4.1$: $2, 3$, that is 2; $p \geq 11$: $121 \cdot 2 > 200$, none. Total $1 + 14 + 7 + 3 + 2 = 27$.

Check. $n = 12 = 2^2 \cdot 3$ has divisors $1, 2, 3, 4, 6, 12$ with product $1728 = 12^3$. The largest number counted is $188 = 2^2 \cdot 47$, whose divisors $1, 2, 4, 47, 94, 188$ pair up as $1 \cdot 188 = 2 \cdot 94 = 4 \cdot 47 = 188$, giving 188^3. A near miss is $196 = 2^2 \cdot 7^2$: it has 9 divisors, so their product is $196^{9/2} = 14^9$, not 196^3.

19. Answer: $-3 + \sqrt{41}$. **The idea.** Put the centers at $O_1 = (0, 0)$ and $O_2 = (6, 0)$. The lens is symmetric about the x-axis and about the line $x = 3$, and it is convex, so the best square can be centered at $(3, 0)$. The subtle point is which circle limits which corner: the right-hand corners are far from O_1, so it is the *left* circle that stops them.

Work it out. (Why centered: if a square fits, so does its mirror image in $x = 3$ and in the x-axis; the lens is convex, so the square halfway between a square and its mirror image, of the same size, also fits, and doing this twice centers it at $(3, 0)$.) Let the half-side be t, so the corners are $(3 \pm t, \pm t)$. The corner $(3+t, t)$ is inside circle O_2 automatically (it is closer to O_2 than the corner $(3 - t, t)$ is), and it lies inside circle O_1 exactly when $(3+t)^2 + t^2 \leq 25$, i.e. $2t^2 + 6t - 16 \leq 0$, i.e. $t^2 + 3t - 8 \leq 0$. The largest t is $\frac{-3+\sqrt{41}}{2}$, and the side is $2t = \sqrt{41} - 3$.

Check. $\sqrt{41} - 3 \approx 3.40$, so the square spans x from about 1.30 to 4.70 and y from -1.70 to 1.70, inside the lens, which spans $x \in [1, 5]$ and $y \in [-4, 4]$. The corner $(4.70, 1.70)$ has distance $\sqrt{22.1 + 2.9} = 5.0$ from O_1, right on the circle.

20. Answer: $\frac{176}{1275}$. **The idea.** Count the favourable ordered deals directly: choose which three ranks appear, then which card of each rank. Once the three cards are chosen, only one of their orders climbs.

Work it out. There are $52 \cdot 51 \cdot 50 = 132600$ ordered deals. A climbing deal uses three different ranks, $\binom{13}{3} = 286$ choices, and one of the 4 suits for each card, $4^3 = 64$ choices; the order is then forced (lowest rank first). So there are $286 \cdot 64 = 18304$ climbing deals, and the probability is $\frac{18304}{132600} = \frac{176}{1275}$.

Check. By symmetry: the three ranks are all different with probability $\frac{48}{51} \cdot \frac{44}{50} = \frac{352}{425}$, and then each of the 6 orders is equally likely, so the probability is $\frac{352}{425 \cdot 6} = \frac{176}{1275}$, about 0.138, a bit under $\frac{1}{6}$.

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21. Answer: $\frac{27\sqrt{15}}{4}$. **The idea.** Let M be the midpoint of $\overline{AC}$. In triangle ABM, the bisector of the angle at A is perpendicular to the opposite side $\overline{BM}$. A line from a vertex that is both an angle bisector and an altitude makes the triangle isosceles.

Work it out. The bisector from A meets $\overline{BM}$ at a right angle, so the two small right triangles on either side of it share a leg and have equal angles at A; they are congruent, and $AM = AB = 6$. Hence $AC = 2 \cdot AM = 12$. Now triangle ABC has sides 6, 9, 12, with semiperimeter $\frac{27}{2}$, and Heron's formula gives

$$\sqrt{\tfrac{27}{2} \cdot \tfrac{15}{2} \cdot \tfrac{9}{2} \cdot \tfrac{3}{2}} = \tfrac{\sqrt{10935}}{4} = \tfrac{\sqrt{729 \cdot 15}}{4} = \tfrac{27\sqrt{15}}{4}.$$

Check. By the law of cosines, $\cos A = \frac{36+144-81}{2 \cdot 6 \cdot 12} = \frac{11}{16}$, so $\sin A = \frac{\sqrt{135}}{16} = \frac{3\sqrt{15}}{16}$ and the area is $\frac{1}{2} \cdot 6 \cdot 12 \cdot \frac{3\sqrt{15}}{16} = \frac{27\sqrt{15}}{4} \approx 26.1$. The side lengths 6, 9, 12 do form a triangle, since $6 + 9 > 12$.

22. Answer: 6. **The idea.** In a geometric sequence the middle term satisfies $b^2 = ac$. That makes $a^2 + b^2 + c^2$ factor: it equals $(a + b + c)(a - b + c)$.

Work it out. Expand: $(a+b+c)(a-b+c) = (a+c)^2 - b^2 = a^2 + 2ac + c^2 - b^2 = a^2 + c^2 + b^2$, using $2ac = 2b^2$. So $26 \cdot (a - b + c) = 364$, giving $a - b + c = 14$. Subtracting this from $a + b + c = 26$ gives $2b = 12$, so $b = 6$.

Check. Then $a + c = 20$ and $ac = 36$, so $\{a, c\} = \{2, 18\}$. The sequence 2, 6, 18 has ratio 3, sum 26, and $4 + 36 + 324 = 364$.

23. Answer: 149. **The idea.** A backward hop is very restricted. After a hop back from $i + 2$ to $i + 1$, the stones i and $i+2$ are both used, so the only legal next hop is two forward, to $i+3$. So every route is a sequence of three kinds of moves: forward 1, forward 2, or the zigzag $i \to i+2 \to i+1 \to i+3$, which advances 3.

Work it out. Let $f(n)$ be the number of routes from stone 1 to stone n. Looking at the last move, $f(n) = f(n-1) + f(n-2) + f(n-3)$, with $f(1) = 1$, $f(2) = 1$, $f(3) = 2$. Then $f(4) = 4$, $f(5) = 7$, $f(6) = 13$, $f(7) = 24$, $f(8) = 44$, $f(9) = 81$ and $f(10) = 149$.

Check. For 4 stones the routes are 1 2 3 4, 1 2 4, 1 3 4 and the zigzag 1 3 2 4: four, matching $f(4) = 4$. A zigzag never lands on stone 10 early, because it would have to pass through $i + 2 \le 9$ first.

24. Answer: $-2 - 3\sqrt{3}$. **The idea.** Name the integer part: $\lfloor x \rfloor = n$. Then $x^2 = 15 - 2n$ is known, so $x = \pm\sqrt{15 - 2n}$, and the only thing left is to check which choices really have floor n, i.e. lie in $[n, n + 1)$. Watch the endpoints: a solution can be an integer.

Work it out. Positive x (so $n \ge 0$): need $n \le \sqrt{15 - 2n} < n+1$, i.e. $n^2 \le 15 - 2n < (n+1)^2$. The left says $n^2 + 2n - 15 \le 0$, $n \le 3$; the right says $n^2 + 4n - 14 > 0$, $n \ge 3$. So $n = 3$, $x = 3$ (indeed $9 + 6 = 15$). Negative x (so $n \le -1$): need $n \le -\sqrt{15 - 2n} < n + 1$. The left means $n^2 \ge 15 - 2n$, so $n^2 + 2n - 15 \ge 0$, $n \le -5$. The right means $\sqrt{15 - 2n} > -(n + 1)$, i.e. $15 - 2n > (n + 1)^2$, so $n^2 + 4n - 14 < 0$, $n \ge -6$. So $n = -5$ ($x = -\sqrt{25} = -5$) or $n = -6$ ($x = -\sqrt{27} = -3\sqrt{3} \approx -5.20$). The solutions are 3, -5, $-3\sqrt{3}$, with sum $-2 - 3\sqrt{3}$.

Check. $x = -5$: $25 - 10 = 15$. $x = -3\sqrt{3} \approx -5.196$ has floor -6: $27 - 12 = 15$. And $x = 0$ gives no solution since $x^2 = 15$ would force $|x| > 3$.

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25. Answer: 9. **The idea.** Every vertex of a cube lies on 3 edges. If exactly 2 of them are chosen, exactly 1 is *not* chosen. So the unchosen edges are a set in which every vertex is an endpoint of exactly one edge: a perfect matching of the cube. Count those instead.

Work it out. Call the four edges joining the top face to the bottom face vertical, and let k be the number of vertical edges in the matching. The top vertices not on a vertical matching edge must be matched among themselves by top edges, so $4-k$ is even. $k=4$: 1 way. $k=2$: the two top vertices left over must be joined by a top edge (4 choices of edge), the other two top vertices go straight down, and the bottom is then forced: 4 ways. $k=0$: the top face is matched by two parallel edges (2 ways) and so is the bottom (2 ways): 4 ways. Total $1+4+4=9$ matchings, hence 9 sets of chosen edges.

Check. Look at the chosen edges directly: they form closed loops through all 8 vertices, and every loop in a cube has even length at least 4, the 4-loops being faces. So the choice is either two faces with no common vertex (3 pairs of opposite faces) or one loop through all 8 vertices. The 3 pairs of opposite faces are the complements of the 3 matchings whose four edges are all parallel, and the other 6 matchings leave the 6 loops through all 8 vertices: $3+6=9$.

26. Answer: $\frac{256}{129}$. **The idea.** Don't chase coordinates of P and Q. The power of the point O gives $OP\cdot OQ$ directly, the midpoint condition turns that into the chord length PQ, and the chord length tells how far the line is from the center. That becomes one quadratic in the slope, and Vieta gives the sum.

Work it out. The center $C=(8,6)$ is 10 from O, so the power of O is $OP\cdot OQ=10^2-5^2=75$. With $OQ=2\,OP$: $2\,OP^2=75$, so $PQ=OP=\sqrt{75/2}$. A chord of length L in a circle of radius 5 is at distance d from the center with $d^2=25-(L/2)^2=25-\frac{75}{8}=\frac{125}{8}$. The line $y=mx$ (it is not vertical: the line $x=0$ misses ω) is at distance $\frac{|8m-6|}{\sqrt{1+m^2}}$ from C, so $8(8m-6)^2=125(1+m^2)$, i.e. $512m^2-768m+288=125+125m^2$, or $387m^2-768m+163=0$. Its discriminant $768^2-4\cdot387\cdot163=337500$ is positive, giving the two lines, and the sum of the slopes is $\frac{768}{387}=\frac{256}{129}$.

Check. Substituting $y=mx$ into the circle gives $(1+m^2)x^2-(16+12m)x+75=0$ with roots x_P and $x_Q=2x_P$. Then $2x_P^2=\frac{75}{1+m^2}$ and $3x_P=\frac{16+12m}{1+m^2}$; squaring the second and dividing by the first eliminates x_P: $2(16+12m)^2=675(1+m^2)$, i.e. $288m^2+768m+512=675m^2+675$, which is again $387m^2-768m+163=0$.

27. Answer: 72. **The idea.** The two legs meet at a right angle, so reflecting A across line BC and B across line CA is the same as reflecting each of them through the point C. So $\overline{DE}$ is the hypotenuse $\overline{AB}$ turned half a turn about C: parallel to $\overline{AB}$, of length 10, on the far side of C. The height of triangle DEF is then a multiple of the altitude from C.

Work it out. The hypotenuse is 10 and the altitude from C to $\overline{AB}$ is $h=\frac{6\cdot8}{10}=\frac{24}{5}$. Line DE is parallel to AB at distance h on the other side of C. The point F, the mirror image of C in line AB, lies at distance h beyond AB. So F is $h+h+h=3h$ from line DE, and the area of triangle DEF is $\frac{1}{2}\cdot10\cdot3h=3\cdot24=72$, three times the area of ABC.

Check. In coordinates, $C=(0,0)$, $A=(6,0)$, $B=(0,8)$: then $D=(-6,0)$, $E=(0,-8)$, and the foot of the altitude from C is $\left(\frac{96}{25},\frac{72}{25}\right)$, so $F=\left(\frac{192}{25},\frac{144}{25}\right)$. The shoelace formula gives $\frac{1}{2}\left|(-6)(-8)-0+0\cdot\frac{144}{25}-(-8)\cdot\frac{192}{25}+\frac{192}{25}\cdot0-\frac{144}{25}\cdot(-6)\right|=\frac{1}{2}\left(48+\frac{1536}{25}+\frac{864}{25}\right)=\frac{1}{2}(48+96)=72$.

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28. Answer: $\frac{67\sqrt{3}}{4}$**. The idea.** Extending every other side of a 120° hexagon produces an equilateral triangle with three small equilateral triangles cut off at its corners. That picture gives both the condition on the sides and the area.

Work it out. Call the sides a, b, c, d, e, f in order. Extend sides a, c, e; the corners cut off are equilateral triangles with sides b, d, f, and the big triangle's sides are $f+a+b$, $b+c+d$, $d+e+f$. These are equal, so $a+f=c+d$ and $b+c=e+f$, i.e. $a-d=c-f=e-b$. So the three pairs of opposite sides all have the same difference. With the lengths 1–6: difference 1 forces the pairs $\{1,2\}, \{3,4\}, \{5,6\}$; difference 3 forces $\{1,4\}, \{2,5\}, \{3,6\}$; differences 2, 4, 5 are impossible (for 2: 1 pairs with 3, then 2 with 4, leaving 5, 6). Also, since $b-e=f-c=d-a$, the alternate sides b, d, f are either all the larger members of their pairs or all the smaller ones. The area is the big triangle minus the corners: $\frac{\sqrt{3}}{4}\left[T^2-b^2-d^2-f^2\right]$, where adding the three big sides gives $3T=(a+\cdots+f)+(b+d+f)=21+(b+d+f)$. *Pairs differing by 1:* $\{b,d,f\}=\{2,4,6\}$ gives $T=11$ and $\frac{\sqrt{3}}{4}(121-56)=\frac{65\sqrt{3}}{4}$; $\{1,3,5\}$ gives $T=10$ and $\frac{\sqrt{3}}{4}(100-35)$, the same. *Pairs differing by 3:* $\{4,5,6\}$ gives $T=12$ and $\frac{\sqrt{3}}{4}(144-77)=\frac{67\sqrt{3}}{4}$; $\{1,2,3\}$ gives $T=9$ and $\frac{\sqrt{3}}{4}(81-14)$, the same. Such hexagons exist (the sides $1,5,3,4,2,6$ in order satisfy $a-d=c-f=e-b=-3$), so the largest area is $\frac{67\sqrt{3}}{4}$.

Check. Walk the hexagon $1,5,3,4,2,6$ turning 60° each time: with unit steps u_k at $0°, 60°, \ldots$, opposite directions cancel as $(1-4)u_0+(5-2)u_1+(3-6)u_2=-3(u_0-u_1+u_2)=0$ since $u_0+u_2=u_1$. So it closes, and its area by shoelace is $\frac{67\sqrt{3}}{4}\approx 29.0$, beating $\frac{65\sqrt{3}}{4}$.

29. Answer: 1487**. The idea.** Draw a dot for each number and join two dots when their sum is a perfect square; a good subset is a set of dots with no two joined. The sums are at most 14+15 = 29, so the only squares that matter are 4, 9, 16, 25, and the resulting picture turns out to be one loop of five dots with two "tails" hanging off it. Counting sets with no two joined dots along a path is a Fibonacci count, and one case split on the dot where the long tail meets the loop finishes it.

Work it out. The joined pairs are: sum 4: 1–3; sum 9: 1–8, 2–7, 3–6, 4–5; sum 16: 1–15, 2–14, 3–13, 4–12, 5–11, 6–10, 7–9; sum 25: 10–15, 11–14, 12–13. Follow them: 1–3–6–10–15–1 is a loop of five; 8 hangs off 1 alone; and from 3 a single path runs 3–13–12–4–5–11–14–2–7–9. All 15 numbers and all 15 pairs are used. On a path of n dots the number of allowed sets is the Fibonacci number F_{n+2} (the last dot is either out, leaving a path of $n-1$, or in, forcing its neighbour out, leaving a path of $n-2$): $2, 3, 5, 8, 13, 21, 34, 55, 89$ for $n=1,\ldots,9$. *Case 1:* 3 *is not chosen.* The tail $13, 12, 4, 5, 11, 14, 2, 7, 9$ is a free path of 9 dots: 89 sets. The rest is the path 6–10–15–1 with 8 hanging on 1: if 1 is chosen, 8 and 15 are out and 6–10 gives 3; if not, 8 is free (2) and 6–10–15 gives 5, for 10; so 13 sets. Case total $13\cdot 89=1157$. *Case 2:* 3 *is chosen.* Then 1, 6 and 13 are out. What is left is the path 10–15 (3 sets), the lone dot 8 (2), and the path 12–4–5–11–14–2–7–9 of 8 dots (55): $3\cdot 2\cdot 55=330$. The total is $1157+330=1487$.

Check. The same picture for $\{1,\ldots,8\}$ keeps only the pairs 1–3, 1–8, 2–7, 3–6, 4–5, three separate pieces: the path 8–1–3–6 (8 sets), 2–7 (3) and 4–5 (3), so 72 subsets, which a direct computer count confirms. For $\{1,\ldots,15\}$ a computer check of all 2^{15} subsets also gives 1487.

30. Answer: 10**. The idea.** Group the positive integers into bands: band k is $k^2, k^2+1, \ldots, k^2+2k$, the numbers whose square root has integer part k. Inside band k the sequence steps by exactly k, so all that matters is the *offset* $r=a_n-k^2$ with which it enters each band. That offset turns out to drop by exactly 1 per band, so the sequence hits a square (offset 0) only rarely, and in a pattern we can predict exactly.

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Work it out. Suppose a term is $k^2 + r$ with $1 \le r \le k - 1$. The next terms are $k^2 + r + k$ (still in band k, since $r + k \le 2k$) and then $k^2 + r + 2k = (k + 1)^2 + (r - 1)$. So the sequence spends exactly 2 terms in band k, neither of them a square (their offsets r and $r + k$ are not 0), and enters band $k + 1$ with offset $r - 1$. Now suppose a term is the square m^2. The next terms are $m^2 + m$, $m^2 + 2m$ (both in band m) and then $m^2 + 3m = (m + 1)^2 + (m - 1)$. So after 3 terms in band m it enters band $m + 1$ with offset $m - 1$. Then the offset falls $m - 1, m - 2, \ldots, 1$ through bands $m + 1, \ldots, 2m - 1$ (each time $r \le k - 1$ holds), 2 terms per band, and reaches offset 0 in band $2m$: the term $(2m)^2$. (For $m = 1$ the sequence $1, 2, 3$ goes straight to 4, which fits the same count.) So the squares in the sequence are exactly $1, 4, 16, 64, \ldots$, the powers of 4, and from m^2 to the next square $(2m)^2$ there are $3 + 2(m - 1) = 2m + 1$ steps. With $m = 2^{j-1}$, the square 4^j comes $2^j + 1$ terms after 4^{j-1}. Starting from $a_1 = 1$, the squares are at positions 1, $1 + 3 = 4$, $4 + 5 = 9$, $9 + 9 = 18$, 35, 68, 133, 262, 519, 1032, and then $1032 + 1025 = 2057 > 2026$. So 10 of the first 2026 terms are squares, namely $4^0, 4^1, \ldots, 4^9$.

Check. Writing out the start: $1, 2, 3, 4, 6, 8, 10, 13, 16, 20, 24, 28, 33, 38, 44, 50, 57, 64, \ldots$. The squares $1, 4, 16, 64$ sit at positions $1, 4, 9, 18$, as predicted. The offsets behave as claimed too: after 16 come 20, 24 (band 4), then $28 = 25 + 3$ enters band 5 with offset $3 = m - 1$; then $38 = 36 + 2$, $50 = 49 + 1$, and finally $64 = 8^2$ with offset 0. A direct computer run of the first 2026 terms also finds exactly 10 squares.

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