MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A two-digit positive integer is exactly 11 more than the product of its two digits. What is the integer?

1. _____

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2. What is the smallest positive integer whose digits are all different from one another and add up to 40?

2. _____

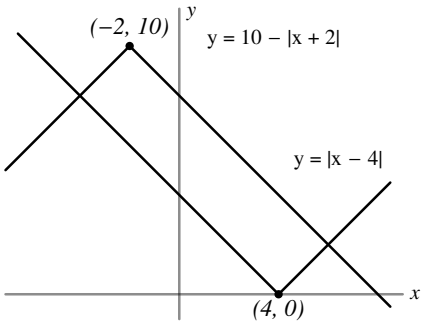
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3. The five letters of the word LEVEL are arranged in a row. Two arrangements that differ only by swapping the two L's, or only by swapping the two E's, count as the same arrangement. In how many arrangements are the two L's not next to each other *and* the two E's not next to each other?

3. _____

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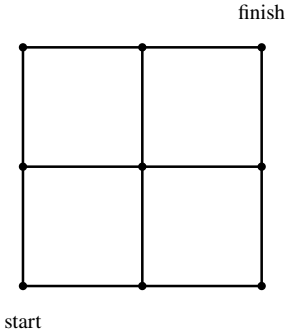
4. In the coordinate plane, the graphs of $y = |x - 4|$ and $y = 10 - |x + 2|$ enclose a region. What is the area of this region, in square units?



4. _____

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5. Nine points are arranged in a 3 by 3 square array, 1 unit apart, and each pair of points that are next to each other in the same row or the same column is joined by a segment (12 segments in all). A path starts at the bottom-left point, moves along the segments, and ends at the top-right point, never visiting any point more than once. How many such paths are there?

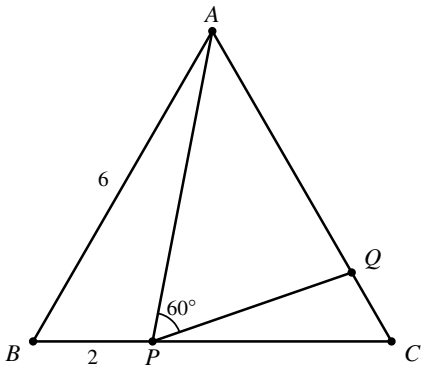


5. _____

6. For a positive integer n, let S_n be the number formed by writing the integers $1, 2, 3, \ldots, n$ one after another in increasing order, with no spaces. For example, $S_{12} = 123456789101112$. For how many integers n with $1 \le n \le 200$ is S_n divisible by 36?

6. _____

7. Equilateral triangle ABC has side length 6. Point P lies on side BC with $BP = 2$, and point Q lies on side CA so that $\angle APQ = 60°$. What is the length of PQ? Express your answer as a common fraction in simplest radical form.



7. _____

8. Three fair six-sided dice, one red, one white and one blue, are rolled. You are told that the three numbers rolled add up to 10. Given this information, what is the probability that the three numbers are all different? Express your answer as a common fraction.

8. _____

9. A sequence starts with $a_1 = 2025$. Each later term is the sum of the squares of the digits of the term before it; for example, $a_2 = 2^2 + 0^2 + 2^2 + 5^2 = 33$. What is a_{2026}?

9. _____

10. $ABCDE$ is a regular pentagon with its vertices labelled in order around it. Square $ABFG$ is drawn on side $\overline{AB}$, on the inside of the pentagon, so that F is the vertex of the square next to B and G is the vertex of the square next to A. What is the degree measure of $\angle CEG$?

10. _____

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11. Five people get into an empty elevator on the ground floor of a building. The building has 5 floors above the ground floor, and each person, independently of the others, gets off at one of these 5 floors, chosen at random with each floor equally likely. The elevator stops at a floor exactly when at least one person gets off there. What is the probability that the elevator stops at exactly 3 floors? Express your answer as a common fraction.

11. _____

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12. The 11 letters of the word MISSISSIPPI are shuffled, and all the different arrangements of the letters are equally likely. (Arrangements that differ only by swapping two identical letters are the same arrangement.) What is the probability that the arrangement is a palindrome, that is, it reads the same forwards and backwards? Express your answer as a common fraction.

12. _____

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13. Call a three-digit positive integer N *trimmable* if deleting its middle digit leaves a two-digit number that divides N. For example, 105 is trimmable, because deleting the 0 leaves 15, and $105 = 7 \cdot 15$. How many three-digit positive integers are trimmable?

13. _____

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14. Call a positive integer a *duo* if it is the product of two different primes; for example, $15 = 3 \cdot 5$ is a duo, but 12 and 49 are not. The integers $33 = 3 \cdot 11$, $34 = 2 \cdot 17$ and $35 = 5 \cdot 7$ are three consecutive duos, and 33 is the least positive integer n for which n, $n + 1$ and $n + 2$ are all duos. What is the next smallest such n?

14. _____

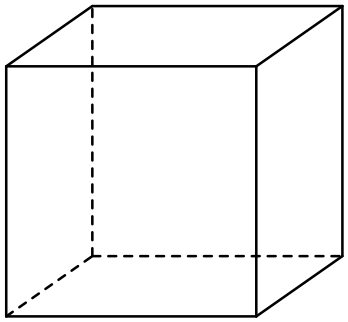
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15. In the year 2026, January 1 falls on a Thursday, and the 13th day of a month is a Friday in exactly three months: February, March and November. What is the next year after 2026 in which the 13th day of a month is a Friday in exactly the months February, March and November, and in no other month? (The years 2028, 2032 and 2036 are leap years with 366 days, since February has 29 days; every other year from 2027 to 2040 has 365 days.)

15. _____

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16. Each of the 12 edges of a cube is painted either red or blue so that every one of the 6 faces has exactly two red edges on its boundary. In how many ways can this be done? (The cube is fixed in place, so two colourings that differ by a rotation of the cube count as different.)



16. _____

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17. Alice rolls two fair six-sided dice and Bob rolls one fair six-sided die. What is the probability that Bob's number is strictly between Alice's two numbers (greater than the smaller of Alice's numbers and less than the larger)? Express your answer as a common fraction.

17. _____

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18. A circle passes through the points $(0, 0)$ and $(8, 0)$ and is tangent to the line $y = x$. What is the radius of the circle? Express your answer in simplest radical form.

18. _____

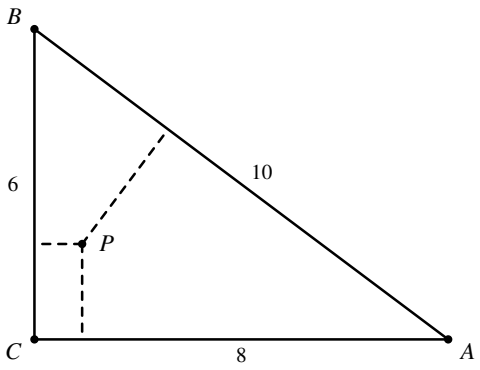
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19. For an integer $b \geq 2$, the number 2026 can be written in base b. Find the sum of all bases b for which the base-b representation of 2026 has exactly three digits and its last (units) digit is 1.

19. _____

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20. Triangle ABC has $BC = 6$, $CA = 8$ and $AB = 10$. A point P inside the triangle is at distances from the lines BC, CA and AB that are in the ratio $1 : 2 : 3$, in that order. What is the distance PC? Express your answer in simplest radical form.



20. _____

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21. A finite arithmetic sequence has at least 3 terms, all of its terms are positive integers, and its common difference is a positive integer. The sum of its terms is 2026. How many such sequences are there?

21. _____

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22. What is the least positive integer n for which the sum $1+2+3+\cdots+n$ is a multiple of 2025?

22. _____

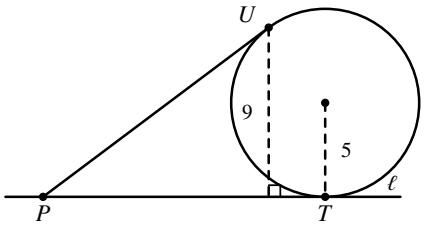
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23. What is the least positive integer N such that N^2 has exactly three times as many positive divisors as N has?

23. _____

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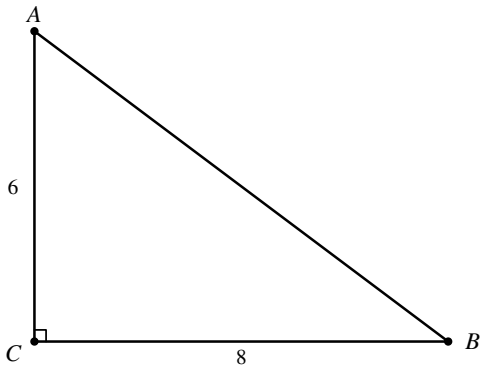
24. A circle of radius 5 is tangent to line ℓ at point T. Point P lies on ℓ, and the other tangent line from P to the circle touches the circle at U. If U is at distance 9 from line ℓ, what is the length PT?



24. _____

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25. A paper right triangle ABC has its right angle at C, with $CA = 6$ and $CB = 8$. The paper is folded along a straight line so that vertex A lands exactly on vertex B, and it is pressed flat. What is the area of the resulting flat shape (the region of the table covered by the folded paper)? Express your answer as a common fraction.



25. _____

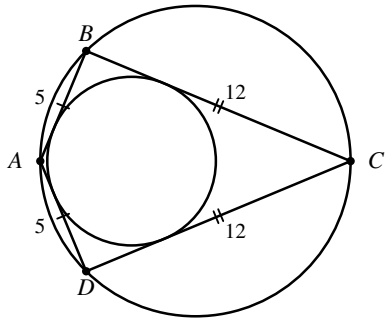
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26. The numbers $1, 2, 3, \ldots, 9$ are placed in the nine cells of a 3×3 grid, one number in each cell. In how many placements is the sum of the three numbers in each row odd *and* the sum of the three numbers in each column odd?

26. _____

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27. Kite $ABCD$ has $AB = AD = 5$ and $CB = CD = 12$, and all four of its vertices lie on one circle. A second circle lies inside the kite and is tangent to all four of its sides. What is the distance between the centers of the two circles? Express your answer as a common fraction.



27. _____

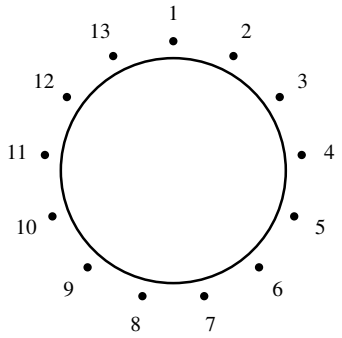
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28. A seven-digit palindrome is a number of the form $\overline{abcdcba}$ with digits a, b, c, d and $a \neq 0$; for example, 3014103. How many seven-digit palindromes are divisible by 7?

28. _____

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29. Thirteen chairs are equally spaced around a round table and numbered 1 through 13 in order, so chair 13 is next to chair 1. How many ways are there to choose a nonempty set of the chairs so that no two chosen chairs have exactly one chair between them? (For example, chairs 12 and 1 may not both be chosen, since chair 13 is the only chair between them.)



29. _____

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30. Each of the nine cells of a 3×3 grid is filled with one of the numbers 1, 2 or 3 (numbers may repeat, and not every number has to be used). The filling must be such that in every row the numbers never decrease from left to right, and in every column the numbers never decrease from top to bottom. How many such fillings are there?

30. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — MathCounts Nationals Sprint Practice Contest 3

1. 29

2. 1456789

3. 12

4. 32

5. 12

6. 10

7. $\frac{4\sqrt{7}}{3}$

8. $\frac{2}{3}$

9. 42

10. 9

11. $\frac{12}{25}$

12. $\frac{1}{1155}$

13. 76

14. 85

15. 2037

16. 20

17. $\frac{5}{27}$

18. $4\sqrt{2}$

19. 112

20. $\frac{12\sqrt{5}}{13}$

21. 169

22. 324

23. 144

24. 15

25. $\frac{117}{8}$

26. 25920

27. $\frac{91}{34}$

28. 1285

29. 520

30. 175

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Nationals Sprint Practice Contest 3

1. Answer: 29**. The idea.** Write the number as $10a + b$ with tens digit a and units digit b. The equation $10a + b = ab + 11$ has a product term, and the way to handle it is to factor: rearranged, it becomes a product of two integers equal to 1.

Work it out. Move everything to one side: $10a - ab + b - 11 = 0$. Group as $a(10-b)-(10-b) = 1$, that is, $(a-1)(10-b) = 1$. Since $b \le 9$, the factor $10 - b$ is a positive integer, so both factors equal 1: $a = 2$ and $b = 9$. The integer is 29.

Check. $2 \cdot 9 + 11 = 29$. No other number works, because 1 has only one factorisation into positive integers; trying digits blindly, 19 gives $9 + 11 = 20$ and 39 gives $27 + 11 = 38$, both misses.

2. Answer: 1456789**. The idea.** Fewer digits means a smaller number, so first find the fewest digits possible, then choose which digits and put them in increasing order. The complement is the key: all ten digits add up to 45, so the digits we leave out must add up to exactly $45 - 40 = 5$.

Work it out. Six different digits add up to at most $9+8+7+6+5+4 = 39 < 40$, so at least seven digits are needed. A seven-digit number cannot contain 0: the other six digits would then have to add up to 40, again more than 39. So the digits are seven of $1, 2, \ldots, 9$, and the two omitted digits add up to $45 - 40 = 5$: either $\{1, 4\}$ or $\{2, 3\}$. Omitting $\{2, 3\}$ leaves 1 available as the leading digit, while omitting $\{1, 4\}$ forces a leading 2. With digits $1, 4, 5, 6, 7, 8, 9$ in increasing order the number is 1456789.

Check. $1 + 4 + 5 + 6 + 7 + 8 + 9 = 40$ and the digits are distinct. The only other seven-digit candidate set gives 2356789, which is larger, and every eight-digit number is larger still.

3. Answer: 12**. The idea.** Two "not next to each other" conditions at once call for inclusion–exclusion: count all arrangements, subtract those with the L's together and those with the E's together, and add back those with both pairs together, which were subtracted twice.

Work it out. All arrangements: $\frac{5!}{2!\,2!} = 30$. L's together: glue them into one block LL; arranging LL, E, E, V gives $\frac{4!}{2!} = 12$. E's together: likewise 12. Both pairs together: arrange the blocks LL, EE and V in $3! = 6$ ways. So the count is $30 - 12 - 12 + 6 = 12$.

Check. Count by the position of V. If V is at an end, the other four places form a run that must alternate: LELE or ELEL, so 2 ways for each end, 4 in all. If V is in the middle, each side has two places holding one L and one E: $2 \cdot 2 = 4$. If V is second, the run in places 3–5 must be LEL or ELE, which fixes place 1: 2 ways, and the same for V fourth. Total $4 + 4 + 2 + 2 = 12$.

4. Answer: 32**. The idea.** Every piece of both graphs is a line of slope 1 or -1. So the sides of the enclosed region are parallel or perpendicular in pairs: the region is a rectangle tilted 45°. Find its four corners and multiply two adjacent sides.

Work it out. Two corners are the tips of the V's: $(4, 0)$ for $y = |x - 4|$ and $(-2, 10)$ for $y = 10 - |x + 2|$. The other two are where the graphs cross. For $x < -2$ the graphs are $y = 4 - x$ and $y = 12 + x$, which meet at $x = -4$, $y = 8$. For $x > 4$ they are $y = x - 4$ and $y = 8 - x$, meeting at $x = 6$, $y = 2$. (For $-2 \le x \le 4$ the upper graph is $y = 8 - x$, always 4 above $y = 4 - x$, so they do not cross there.) The corners are $(4, 0)$, $(6, 2)$, $(-2, 10)$, $(-4, 8)$. Adjacent sides have lengths $\sqrt{2^2 + 2^2} = 2\sqrt{2}$ (from $(4, 0)$ to $(6, 2)$) and $\sqrt{8^2 + 8^2} = 8\sqrt{2}$ (from $(6, 2)$ to $(-2, 10)$). The area is $2\sqrt{2} \cdot 8\sqrt{2} = 32$.

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Check. The shoelace formula on the corners in order: $(4 \cdot 2 - 6 \cdot 0) + (6 \cdot 10 - (-2) \cdot 2) + ((-2) \cdot 8 - (-4) \cdot 10) + ((-4) \cdot 0 - 4 \cdot 8) = 8 + 64 + 24 - 32 = 64$, and half of 64 is 32.

5. Answer: 12**. The idea.** The array is symmetric about the diagonal through the start and the finish, and that reflection swaps paths whose first step is to the right with paths whose first step is up. So count the paths that begin with a step to the right and double. Organise that count by where the path goes next.

Work it out. Use coordinates $(0,0)$ to $(2,2)$ and begin with $(0,0) \to (1,0)$.

Case 1: next $(1,1)$. From the centre the path can go to $(2,1)$ and then must finish at $(2,2)$; or to $(1,2)$ and finish at $(2,2)$; or to $(0,1)$, then $(0,2)$, $(1,2)$, $(2,2)$. (Going $(1,1) \to (1,2) \to (0,2)$ strands the path at $(0,1)$, and $(2,1) \to (2,0)$ is a dead end.) That gives 3 paths.

Case 2: next $(2,0)$, forced on to $(2,1)$. Stepping to $(2,2)$ now would end the path, but that is allowed: 1 path. Otherwise go to $(1,1)$, then either $(1,2) \to (2,2)$, or $(0,1) \to (0,2) \to (1,2) \to (2,2)$: 2 more paths. (From $(1,1)$ the move to $(1,0)$ repeats a point.) That gives 3 paths.

So 6 paths start to the right, and by symmetry 6 start upward: 12 in all.

Check. Colour the points like a chessboard. Each step changes colour and the two corners have the same colour, so every path has an even number of steps: 4, 6 or 8. The 4-step paths are the $\binom{4}{2} = 6$ shortest routes; the list above also contains 4 paths of 6 steps and 2 of 8 (the two snakes through all nine points): $6 + 4 + 2 = 12$.

6. Answer: 10**. The idea.** Split $36 = 4 \cdot 9$. Divisibility by 9 depends only on the digit sum, and every number has the same remainder mod 9 as its own digit sum, so $S_n \equiv 1+2+\cdots+n \pmod 9$. Divisibility by 4 depends only on the last two digits, which for $n \geq 10$ are just the last two digits of n.

Work it out. Mod 9: we need $9 \mid \frac{n(n+1)}{2}$, which is the same as $9 \mid n(n+1)$. Consecutive integers cannot both be multiples of 3, so $9 \mid n$ or $9 \mid n+1$: $n \equiv 0$ or $8 \pmod 9$.

Mod 4: for $n \geq 10$ we need $4 \mid n$. Combining, $n \equiv 0 \pmod{36}$ or $n \equiv 8 \pmod{36}$ (the number 8 is both a multiple of 4 and 8 more than a multiple of 9). For $n \leq 9$ the only candidates are $n = 8$ and $n = 9$, but $S_8 = 12345678$ ends in 78 and S_9 ends in 89, and neither is a multiple of 4.

So n is one of 36, 72, 108, 144, 180 or 44, 80, 116, 152, 188: 10 values.

Check. S_{36} has digit sum congruent to $1 + 2 + \cdots + 36 = 666 = 9 \cdot 74$ and ends in 36, so $36 \mid S_{36}$. The trap is $n = 8$: it passes the mod 9 test ($1 + \cdots + 8 = 36$), but S_8 ends in 78, not 8 alone, so counting it gives 11.

7. Answer: $\frac{4\sqrt{7}}{3}$**. The idea.** The 60° angle at P matches the 60° angles at B and C, and that makes triangles ABP and PCQ similar. Then PQ is just AP scaled by the similarity ratio.

Work it out. The three angles at P along line BC add to 180°: $\angle APB + 60° + \angle QPC = 180°$, so $\angle QPC = 120° - \angle APB$. In triangle ABP, $\angle BAP = 180° - 60° - \angle APB = 120° - \angle APB$ too. So $\angle BAP = \angle CPQ$, and with $\angle B = \angle C = 60°$, triangle ABP is similar to triangle PCQ ($A \leftrightarrow P$, $B \leftrightarrow C$, $P \leftrightarrow Q$). The ratio is $\frac{PC}{AB} = \frac{4}{6} = \frac{2}{3}$, so $PQ = \frac{2}{3}AP$. The law of cosines in triangle ABP gives $AP^2 = 6^2 + 2^2 - 2 \cdot 6 \cdot 2\cos 60° = 28$, so $AP = 2\sqrt{7}$ and $PQ = \frac{4\sqrt{7}}{3}$.

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Check. The similarity also gives $CQ = \frac{2}{3}BP = \frac{4}{3}$. Put $B = (0,0)$, $C = (6,0)$, $A = (3, 3\sqrt{3})$, $P = (2,0)$. Moving $\frac{4}{3}$ from C toward A gives $Q = \left(\frac{16}{3}, \frac{2\sqrt{3}}{3}\right)$, and $PQ^2 = \left(\frac{10}{3}\right)^2 + \left(\frac{2\sqrt{3}}{3}\right)^2 = \frac{112}{9}$, so $PQ = \frac{4\sqrt{7}}{3}$. Also $\overrightarrow{PA} \cdot \overrightarrow{PQ} = 1 \cdot \frac{10}{3} + 3\sqrt{3} \cdot \frac{2\sqrt{3}}{3} = \frac{28}{3}$, which is half of $AP \cdot PQ = \frac{56}{3}$, confirming the 60° angle.

8. Answer: $\frac{2}{3}$. The idea. Conditioning on the sum means the sample space shrinks to the equally likely ordered rolls with sum 10. List the possible multisets of values and count the orderings of each.

Work it out. Multisets of three numbers from 1 to 6 summing to 10: with all different values, $\{1,3,6\}$, $\{1,4,5\}$, $\{2,3,5\}$, each giving $3! = 6$ ordered rolls, for 18. With a repeated value, $\{2,2,6\}$, $\{3,3,4\}$, $\{2,4,4\}$, each giving 3 ordered rolls, for 9. (Three equal values would need $3a = 10$, impossible; $\{1,1,8\}$ and $\{5,5,0\}$ are not allowed.) There are 27 equally likely rolls with sum 10, and 18 of them have all different numbers, so the probability is $\frac{18}{27} = \frac{2}{3}$.

Check. The number of ordered rolls with sum 10 is the coefficient of x^{10} in $(x + \cdots + x^6)^3$, which is $\binom{9}{2} - 3\binom{3}{2} = 36 - 9 = 27$, matching.

9. Answer: 42**. The idea.** Once the terms are small the sequence must eventually repeat, since it only takes finitely many values. Compute until a value repeats, find where the cycle begins and its length, then reduce 2026 modulo the cycle length.

Work it out. $a_1 = 2025$, $a_2 = 33$, $a_3 = 9 + 9 = 18$, $a_4 = 1 + 64 = 65$, $a_5 = 36 + 25 = 61$, $a_6 = 36 + 1 = 37$, $a_7 = 9 + 49 = 58$, $a_8 = 25 + 64 = 89$, $a_9 = 64 + 81 = 145$, $a_{10} = 1 + 16 + 25 = 42$, $a_{11} = 16 + 4 = 20$, $a_{12} = 4$, $a_{13} = 16$, $a_{14} = 1 + 36 = 37 = a_6$.

So from a_6 on the sequence cycles through $37, 58, 89, 145, 42, 20, 4, 16$ with period 8, and a_k depends on $(k - 6) \bmod 8$. Since $2026 - 6 = 2020 = 8 \cdot 252 + 4$, $a_{2026} = a_{6+4} = a_{10} = 42$.

Check. $a_{14} = a_6$, $a_{22} = a_6$, and in general $a_{6+8j} = 37$; $2022 = 6 + 8 \cdot 252$, so $a_{2022} = 37$ and four steps later $(58, 89, 145, 42)$ gives $a_{2026} = 42$.

10. Answer: 9**. The idea.** Both angles at E that we need come from isosceles triangles with apex at A or D: the square's side AG equals the pentagon's side AE, and the pentagon's sides $DE = DC$.

Work it out. Each interior angle of a regular pentagon is 108°.

Since $\angle BAG = 90^\circ$, $\angle EAG = 108^\circ - 90^\circ = 18^\circ$. Triangle AEG has $AE = AG$ (both equal the side length), so its base angles are $\frac{180^\circ - 18^\circ}{2} = 81^\circ$; thus $\angle AEG = 81^\circ$.

Triangle DEC has $DE = DC$ and apex angle 108°, so $\angle DEC = 36^\circ$ and $\angle AEC = 108^\circ - 36^\circ = 72^\circ$.

Both rays EC and EG lie inside angle AED, measured from EA at 72° and 81°, so $\angle CEG = 81^\circ - 72^\circ = 9^\circ$.

Check. From the other side, $\angle DEG = 108^\circ - 81^\circ = 27^\circ$, and $\angle DEC - \angle DEG = 36^\circ - 27^\circ = 9^\circ$. A coordinate computation agrees.

11. Answer: $\frac{12}{25}$. The idea. There are $5^5 = 3125$ equally likely ways to assign the people to floors. To stop at exactly 3 floors, first choose the 3 floors, then count the assignments of the 5 people onto those 3 floors that use every one of them, using inclusion–exclusion.

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Work it out. Choose the floors: $\binom{5}{3} = 10$ ways. Assignments of 5 people to 3 given floors: $3^5 = 243$. Those missing a particular floor: $2^5 = 32$, for each of 3 floors; those missing two particular floors: 1, for each of 3 pairs. So the assignments using all three floors number $243 - 3 \cdot 32 + 3 \cdot 1 = 150$. The favourable count is $10 \cdot 150 = 1500$, and the probability is $\frac{1500}{3125} = \frac{12}{25}$.

Check. Count every possible number of stops the same way: 1 stop, 5 ways; 2 stops, $\binom{5}{2}(2^5 - 2) = 300$; 3 stops, 1500; 4 stops, $\binom{5}{4}(4^5 - 4 \cdot 3^5 + 6 \cdot 2^5 - 4) = 5 \cdot 240 = 1200$; 5 stops, $5! = 120$. These add up to $5 + 300 + 1500 + 1200 + 120 = 3125 = 5^5$.

12. Answer: $\frac{1}{1155}$**. The idea.** A palindrome of odd length is determined by its middle letter and its first half. Every letter except the middle one appears in mirror pairs, so the one letter with an odd count must sit in the middle.

Work it out. The letters are M once, I four times, S four times and P twice. The number of distinct arrangements is $\frac{11!}{4!\,4!\,2!} = \frac{39916800}{1152} = 34650$.

In a palindrome the 6th letter is the middle and letters 1–5 mirror letters 7–11. Only M appears an odd number of times, so M is in the middle, and the first half uses half of each other letter: two I's, two S's and one P. These can be arranged in $\frac{5!}{2!\,2!\,1!} = 30$ ways, each giving exactly one palindrome. The probability is $\frac{30}{34650} = \frac{1}{1155}$.

Check. Think of placing the letters in two stages. The middle letter is M with probability $\frac{1}{11}$. Given that, the other 10 letters (four I's, four S's, two P's) are in one of $\frac{10!}{4!\,4!\,2!} = 3150$ equally likely arrangements, of which 30 are mirror-symmetric, a probability of $\frac{30}{3150} = \frac{1}{105}$. Multiplying, $\frac{1}{11} \cdot \frac{1}{105} = \frac{1}{1155}$.

13. Answer: 76**. The idea.** Write $N = 100a + 10b + c$ and let $M = 10a + c$ be the number left after deleting b. Compare N with $10M = 100a + 10c$: $N = 10M + (10b - 9c)$. So $M \mid N$ exactly when M divides the small number $v = 10b - 9c$, which lies between $-9c$ and $90 - 9c$. Write $v = kM$ and work out which multipliers k are possible.

Work it out. If $c = 0$: $M = 10a$ and $v = 10b$, so we need $a \mid b$. Every a allows $b = 0$ (9 numbers), and $b \in \{1, \ldots, 9\}$ gives $9 + 4 + 3 + 2 + 1 + 1 + 1 + 1 + 1 = 23$ more: 32 in all.

If $c \geq 1$: $v \neq 0$ (as $10b = 9c$ would force $c = 0$). Also $v \equiv c \pmod{10}$ and $kM \equiv kc \pmod{10}$, so $c(k - 1) \equiv 0 \pmod{10}$.

For $k = 1$: $10b - 9c = 10a + c$ gives $b = a + c$, possible for $1 \leq a \leq 9 - c$: that is $9 - c$ numbers, $8 + 7 + \cdots + 1 = 36$ over $c = 1, \ldots, 8$ (such as 121 and 891).

Other k must satisfy $-9c \leq k(10a + c) \leq 90 - 9c$, and $c(k - 1) \equiv 0 \pmod{10}$ needs $k \equiv 1 \pmod{10}$ for odd $c \neq 5$, $k \equiv 1 \pmod{5}$ for even c, and k odd for $c = 5$. The bounds leave only: $c = 2$, $k = 6$: $b = 6a + 3$, giving 192. $c = 8$, $k = -4$: $b = 4 - 4a$, giving 108. $c = 5$, $k = 3$: $b = 3a + 6$, giving 195; $k = -1$: $b = 4 - a$, giving 135, 225, 315, 405; $k = -3$: $b = 3 - 3a$, giving 105. That is 8 more.

The total is $32 + 36 + 8 = 76$.

Check. Test a few: $192 = 12 \cdot 16$, $108 = 18 \cdot 6$, $195 = 15 \cdot 13$, $405 = 45 \cdot 9$, $891 = 81 \cdot 11$. And a near miss: 981 trims to 91, but $981 = 10 \cdot 91 + 71$, and $91 \nmid 71$.

14. Answer: 85**. The idea.** Use the small primes to prune the search. A duo is never divisible by 4 or by 9. Among three consecutive integers, one or two are even, and every even duo has the form

$2p$, which is $\equiv 2 \pmod 4$; so there must be exactly one even number, in the middle, and n is odd with $n+1 \equiv 2 \pmod 4$. That means $n \equiv 1 \pmod 4$. Also one of the three is a multiple of 3, and it must not be a multiple of 9.

Work it out. Candidates after 33 with $n \equiv 1 \pmod 4$: 37 (prime, fails), 41 (prime), $45 = 9 \cdot 5$ (fails), $49 = 7^2$ (fails), 53 (prime), $57 = 3 \cdot 19$, $58 = 2 \cdot 29$, 59 prime (fails), 61 (prime), $65 = 5 \cdot 13$, $66 = 2 \cdot 3 \cdot 11$ (three primes, fails), $69 = 3 \cdot 23$, $70 = 2 \cdot 5 \cdot 7$ (fails), 73 (prime), $77 = 7 \cdot 11$, $78 = 2 \cdot 3 \cdot 13$ (fails), 81 (fails), $85 = 5 \cdot 17$, $86 = 2 \cdot 43$, $87 = 3 \cdot 29$. All three are duos, so the answer is 85.

Check. The first run fits the pattern too: $33 \equiv 1 \pmod 4$ and $34 = 2 \cdot 17$. A computer search over all n confirms 33 and 85 are the first two.

15. Answer: 2037**. The idea.** Everything is arithmetic modulo 7. First decide what kind of year can have exactly this pattern; then track the weekday of January 1 from year to year: a 365-day year moves it forward 1 day ($365 = 7 \cdot 52 + 1$) and a leap year moves it forward 2.

Work it out. February 13 and March 13 are on the same weekday only if February has 28 days (a multiple of 7), so the year must be a 365-day year. In such a year, February 13 is $31 \equiv 3 \pmod 7$ days after January 13, so Friday on February 13 forces January 13 to be a Tuesday and January 1, $12 \equiv 5$ days earlier, to be a Thursday. Conversely, a 365-day year starting on Thursday has exactly the same calendar as 2026, so the same Friday-the-13th months.

Now step forward from Thursday in 2026: 2027 Friday; 2028 Saturday; 2029 Monday (after the leap year); 2030 Tuesday; 2031 Wednesday; 2032 Thursday, but 2032 is a leap year, so it fails; 2033 Saturday; 2034 Sunday; 2035 Monday; 2036 Tuesday; 2037 Thursday, a 365-day year. The answer is 2037.

Check. From January 1, 2026 to January 1, 2037 there are 11 years containing the leap days of 2028, 2032 and 2036: $11 \cdot 365 + 3 = 4018 = 7 \cdot 574$ days, a whole number of weeks. In 2032, February 13 is a Friday but March 13 is a Saturday, which is the trap.

16. Answer: 20**. The idea.** Sort the edges by direction. The two faces perpendicular to the z-direction (top and bottom) together contain every x-edge and every y-edge exactly once, so they force (red x-edges) + (red y-edges) = 4. Doing this for all three directions shows there are exactly 2 red edges in each direction. Then each face condition becomes a condition on how those pairs are placed.

Work it out. Let r_x, r_y, r_z count the red edges in each direction. Top + bottom gives $r_x + r_y = 4$, and similarly $r_y + r_z = 4$ and $r_z + r_x = 4$, so $r_x = r_y = r_z = 2$. Now let u be the number of red x-edges on the face $y = 0$ (the other $2 - u$ are on $y = 1$), and let v be the number of red x-edges on the face $z = 0$. Face $y = 0$ holds only x- and z-edges, so it needs $2 - u$ red z-edges on it; likewise face $z = 0$ needs $2 - v$ red y-edges. Let w be the number of red y-edges on face $x = 0$; then face $x = 0$ needs $2 - w$ red z-edges. (The opposite faces then take care of themselves, since each direction has exactly 2 red edges.)

The four edges of one direction sit at the corners of a 2×2 square when seen end-on, and a pair of them is either on a common face (a side of the square) or not (a diagonal). A side pair has its two edges together in one coordinate (counts 0 or 2) and split in the other (count 1); a diagonal pair has count 1 in both coordinates, and there are 2 diagonals. So every pair needs at least one of its two counts equal to 1. The x-pair has counts (u, v), the y-pair $(w, 2 - v)$, the z-pair $(2 - w, 2 - u)$.

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If $u = v = w = 1$, all three pairs are diagonals: $2 \cdot 2 \cdot 2 = 8$ ways. If $u \neq 1$, the x-pair forces $v = 1$ and the z-pair forces $w = 1$; then $u \in \{0, 2\}$, the x- and z-pairs are determined sides and the y-pair is one of 2 diagonals: $2 \cdot 2 = 4$ ways. In the same way $v \neq 1$ gives 4 and $w \neq 1$ gives 4, and two of u, v, w can never both differ from 1. The total is $8 + 4 + 4 + 4 = 20$.

Check. The 20 colourings have recognisable shapes. The 8 all-diagonal ones: 4 are six-edge loops (the "equator" hexagons that avoid a pair of opposite corners, one per long diagonal) and 4 are two "claws" of three edges at a pair of opposite corners. The other 12 are pairs of three-edge paths. For a claw pair, each face through one of the two corners holds exactly two edges of that claw and none of the other, as required; and every one of the 20 uses exactly two edges in each direction.

17. Answer: $\frac{5}{27}$**. The idea.** If Alice's numbers differ by d, exactly $d - 1$ values lie strictly between them. So sort Alice's 36 ordered outcomes by their difference.

Work it out. A difference of $d \geq 1$ happens in $2(6-d)$ of Alice's ordered outcomes, and then Bob has $d-1$ good values. The good outcomes number $2{\cdot}4{\cdot}1{+}2{\cdot}3{\cdot}2{+}2{\cdot}2{\cdot}3{+}2{\cdot}1{\cdot}4 = 8{+}12{+}12{+}8 = 40$ (for $d = 2, 3, 4, 5$; $d = 0$ or 1 leaves no room). Out of $6^3 = 216$ equally likely outcomes, the probability is $\frac{40}{216} = \frac{5}{27}$.

Check. Bob can only be strictly between if all three dice differ, which happens in $6 \cdot 5 \cdot 4 = 120$ outcomes. By symmetry each of the three dice is equally likely to be the middle one, so Bob is in the middle in $\frac{120}{3} = 40$ outcomes.

18. Answer: $4\sqrt{2}$**. The idea.** The line $y = x$ already passes through $(0, 0)$, a point of the circle. A tangent line meets the circle only once, so the point of tangency must be the origin itself. Then the angle between the tangent and the chord to $(8, 0)$ equals the inscribed angle on that chord.

Work it out. By the tangent-chord angle, the chord from $(0, 0)$ to $(8, 0)$ makes a $45°$ angle with the tangent $y = x$, so it subtends an inscribed angle of $45°$ (a central angle of $90°$). A chord of length 8 under a central right angle is the hypotenuse of an isosceles right triangle with legs r, so $r\sqrt{2} = 8$ and $r = 4\sqrt{2}$.

Check. Coordinates: the center is $(4, k)$ with $r^2 = 16 + k^2$, and tangency to $y = x$ says $\frac{(4-k)^2}{2} = 16 + k^2$, which simplifies to $k^2 + 8k + 16 = 0$, i.e. $(k + 4)^2 = 0$. The double root $k = -4$ gives $r^2 = 32$, $r = 4\sqrt{2}$. The radius to the origin has slope $\frac{-4}{4} = -1$, perpendicular to $y = x$, confirming the tangency is at $(0, 0)$.

19. Answer: 112**. The idea.** The units digit in base b is the remainder on division by b, and the number of digits is fixed by which powers of b the number lies between.

Work it out. Units digit 1 means $2026 \equiv 1 \pmod{b}$, that is, b divides $2025 = 3^4 \cdot 5^2$. Exactly three digits means $b^2 \leq 2026 < b^3$. Since $12^3 = 1728 < 2026 < 2197 = 13^3$ and $45^2 = 2025 \leq 2026 < 46^2$, this is $13 \leq b \leq 45$. The divisors of 2025 are $1, 3, 5, 9, 15, 25, 27, 45, 75, 81, \ldots$, and those between 13 and 45 are $15, 25, 27, 45$. Their sum is $15 + 25 + 27 + 45 = 112$.

Check. In base 45: $2026 = 1 \cdot 45^2 + 0 \cdot 45 + 1$, written 101. In base 15: $2026 = 9 \cdot 225 + 0 \cdot 15 + 1$, written 901. Base 9 also ends in 1, but $9^3 = 729 \leq 2026$, so it needs four digits.

20. Answer: $\frac{12\sqrt{5}}{13}$**. The idea.** Joining P to the three vertices cuts the triangle into three smaller triangles whose heights are exactly the three distances. Their areas add up to the whole area,

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which fixes the scale of the ratio. Then, because the angle at C is a right angle, the distances to the two legs are just coordinates of P.

Work it out. Since $6^2+8^2=10^2$, the angle at C is a right angle and the area is $\frac{1}{2}\cdot 6\cdot 8=24$. Let the three distances be x, $2x$, $3x$. Triangles PBC, PCA, PAB fill triangle ABC, so $\frac{1}{2}(6\cdot x+8\cdot 2x+10\cdot 3x)=24$, that is, $26x=24$ and $x=\frac{12}{13}$. Put C at the origin with $A=(8,0)$ and $B=(0,6)$. The distance from P to line CA (the x-axis) is its y-coordinate and the distance to line BC (the y-axis) is its x-coordinate, so $P=\left(\frac{12}{13},\frac{24}{13}\right)$ and $PC=\sqrt{x^2+(2x)^2}=x\sqrt{5}=\frac{12\sqrt{5}}{13}$.

Check. Line AB is $6x+8y=48$, and the distance from P to it is $\frac{48-6\cdot\frac{12}{13}-8\cdot\frac{24}{13}}{10}=\frac{624-72-192}{130}=\frac{360}{130}=\frac{36}{13}$, which is $3\cdot\frac{12}{13}$ as required. Also $6\cdot\frac{12}{13}+8\cdot\frac{24}{13}=\frac{264}{13}<48$, so P really is inside the triangle.

21. Answer: 169**. The idea.** With n terms, first term a and difference d, the sum is $\frac{n(2a+(n-1)d)}{2}$. So $n\big(2a+(n-1)d\big)=4052=2^2\cdot 1013$, and 1013 is prime. That leaves almost no choice for n, and the size of the second factor rules out the big ones.

Work it out. Since $n\geq 3$ divides $4\cdot 1013$, n is 4, 1013, 2026 or 4052. The second factor $2a+(n-1)d$ is at least $2+(n-1)=n+1$, so $n(n+1)\leq 4052$, giving $n\leq 63$. Hence $n=4$, and $2a+3d=1013$. For this to have a positive integer a, $3d$ must be odd and at most 1011: d is odd with $1\leq d\leq 337$. There are $\frac{337+1}{2}=169$ such d, each giving exactly one sequence.

Check. $d=1$ gives $a=505$: $505+506+507+508=2026$. $d=337$ gives $a=1$: $1+338+675+1012=2026$. And $d=339$ would need $2a=1013-1017<0$, so 337 really is the largest.

22. Answer: 324**. The idea.** The sum is $\frac{n(n+1)}{2}$, and $2025=3^4\cdot 5^2=81\cdot 25$ is odd, so we need $81\cdot 25$ to divide the product $n(n+1)$. Two consecutive integers share no factor, so all of the 81 goes into one of them and all of the 25 goes into one of them. That leaves four cases.

Work it out. Since 2025 is odd, 2025 divides $\frac{n(n+1)}{2}$ exactly when it divides $n(n+1)$. If 81 and 25 both divide the same one of n, $n+1$, then that number is at least 2025, so $n\geq 2024$. *Case* $81\mid n$*,* $25\mid n+1$*:* try $n=81,162,243,324$; then $n+1=82,163,244,325$, and $325=13\cdot 25$ is the first multiple of 25. So $n=324$ works. *Case* $25\mid n$*,* $81\mid n+1$*:* $n=81k-1$ must end in 00, 25, 50 or 75; $81k\equiv 1 \pmod{25}$ means $6k\equiv 1 \pmod{25}$, so $k\equiv 21$ and the least is $n=81\cdot 21-1=1700$. The smallest of all the cases is $n=324$.

Check. $1+2+\cdots+324=\frac{324\cdot 325}{2}=162\cdot 325=52650=2025\cdot 26$. For any smaller n, neither n nor $n+1$ reaches 2025, so one of them must be a multiple of 81 and the other a multiple of 25, and the case work shows the first time that happens is at 324.

23. Answer: 144**. The idea.** If $N=p_1^{e_1}\cdots p_k^{e_k}$, then N has $\prod(e_i+1)$ divisors and N^2 has $\prod(2e_i+1)$. So we need $\prod\frac{2e_i+1}{e_i+1}=3$. Each factor is at least $\frac{3}{2}$ and less than 2, which pins down how many primes there are.

Work it out. One prime is impossible, since a single factor is less than 2. Three or more primes are impossible, since the product would be at least $\left(\frac{3}{2}\right)^3=\frac{27}{8}>3$. So N has exactly two prime factors, and $(2a+1)(2b+1)=3(a+1)(b+1)$. Try $a=1,2,3,\ldots$: $a=1$ gives $6b+3=6b+6$ (no); $a=2$ gives $10b+5=9b+9$, so $b=4$; $a=3$ gives $14b+7=12b+12$ (no); $a=4$ gives $b=2$; beyond that, $\frac{2a+1}{a+1}\geq\frac{11}{6}$ would need $\frac{2b+1}{b+1}\leq\frac{18}{11}$, which forces $b=1$, and $b=1$ fails

just as $a = 1$ did. So the exponents are 2 and 4. The smallest choice puts the 4 on the prime 2: $N = 2^4 \cdot 3^2 = 144$.

Check. 144 has $5 \cdot 3 = 15$ divisors and $144^2 = 2^8 \cdot 3^4$ has $9 \cdot 5 = 45 = 3 \cdot 15$. The other arrangement $2^2 \cdot 3^4 = 324$ is larger.

24. Answer: 15**. The idea.** Two facts meet at the chord TU. First, if T' is the point of the circle opposite T, then TT' is a diameter perpendicular to ℓ and angle TUT' is right, so projecting U onto TT' gives $TU^2 = TT' \cdot (\text{height of } U) = 2r \cdot h$. Second, $PTOU$ is a kite (with O the center), so TU is twice the altitude from T in right triangle PTO.

Work it out. With $r = 5$ and $h = 9$: $TU^2 = 10 \cdot 9 = 90$. Let $x = PT$. In right triangle PTO (legs x and 5, hypotenuse $\sqrt{x^2+25}$), the altitude from T is $\frac{5x}{\sqrt{x^2+25}}$, and the diagonal OP of the kite bisects TU at right angles, so $TU = \frac{10x}{\sqrt{x^2+25}}$. Then $\frac{100x^2}{x^2+25} = 90$, so $100x^2 = 90x^2 + 2250$, $x^2 = 225$ and $PT = 15$.

Check. With $PT = 15$, the angle $\alpha = \angle TPO$ has $\tan\alpha = \frac{5}{15} = \frac{1}{3}$, so $\sin 2\alpha = \frac{2 \cdot \frac{1}{3}}{1+\frac{1}{9}} = \frac{3}{5}$. Since $PU = PT = 15$ and PU makes angle 2α with ℓ, the height of U is $15 \cdot \frac{3}{5} = 9$. As P moves away from T the height of U grows steadily from 0 toward 10, so 15 is the only answer.

25. Answer: $\frac{117}{8}$**. The idea.** The fold line is the perpendicular bisector of $\overline{AB}$. The flap that moves lands so that it completely covers the small triangle on B's side, so the covered area is the whole triangle minus that small triangle.

Work it out. $AB = 10$; let M be its midpoint, so $BM = 5$. The fold line passes through M perpendicular to AB and meets the longer leg CB at P. Triangles BMP and BCA both have a right angle and share angle B, so $\frac{MP}{BM} = \frac{CA}{CB} = \frac{3}{4}$, giving $MP = \frac{15}{4}$ and $[BMP] = \frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$. Also $BP = \frac{25}{4} < 8$, so P really lies on side CB.

The triangle BMP stays still. The other piece, quadrilateral $AMPC$, flips over the line MP. Inside it lies triangle AMP, the mirror image of BMP, so after the fold AMP lies exactly on top of BMP. The rest of the flap lands outside BMP. So the shape covers the area of the flap alone: $[AMPC] = 24 - \frac{75}{8} = \frac{117}{8}$.

Check. Split $AMPC$ along AP: $[APC] = \frac{1}{2} \cdot CP \cdot CA = \frac{1}{2} \cdot \frac{7}{4} \cdot 6 = \frac{21}{4}$ and $[AMP] = \frac{75}{8}$, which total $\frac{42+75}{8} = \frac{117}{8}$.

26. Answer: 25920**. The idea.** Only the positions of the odd numbers matter for the sums' parities. A line (row or column) has an odd sum exactly when it holds an odd count of odd numbers, 1 or 3. There are five odd numbers, and that forces their positions into a very rigid shape.

Work it out. The odd numbers are $1, 3, 5, 7, 9$. The three rows hold 1 or 3 of them each and 5 in total, so one row is entirely odd and the other two rows hold one odd number each. The same holds for columns: one column is entirely odd. A full row and a full column together already use $3 + 3 - 1 = 5$ cells, so the odd numbers fill exactly a "cross": one whole row plus one whole column. Every cross works (the full row has 3 odds, every other row has 1, and likewise for columns). There are $3 \cdot 3 = 9$ crosses. For each, the 5 odd numbers fill the cross in $5! = 120$ ways and the 4 even numbers fill the rest in $4! = 24$ ways, so the count is $9 \cdot 120 \cdot 24 = 25920$.

Check. Count with rows alone first: choose the all-odd row (3 ways) and one cell in each other row ($3 \cdot 3$ ways), giving 27 patterns and $27 \cdot 2880 = 77760$ placements. The columns are then

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all odd exactly when the two single odd cells sit in the same column, which happens in 3 of the 9 choices, one third of the time. And $\frac{77760}{3} = 25920$.

27. Answer: $\frac{91}{34}$**. The idea.** Being cyclic forces right angles at B and D, and then the diagonal AC is a diameter. The inscribed circle's center is on AC too (by symmetry), where the bisector of angle B crosses it, so the angle bisector theorem places it.

Work it out. The kite is symmetric about AC, so triangles ABC and ADC are congruent and $\angle B = \angle D$. In a cyclic quadrilateral opposite angles add to 180°, so $\angle B = \angle D = 90°$. Then $AC = \sqrt{5^2 + 12^2} = 13$ is a diameter, and the circumcenter O is the midpoint of AC: $AO = \frac{13}{2}$.

The incenter I lies on the symmetry line AC. It is also equidistant from BA and BC, so it lies on the bisector of angle ABC. In triangle ABC that bisector cuts AC in the ratio $AI : IC = BA : BC = 5 : 12$, so $AI = \frac{5}{17} \cdot 13 = \frac{65}{17}$. Hence $OI = \frac{13}{2} - \frac{65}{17} = \frac{221-130}{34} = \frac{91}{34}$.

Check. The inradius from the area is $r = \frac{\text{area}}{\text{semiperimeter}} = \frac{60}{17}$. From I: $r = AI \sin \angle BAC = \frac{65}{17} \cdot \frac{12}{13} = \frac{60}{17}$. The two agree.

28. Answer: 1285**. The idea.** Reduce mod 7. The palindrome is $1000001a + 100010b + 10100c + 1000d$, and since $10^6 \equiv 1$, $10^5 \equiv 5$, $10^4 \equiv 4$, $10^3 \equiv 6$, $10^2 \equiv 2$, $10 \equiv 3 \pmod 7$, it is $\equiv 2a + b + 6c + 6d \pmod 7$. For each choice of a, b, c this fixes d modulo 7, and a residue class has two digits (0/7, 1/8, 2/9) or one. So the answer is 900 plus the number of triples whose required d is 0, 1 or 2 (mod 7).

Work it out. $6d \equiv -(2a + b + 6c)$, and multiplying by 6 (its own inverse, as $36 \equiv 1$) gives $d \equiv -6(2a + b + 6c) \equiv 2a + b - c \pmod 7$. Count triples with $2a + b - c \bmod 7 \in \{0, 1, 2\}$. Split each digit's range into a full set of residues plus leftovers: $a = 1, \ldots, 7$ plus $a \in \{8, 9\}$ (giving $2a \equiv 2, 4$); $b = 0, \ldots, 6$ plus $b \in \{7, 8, 9\}$ ($b \equiv 0, 1, 2$); c likewise ($-c \equiv 0, 6, 5$). Any product that uses a full set of residues for at least one digit is spread evenly over the 7 residues. Only the $2 \cdot 3 \cdot 3 = 18$ all-leftover triples are not, so the other $900 - 18 = 882$ triples put 126 in each residue, 378 in $\{0, 1, 2\}$. For the leftovers, $b - c$ takes the values $-2, -1, 0, 1, 2$ in $1, 2, 3, 2, 1$ ways. Adding 2 gives $0, 1, 2, 3, 4$: $1 + 2 + 3 = 6$ land in $\{0, 1, 2\}$. Adding 4 gives $2, 3, 4, 5, 6$: just 1. So $378 + 7 = 385$ triples get two choices of d, and the count is $900 + 385 = 1285$.

Check. The example $3014103 = 7 \cdot 430586 + 1$ is not a multiple of 7, and indeed $2a + b - c = 6 + 0 - 1 = 5 \not\equiv d = 4$. Changing d to 5 gives $3015103 = 7 \cdot 430729$. As a sanity check on size, about one number in seven should work, and $\frac{9000}{7} \approx 1285.7$.

29. Answer: 520**. The idea.** The forbidden pairs are chairs two apart. Relist the chairs by jumping two at a time: $1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12$, and then back to 1. Because 13 is odd, this single loop visits every chair, and two chairs are two apart around the table exactly when they are neighbours in the new loop. So the question becomes: how many sets of points on a 13-point circle contain no two neighbours?

Work it out. First count sets with no two neighbours in a *row* of n points; call it a_n. Either the last point is unused (a_{n-1} sets) or it is used and the one before it is not (a_{n-2} sets), so $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2, a_2 = 3$: the sequence runs $2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377$ for $n = 1, \ldots, 12$. On the 13-point circle, look at one point. If it is not chosen, the other 12 form a row: $a_{12} = 377$ sets. If it is chosen, its two neighbours are not, and the remaining 10 form a row: $a_{10} = 144$ sets. That is $377 + 144 = 521$ sets, including the empty set, so the answer is $521 - 1 = 520$.

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Check. Try 5 chairs, where the jumping loop is $1, 3, 5, 2, 4$. The rule gives $a_4 + a_2 = 8 + 3 = 11$ sets including the empty one; directly, there are the empty set, 5 single chairs and the 5 pairs of neighbouring chairs (which have no chair between them), again 11. Oddness matters: with 12 chairs the jumps split the table into two separate 6-point loops (odd chairs and even chairs), and the count would be $(a_5 + a_3)^2 - 1 = 18^2 - 1 = 323$ instead.

30. Answer: 175**. The idea.** In a valid filling, the cells holding 1 form a "staircase" in the top-left corner, and the cells holding 1 or 2 form a larger staircase containing it. The boundary of each staircase is a lattice path of 6 unit steps (up or right) from the bottom-left corner of the grid to the top-right corner. So we count pairs of paths in which the first never goes below or to the right of the second. Count all pairs, then subtract the bad pairs with the tail-swapping trick.

Work it out. Put the grid in $[0, 3] \times [0, 3]$. A filling is the same as two up/right paths P (the boundary of the 1s) and Q (the boundary of the numbers at most 2), each from $(0, 0)$ to $(3, 3)$, with P weakly above and left of Q. Each path is one of $\binom{6}{3} = 20$, so there are 400 pairs in all. Now shift P up by 1 and left by 1, calling the result P', a path from $(-1, 1)$ to $(2, 4)$. P is weakly above-left of Q exactly when P' and Q never touch. For a bad pair, find the first lattice point where P' and Q meet and swap the parts after it. This gives a path from $(-1, 1)$ to $(3, 3)$ and a path from $(0, 0)$ to $(2, 4)$. The swap can be undone the same way, so it is a one-to-one matching. Every such crossed pair must meet, so the bad pairs number $\binom{6}{2} \cdot \binom{6}{2} = 15 \cdot 15 = 225$ (the first path has 4 rights and 2 ups; the second has 2 rights and 4 ups). The answer is $400 - 225 = 175$.

Check. Count row by row instead. For each row record (a, b) = (number of 1s, number of entries at most 2), with $0 \le a \le b \le 3$: there are 10 such pairs. Going down the rows, a and b can only stay the same or shrink. Let $g(s)$ be the number of states $t \le s$ (both coordinates), which counts the choices for the row below a row in state s. Then g is $1, 2, 3, 3, 5, 6, 4, 7, 9, 10$ at $(0, 0), (0, 1), (1, 1), (0, 2), (1, 2), (2, 2), (0, 3), (1, 3), (2, 3), (3, 3)$. For a top-row state s, the number of ways to finish the two rows below is the sum of $g(t)$ over the states $t \le s$. This gives $1, 3, 6, 6, 14, 20, 10, 25, 40, 50$, and the total is 175.

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