MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A list of consecutive integers has this property: the mean of the squares of the integers is exactly 10 more than the square of the mean of the integers. How many integers are in the list?

1. _____

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2. How many two-digit positive integers are divisible by the sum of their own digits? (For example, 12 counts, because $1 + 2 = 3$ divides 12.)

2. _____

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3. The numbers 1 through 8 are written at the 8 vertices of a cube, one number at each vertex. The cube is fixed in place, so two arrangements that differ by a rotation count as different. The arrangement must meet two conditions: the two numbers at the ends of every edge have an odd sum, and the four numbers at the corners of every face have a sum of 18. How many arrangements meet both conditions?

3. _____

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4. Three jars, A, B and C, hold a total of 48 liters of water. First, water is poured from A into B until B holds twice as much as it held before. Next, water is poured from B into C until C holds twice as much as it held before. Finally, water is poured from C into A until A holds twice as much as it held before. After these three pours, every jar holds 16 liters. How many liters did jar A hold at the very start?

4. _____

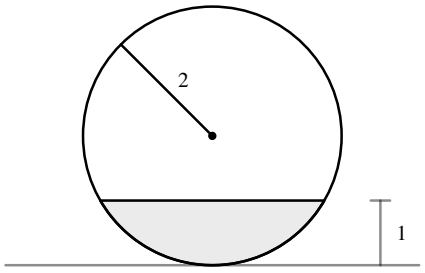
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5. How many positive integers less than 3000 can be written as $3^a + 3^b$, where a and b are integers with $a > b \geq 0$, and are also multiples of 4?

5. _____

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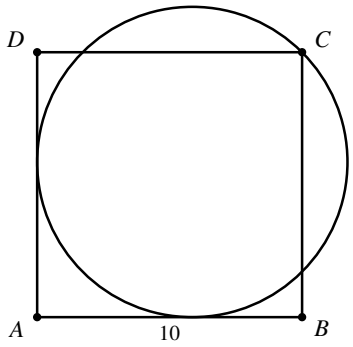
6. A closed cylindrical tank has radius 2 meters and length 5 meters. It lies on its side on level ground, so its two circular ends stand vertically. Water fills the tank to a depth of 1 meter. How many cubic meters of water are in the tank? Express your answer in simplest radical form in terms of π.



6. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

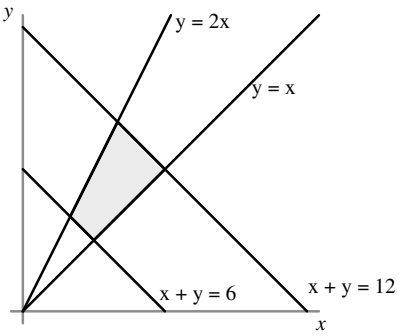
7. Square $ABCD$ has side length 10. A circle is tangent to side $\overline{AB}$ and to side $\overline{AD}$, touching each of them at a point of the side itself, and it passes through vertex C. What is the radius of the circle? Express your answer in simplest radical form.



7. _____

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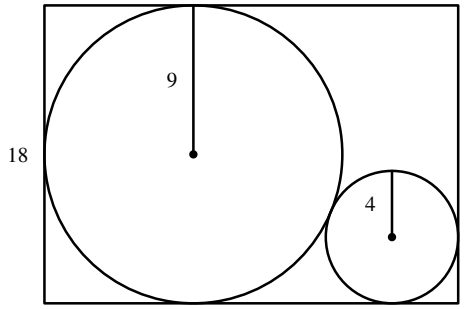
8. In the coordinate plane, the four lines $y = x$, $y = 2x$, $x + y = 6$ and $x + y = 12$ enclose a quadrilateral. What is its area?



8. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

9. Two circular coins lie flat inside a rectangular tray of height 18 (the length of its left and right sides). The large coin has radius 9 and touches the top, bottom and left sides of the tray. The small coin has radius 4 and touches the bottom and right sides of the tray. The two coins touch each other. What is the area of the tray?



9. _____

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10. A real number x has the property that its square is exactly 1 more than the number itself, that is, $x^2 = x + 1$. (There are two such numbers, and the answer is the same for both.) What is the value of $x^{10} - 55x$? 10. _____

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11. How many nonempty subsets of $\{1, 2, 3, \ldots, 10\}$ have the property that their smallest element plus their largest element equals 11? 11. _____

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12. Ann rolls two fair six-sided dice and keeps the larger of her two numbers (if they are equal, she keeps that number). Ben does the same with his own two fair dice. What is the probability that Ann's kept number is strictly greater than Ben's? Express your answer as a common fraction. 12. _____

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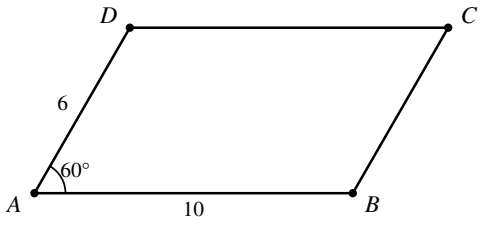
13. A digital clock shows the time in 24-hour form as HH:MM, from 00:00 up to 23:59, always using four digits (so seven minutes past three in the morning is shown as 03:07). For how many of the 1440 minutes in a day are the four digits shown, read from left to right, strictly increasing? (For example, 01:48 and 12:34 count, but 13:35 does not.) 13. _____

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14. What is the least positive multiple of 72 whose decimal digits are all 4s and 0s? (Both digits need not appear.) 14. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

15. Parallelogram $ABCD$ has $AB = 10$, $AD = 6$ and $\angle DAB = 60°$. The bisectors of its four interior angles are drawn, and together they enclose a quadrilateral. What is the area of that quadrilateral? Express your answer in simplest radical form.



15. _____

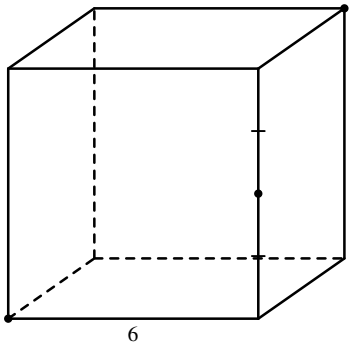
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16. Three different squares are chosen from a 6×6 board. How many such sets of three squares lie all in the same row, all in the same column, or all on the same diagonal? (A diagonal is a line of squares running at 45° from corner to corner of consecutive squares, in either direction, and it may have any length from 1 to 6 squares. The order in which the squares are chosen does not matter.)

16. _____

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17. A solid cube has edge length 6. A flat saw cut passes through two opposite corners of the cube (the two ends of one of its space diagonals) and also through the midpoint of an edge that touches neither of those two corners. The cut splits the cube into two pieces. What is the area of the cut surface? Express your answer in simplest radical form.



17. _____

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18. A theatre sells children's tickets for \$3 each and adult tickets for \$5 each. One day it sold at least one ticket of each kind and took in a whole number T of dollars. For some values of T, there are exactly two possibilities for the pair (number of children's tickets, number of adult tickets) sold that day. How many different values of T have this property?

18. _____

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19. A bag holds 12 balls numbered 1 through 12. Three balls are drawn at random without replacement, and every set of three balls is equally likely. What is the probability that the sum of the three numbers drawn is divisible by 3? Express your answer as a common fraction.

19. _____

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20. A positive integer n has exactly 6 positive divisors that are odd and exactly 12 positive divisors that are even. (For example, $n = 180$ works: its odd divisors are $1, 3, 5, 9, 15, 45$.) How many positive integers n less than 1000 have this property?

20. _____

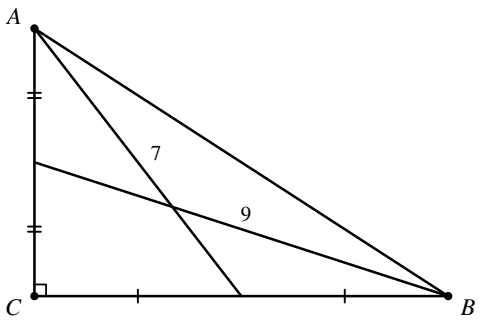
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21. Let $f(x) = x^2 - 1$. Apply f ten times in a row: start with x, replace it by $f(x)$, then by $f(f(x))$, and so on, ten times in all. For how many real numbers x is the result of the ten applications equal to 0?

21. _____

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22. Triangle ABC has a right angle at C. The median from A (the segment from A to the midpoint of $\overline{BC}$) has length 7, and the median from B (the segment from B to the midpoint of $\overline{CA}$) has length 9. What is the length of the median from C to the midpoint of the hypotenuse $\overline{AB}$? Express your answer in simplest radical form.



22. _____

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23. A marching band of n players can form a rectangle of r rows with c players in each row, using every player. Here $r \times c$ and $c \times r$ count as different formations, and the single-row and single-column formations count too. The band has exactly 12 possible formations. If the band doubled in size it would have exactly 16 possible formations, and if instead it tripled in size it would have exactly 15. What is n?

23. _____

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24. A 1×10 strip is to be covered completely, without overlaps, by tiles of three kinds: red 1×1 tiles, blue 1×1 tiles and green 1×2 tiles. There are plenty of tiles of each kind. No two tiles of the same colour may share an edge. (For example, a 1×3 strip can be covered in exactly 6 ways.) In how many ways can the 1×10 strip be covered?

24. _____

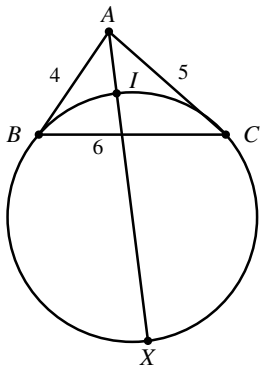
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25. The seven cards numbered 1 through 7 are shuffled, with all orders equally likely, and turned face up one at a time. A card is called a *new high* if its number is larger than the number on every card turned up before it, so the first card is always a new high. What is the probability that exactly 3 of the seven cards are new highs? Express your answer as a common fraction.

25. _____

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26. Triangle ABC has $AB = 4$, $AC = 5$ and $BC = 6$. Let I be the center of the circle inscribed in the triangle. The circle passing through B, C and I meets line AI at a second point X (other than I). What is the length AX?



26. _____

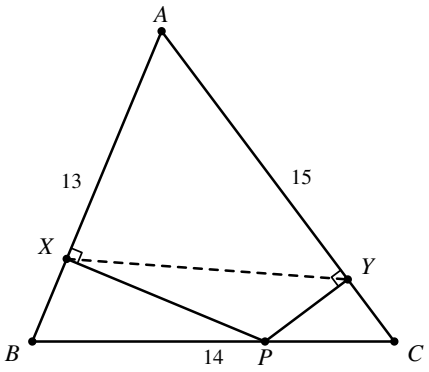
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27. An 8×8 checkerboard is colored in the usual way, with squares alternating black and white along every row and column. How many rectangles made up of whole squares of the board, with sides along the grid lines, contain exactly 4 black squares? (Rectangles of every size and position count, including $1 \times k$ strips.)

27. _____

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28. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. A point P moves along side $\overline{BC}$. Let X be the foot of the perpendicular from P to line AB, and let Y be the foot of the perpendicular from P to line AC. What is the least possible length of $\overline{XY}$? Express your answer as a common fraction.



28. _____

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29. The square of 3 is 9, and $9997^2 = 99940009$, so both 3^2 and 9997^2 have 0009 as their last four digits. What is the smallest integer n greater than 3 whose square n^2 has 0009 as its last four digits?

29. _____

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30. Seven runners wearing bibs numbered 1 through 7 line up in a single row for a photograph. The photographer insists that no two runners whose bib numbers differ by exactly 1 stand next to each other (so 3 may not stand beside 2 or 4, for example). In how many different orders can the seven runners line up?

30. _____

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Answer Key — MathCounts Nationals Sprint Practice Contest 2

1. 11

2. 23

3. 48

4. 22

5. 16

6. $\frac{20\pi}{3} - 5\sqrt{3}$

7. $20 - 10\sqrt{2}$

8. 9

9. 450

10. 34

11. 341

12. $\frac{505}{1296}$

13. 87

14. 4444444440

15. $4\sqrt{3}$

16. 340

17. $18\sqrt{6}$

18. 15

19. $\frac{19}{55}$

20. 12

21. 11

22. $\sqrt{26}$

23. 108

24. 232

25. $\frac{29}{90}$

26. 10

27. 120

28. $\frac{672}{65}$

29. 1253

30. 646

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — MathCounts Nationals Sprint Practice Contest 2

1. Answer: 11**. The idea.** The gap between the mean of the squares and the square of the mean does not change if every number is shifted by the same amount. So slide the list until its mean is 0; then the square of the mean is 0 and the gap is just the mean of the squares of the centred list.

Work it out. Say there are n integers with mean m, and write them as $m + d$. The mean of $(m+d)^2 = m^2 + 2md + d^2$ is $m^2 + 0 +$ (mean of d^2), because the offsets d average to 0. So the gap is the mean of d^2. If n is even, the offsets are $\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm\frac{n-1}{2}$, and the mean of their squares is $\frac{n^2-1}{12}$. That has an odd numerator, so it can never be the whole number 10. If $n = 2k + 1$ is odd, the offsets are $-k, \ldots, k$, and the mean of their squares is $\frac{2(1^2+2^2+\cdots+k^2)}{2k+1} = \frac{k(k+1)}{3}$. Setting $\frac{k(k+1)}{3} = 10$ gives $k(k + 1) = 30$, so $k = 5$ and $n = 11$.

Check. Take $-5, -4, \ldots, 5$: the mean is 0, and the mean of the squares is $\frac{2(1+4+9+16+25)}{11} = \frac{110}{11} = 10$. Shifting to, say, $1, 2, \ldots, 11$: the mean is 6, and $\frac{1^2+\cdots+11^2}{11} = \frac{506}{11} = 46 = 36 + 10$. The list has **11** integers.

2. Answer: 23**. The idea.** Write the number as $10a + b$, with tens digit a and digit sum $s = a + b$. Then $10a + b = 9a + s$. So s divides the number exactly when s divides $9a$. For each tens digit this is a short list of divisors to check, instead of 90 separate divisions.

Work it out. For a fixed a, the digit sum s runs from a (when $b = 0$) to $a + 9$ (when $b = 9$). Count the divisors of $9a$ in that range. $a = 1$: divisors of 9 in $[1, 10]$ are $1, 3, 9$, giving $10, 12, 18$: 3. $a = 2$: divisors of 18 in $[2, 11]$ are $2, 3, 6, 9$: 4. $a = 3$: divisors of 27 in $[3, 12]$ are $3, 9$: 2. $a = 4$: divisors of 36 in $[4, 13]$ are $4, 6, 9, 12$: 4. $a = 5$: divisors of 45 in $[5, 14]$ are $5, 9$: 2. $a = 6$: divisors of 54 in $[6, 15]$ are $6, 9$: 2. $a = 7$: divisors of 63 in $[7, 16]$ are $7, 9$: 2. $a = 8$: divisors of 72 in $[8, 17]$ are $8, 9, 12$: 3. $a = 9$: divisors of 81 in $[9, 18]$ are just 9: 1. The total is $3 + 4 + 2 + 4 + 2 + 2 + 2 + 3 + 1 = 23$.

Check. The list is $10, 12, 18, 20, 21, 24, 27, 30, 36, 40, 42, 45, 48, 50, 54, 60, 63, 70, 72, 80, 81, 84, 90$. Spot checks: $48 = 4 \cdot 12$ and $84 = 7 \cdot 12$, while 11 and 99 fail ($99/18$ is not whole). There are **23**.

3. Answer: 48**. The idea.** Colour the vertices of the cube in two alternating classes of four, so every edge joins the two classes (each class is a regular tetrahedron whose edges are face diagonals). An odd edge sum means neighbours have opposite parity, so one class holds all the odd numbers and the other all the even numbers. Then the face condition turns out to force every even number once the odd numbers are placed.

Work it out. Choose which class is odd (2 ways) and place $1, 3, 5, 7$ in it ($4! = 24$ ways). For an odd number x, let $g(x)$ be the even number at the vertex opposite (antipodal to) x; opposite vertices are always in different classes. A face contains two odd vertices, holding some a and b. Its two even vertices are opposite the two odd vertices of the opposite face, which hold the other two odd numbers e and f. So the face condition reads $a + b + g(e) + g(f) = 18$. Since $a + b = 16 - e - f$, this is $g(e) + g(f) = e + f + 2$. Each of the 6 pairs of odd numbers is the pair on exactly one face, so with $h(x) = g(x) - x$ we need $h(e) + h(f) = 2$ for every pair. From $h(1) + h(3) = h(1) + h(5) = h(3) + h(5) = 2$, all these values equal 1, and then $h(7) = 1$ too. So $g(x) = x + 1$: the evens $2, 4, 6, 8$ go opposite $1, 3, 5, 7$ respectively, with no freedom. That gives $2 \cdot 24 = 48$ arrangements.

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Check. Put $1, 3, 5, 7$ at $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ and $2, 4, 6, 8$ at their opposites $(1,1,1)$, $(1,0,0)$, $(0,1,0$ The face $x = 0$ holds $1, 3, 6, 8$ and the face $x = 1$ holds $5, 7, 2, 4$, both summing to 18, and the other faces work the same way. Without the face condition there would be $2 \cdot 4! \cdot 4! = 1152$ arrangements, and the face condition keeps 1 of the 24 even placements for each. The answer is **48**.

4. Answer: 22**. The idea.** Work backwards. Each pour doubles the receiving jar, so undoing it halves that jar and gives the same amount back to the jar that poured.

Work it out. The end state is $(A, B, C) = (16, 16, 16)$. Undo the pour $C \to A$: A had 8, so C gave 8 and held 24. Before it: $(8, 16, 24)$. Undo the pour $B \to C$: C had 12, so B held $16 + 12 = 28$. Before it: $(8, 28, 12)$. Undo the pour $A \to B$: B had 14, so A held $8 + 14 = 22$. The start is $(22, 14, 12)$.

Check. Run forwards. A gives 14 to B: $(8, 28, 12)$. B gives 12 to C: $(8, 16, 24)$. C gives 8 to A: $(16, 16, 16)$. The total stays 48 throughout. Jar A started with **22** liters.

5. Answer: 16**. The idea.** Factor out the smaller power: $3^a + 3^b = 3^b\left(3^{a-b} + 1\right)$. The factor 3^b is odd, so divisibility by 4 depends only on the gap $k = a - b$. Since $3 \equiv -1 \pmod 4$, we get $3^k + 1 \equiv (-1)^k + 1 \pmod 4$, which is 0 when k is odd and 2 when k is even. So we need the gap $a - b$ to be odd.

Work it out. Different pairs (a, b) give different numbers (in base 3 the number is written with two 1s in positions a and b), so we just count pairs. Since $3^7 = 2187$ and $3^8 > 3000$, we need $a \le 7$. For $a \le 6$ every sum is at most $729+243 = 972 < 3000$, and the number of $b < a$ with $a-b$ odd is $1, 1, 2, 2, 3, 3$ for $a = 1, \ldots, 6$: that is 12 pairs. For $a = 7$ we need $3^b < 3000 - 2187 = 813$, so $b \le 6$, and b must be even: $b = 0, 2, 4, 6$, giving 4 more. The total is $12 + 4 = 16$.

Check. Small cases: $3 + 1 = 4$ works (gap 1), $9 + 1 = 10$ does not (gap 2), $27 + 1 = 28 = 4 \cdot 7$ works (gap 3). The largest one is $2187 + 729 = 2916 = 4 \cdot 729$. The count is **16**.

6. Answer: $\frac{20\pi}{3} - 5\sqrt{3}$**. The idea.** Every vertical slice across the tank looks the same, so the volume is the length times the area of the water's cross-section. That cross-section is a circular segment: the part of a circle of radius 2 below a chord at depth 1. A segment is a sector minus a triangle.

Work it out. Let O be the centre of the circular end. The water surface is a chord PQ at distance $2 - 1 = 1$ below O. In the right triangle formed by O, the midpoint of PQ and P, the hypotenuse is 2 and one leg is 1, so the angle at O is 60° and $\angle POQ = 120^\circ$. The sector POQ has area $\frac{120}{360} \cdot \pi \cdot 2^2 = \frac{4\pi}{3}$. Triangle POQ has area $\frac{1}{2} \cdot 2 \cdot 2 \cdot \sin 120^\circ = \sqrt{3}$. The segment has area $\frac{4\pi}{3} - \sqrt{3}$, and the volume is $5\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{20\pi}{3} - 5\sqrt{3}$.

Check. Numerically this is about $20.94 - 8.66 = 12.28$ cubic meters out of a full $20\pi \approx 62.83$, roughly 20%. That is sensible: the depth is a quarter of the diameter, and the bottom of a lying cylinder is narrow, so it holds less than a quarter of the volume. The answer is $\boldsymbol{\frac{20\pi}{3} - 5\sqrt{3}}$.

7. Answer: $20 - 10\sqrt{2}$**. The idea.** A circle tangent to both sides of the right angle at A has its centre at equal distance r from both sides, so the centre lies on the diagonal $\overline{AC}$. Its distance from C must also be r, and distances along the diagonal are $\sqrt{2}$ times the difference in coordinates.

Work it out. Put $A = (0, 0)$, $B = (10, 0)$, $D = (0, 10)$ and $C = (10, 10)$. The centre is (r, r) and touches $\overline{AB}$ at $(r, 0)$ and $\overline{AD}$ at $(0, r)$. Its distance to C is $\sqrt{2}\,|10 - r|$. For the tangent points to

lie on the sides we need $r \le 10$, so $(10-r)\sqrt{2} = r$, which gives $r = \frac{10\sqrt{2}}{1+\sqrt{2}} = 10\sqrt{2}(\sqrt{2}-1) = 20 - 10\sqrt{2}$.

Check. $r \approx 5.86$, and the centre $(5.86, 5.86)$ is $\sqrt{2} \cdot 4.14 \approx 5.86$ from C. The other root of $2(10-r)^2 = r^2$, namely $r = 20+10\sqrt{2} \approx 34.1$, is tangent to the lines AB and AD only far beyond the square, which the statement rules out. The radius is $\mathbf{20 - 10\sqrt{2}}$.

8. Answer: 9**. The idea.** The lines $y = x$ and $y = 2x$ form a wedge at the origin. Each line $x+y = c$ cuts off a triangle from that wedge, and these triangles are all similar, with sides proportional to c. So their areas are proportional to c^2, and the quadrilateral is the difference of two of them.

Work it out. The line $x + y = c$ meets $y = x$ at $(\frac{c}{2}, \frac{c}{2})$ and meets $y = 2x$ at $(\frac{c}{3}, \frac{2c}{3})$. The triangle with these two points and the origin has area $\frac{1}{2}\left|\frac{c}{2} \cdot \frac{2c}{3} - \frac{c}{3} \cdot \frac{c}{2}\right| = \frac{1}{2} \cdot \frac{c^2}{6} = \frac{c^2}{12}$. The quadrilateral is the triangle for $c = 12$ minus the one for $c = 6$: $\frac{144}{12} - \frac{36}{12} = 12 - 3 = 9$.

Check. Its vertices are $(3, 3)$, $(6, 6)$, $(4, 8)$ and $(2, 4)$. The shoelace formula gives $\frac{1}{2}|(18 - 18) + (48 - 24) + (16 - 16) + (6 - 12)| = \frac{1}{2} \cdot 18 = 9$. The area is **9**.

9. Answer: 450**. The idea.** Put the bottom-left corner of the tray at the origin. The distance between the centres of two touching circles is the sum of their radii, and the centres differ by known vertical and horizontal amounts, so the Pythagorean theorem gives the width.

Work it out. Let the width be W. The large centre is at $(9, 9)$ and the small centre is at $(W-4, 4)$. They are $9 + 4 = 13$ apart, with vertical difference $9 - 4 = 5$, so the horizontal difference is $\sqrt{13^2 - 5^2} = 12$. So $W - 4 - 9 = 12$ and $W = 25$. (The other sign would give $W = 1$, far too narrow for a coin of diameter 18.) The area is $18 \times 25 = 450$.

Check. The small coin reaches up to height $8 < 18$ and fits under the top. The large coin spans x from 0 to $18 \le 25$. The centres $(9, 9)$ and $(21, 4)$ are $\sqrt{144 + 25} = 13$ apart. The area is **450**.

10. Answer: 34**. The idea.** Use $x^2 = x + 1$ to lower powers one step at a time. If $x^n = ax + b$, then $x^{n+1} = ax^2 + bx = a(x + 1) + bx = (a + b)x + a$. So every power of x is a linear expression in x, and its coefficients follow the Fibonacci numbers.

Work it out. Start from $x^1 = 1x + 0$ and apply $(a, b) \to (a + b, a)$: $x^2 = x + 1$, $x^3 = 2x + 1$, $x^4 = 3x + 2$, $x^5 = 5x + 3$, $x^6 = 8x + 5$, $x^7 = 13x + 8$, $x^8 = 21x + 13$, $x^9 = 34x + 21$, $x^{10} = 55x + 34$. Therefore $x^{10} - 55x = 34$, whichever of the two roots x is.

Check. Square $x^5 = 5x + 3$: $x^{10} = 25x^2 + 30x + 9 = 25(x + 1) + 30x + 9 = 55x + 34$, which agrees. As a numerical check, $x \approx 1.618$ gives $x^{10} \approx 122.99$ and $55x \approx 88.99$, a difference of 34. The value is **34**.

11. Answer: 341**. The idea.** Sort the subsets by their smallest element k. The largest element is then $11 - k$, and any numbers strictly between those two can be included or left out freely.

Work it out. We need $k < 11 - k$, so $k \le 5$. A one-element subset would need $2k = 11$, which is impossible. For a given k, there are $(11-k)-k-1 = 10-2k$ numbers strictly between the two ends, giving 2^{10-2k} subsets. Summing over $k = 1, \ldots, 5$: $2^8+2^6+2^4+2^2+2^0 = 256+64+16+4+1 = 341$.

Check. The sum is a geometric series: $\frac{4^5-1}{4-1} = \frac{1023}{3} = 341$. The small case $\{1, 2, 3, 4\}$ with ends adding to 5 gives $2^2 + 2^0 = 5$, namely $\{1, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$, $\{2, 3\}$, which is right. The answer is **341**.

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12. Answer: $\frac{505}{1296}$**. The idea.** Ann and Ben play by identical rules, so "Ann higher" and "Ben higher" have the same probability. It is enough to find the probability of a tie, and for that we need the distribution of the larger of two dice.

Work it out. The larger of two dice is at most k exactly when both dice are at most k, which has probability $\frac{k^2}{36}$. So the larger is exactly k with probability $\frac{k^2-(k-1)^2}{36} = \frac{2k-1}{36}$: the values $1, \ldots, 6$ have probabilities $\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}$. A tie has probability $\frac{1^2+3^2+5^2+7^2+9^2+11^2}{36^2} = \frac{286}{1296}$. The rest, $\frac{1010}{1296}$, splits evenly between Ann winning and Ben winning, so Ann wins with probability $\frac{505}{1296}$.

Check. Count Ann's wins directly: when Ann's larger number is k, Ben's larger must be at most $k-1$, with probability $\frac{(k-1)^2}{36}$. So the count of winning outcomes out of 1296 is $\sum_{k=1}^{6}(2k-1)(k-1)^2 = 3\cdot 1+5\cdot 4+7\cdot 9+9\cdot 16+11\cdot 25 = 3+20+63+144+275 = 505$, matching. The probability is $\frac{\mathbf{505}}{\mathbf{1296}}$.

13. Answer: 87**. The idea.** Four strictly increasing digits $a < b < c < d$ are fixed once we choose which four digits appear. So we count 4-element sets of digits that fit the clock's rules: the first digit a is 0, 1 or 2, the minutes' tens digit c is at most 5, and when $a = 2$ the hour's second digit b is at most 3.

Work it out. Fix c and count choices of b and d. For $a = 0$, b ranges over $1, \ldots, c-1$ and d over $c+1, \ldots, 9$, so the count is $\sum_{c=2}^{5}(c-1)(9-c) = 7+12+15+16 = 50$. For $a = 1$, b ranges over $2, \ldots, c-1$: $\sum_{c=3}^{5}(c-2)(9-c) = 6+10+12 = 28$. For $a = 2$ we need $2 < b \le 3$, so $b = 3$ and c is 4 or 5, with 5 and 4 choices of d: 9 times. In total, $50 + 28 + 9 = 87$.

Check. Count the $a = 0$ case another way: choose $\{b, c\} \subset \{1, \ldots, 5\}$ and then $d > c$. The pairs with $c = 5$ give $4 \cdot 4 = 16$, with $c = 4$ give $3 \cdot 5 = 15$, and so on, matching the 50 above. A direct run through all 1440 times also gives **87**.

14. Answer: 4444444440**. The idea.** $72 = 8 \cdot 9$. Divisibility by 9 controls how many 4s there are, and divisibility by 8 controls only the last three digits.

Work it out. The digit sum is $4k$, where k is the number of 4s. For $9 \mid 4k$ we need $9 \mid k$, so there are at least nine 4s. For divisibility by 8, the last three digits must form a multiple of 8 using only 4 and 0. The candidates are 000, 004, 040, 044, 400, 404, 440, 444, and the multiples of 8 among them are 000, 040, 400, 440. So 444 cannot end the number, which rules out the nine-digit number 444444444. With exactly one 0, the ending must contain just one zero, so it is 440. That forces the 0 to be last: 4444444440. Any other valid number has more digits, so this is the least.

Check. $4444444440 = 72 \times 61728395$. A computer search over all numbers made of 4s and 0s, in increasing order, finds this as the first multiple of 72. The answer is **4444444440**.

15. Answer: $4\sqrt{3}$**. The idea.** Consecutive angles of a parallelogram add to 180°, so the bisectors of two consecutive angles meet at 90°. The enclosed quadrilateral therefore has four right angles: it is a rectangle. Its sides are differences of lengths along the bisectors, which come from small 30-60-90 triangles.

Work it out. The angles are 60° at A and C and 120° at B and D, so the bisectors make angles of 30° and 60° with the sides. Let the bisector from A meet the bisectors from B and D at P and R. Triangle APB has angles 30°, 60°, 90° at A, B, P and hypotenuse $AB = 10$, so $AP = 5\sqrt{3}$ and $BP = 5$. Triangle ARD has angles 30°, 60°, 90° at A, D, R and hypotenuse $AD = 6$, so

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$AR = 3\sqrt{3}$. One side of the rectangle is $PR = 5\sqrt{3} - 3\sqrt{3} = 2\sqrt{3}$. Similarly, the bisector from C meets the bisector from B at S with $BS = 6 \cdot \frac{1}{2} = 3$ (triangle BSC has hypotenuse $BC = 6$ and a $60°$ angle at B), so the other side is $PS = BP - BS = 5 - 3 = 2$. The area is $2\sqrt{3} \cdot 2 = 4\sqrt{3}$.

Check. Use coordinates with $A = (0, 0)$, $B = (10, 0)$ and $D = (3, 3\sqrt{3})$. Then $R = A + 3\sqrt{3}(\cos 30°, \sin 30°) = (\frac{9}{2}, \frac{3\sqrt{3}}{2})$ and $S = B + 3(\cos 120°, \sin 120°) = (\frac{17}{2}, \frac{3\sqrt{3}}{2})$. So the diagonal RS has length 4, and indeed $(2\sqrt{3})^2 + 2^2 = 16 = 4^2$. The area is $\mathbf{4\sqrt{3}}$.

16. Answer: 340**. The idea.** Every line of squares (row, column or diagonal) with L squares holds $\binom{L}{3}$ sets of three, and no set of three squares lies on two different lines, so we just add $\binom{L}{3}$ over all lines. Rows and columns all have length 6; the diagonals have lengths $1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1$ in each direction, and the hockey-stick identity adds up their $\binom{L}{3}$ quickly.

Work it out. The 6 rows and 6 columns give $12\binom{6}{3} = 12 \cdot 20 = 240$. In one diagonal direction, the lines of length 1 and 2 contribute nothing, and the rest give $2\left[\binom{3}{3} + \binom{4}{3} + \binom{5}{3}\right] + \binom{6}{3}$. By the hockey stick, $\binom{3}{3}+\binom{4}{3}+\binom{5}{3} = \binom{6}{4} = 15$, so one direction gives $2 \cdot 15 + 20 = 50$ (that is, 1+4+10+20+10+4+1). The two directions give 100, and the total is $240 + 100 = 340$.

Check. Count one diagonal direction another way: the full hockey stick gives $\binom{1}{3} + \cdots + \binom{6}{3} = \binom{7}{4} = 35$ for the lengths 1 through 6, and the lengths 1 through 5 appear again on the other side, adding $\binom{6}{4} = 15$. That is 50 again. Also, three squares in one row cannot also share a column or a diagonal, so nothing was counted twice.

17. Answer: $18\sqrt{6}$**. The idea.** Use coordinates. The cut turns out to go through a second edge midpoint as well, and the cross-section is a rhombus whose diagonals are the space diagonal and the segment joining the two edge midpoints. A rhombus has area half the product of its diagonals.

Work it out. Put the cube at $0 \le x, y, z \le 6$ with the chosen corners $O = (0, 0, 0)$ and $G = (6, 6, 6)$. By symmetry any allowed edge gives the same cut, so take the edge from $(6, 0, 0)$ to $(6, 0, 6)$, with midpoint $M = (6, 0, 3)$. The plane through O, G, M is $x + y = 2z$ (check: $0 = 0$, $12 = 12$, $6 = 6$). It also contains $N = (0, 6, 3)$, the midpoint of the opposite edge from $(0, 6, 0)$ to $(0, 6, 6)$. Going round the section, the corners are O, M, G, N, and $OM = MG = GN = NO = \sqrt{36 + 9} = 3\sqrt{5}$, so the section is a rhombus. Its diagonals are $OG = 6\sqrt{3}$ and $MN = \sqrt{36 + 36} = 6\sqrt{2}$, so its area is $\frac{1}{2} \cdot 6\sqrt{3} \cdot 6\sqrt{2} = 18\sqrt{6}$.

Check. $G = M + N$ as vectors, so $OMGN$ is the parallelogram spanned by $\vec{OM} = (6, 0, 3)$ and $\vec{ON} = (0, 6, 3)$. Its area is the length of the cross product $(-18, -18, 36)$, which is $\sqrt{324 + 324 + 1296} = \sqrt{1944} = 18\sqrt{6}$.

18. Answer: 15**. The idea.** Remove the compulsory first ticket of each kind, then look at how many ways the rest can be split. Any two splits of the same amount differ by trading 5 children's tickets for 3 adult tickets, so the splits form a chain, and the number of adult tickets is stuck in one residue class mod 3. Counting the ways then becomes a floor-of-a-division question.

Work it out. Write the numbers sold as $x+1$ children's and $y+1$ adult tickets, with $x, y \ge 0$. Then $3x + 5y = M$, where $M = T - 8$. For a solution, $5y \equiv M \pmod 3$, so y must be congruent mod 3 to a fixed $y_0 \in \{0, 1, 2\}$ (namely $y_0 = 0, 1, 2$ when $M \equiv 0, 2, 1 \pmod 3$). The allowed values are $y = y_0, y_0 + 3, y_0 + 6, \ldots$ as long as $5y \le M$, and each gives exactly one x. So there are exactly two solutions when $y_0 + 3$ works but $y_0 + 6$ does not, that is, when $15 \le M - 5y_0 \le 29$. If $y_0 = 0$:

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$M \in \{15, 18, 21, 24, 27\}$. If $y_0 = 1$: $20 \le M \le 34$ with $M \equiv 2$, so $M \in \{20, 23, 26, 29, 32\}$. If $y_0 = 2$: $25 \le M \le 39$ with $M \equiv 1$, so $M \in \{25, 28, 31, 34, 37\}$. These 15 values of M are different, so there are 15 values of $T = M + 8$, from 23 to 45.

Check. $T = 45$: $3 \cdot 10 + 5 \cdot 3 = 3 \cdot 5 + 5 \cdot 6 = 45$, while $(15, 0)$ and $(0, 9)$ are ruled out because each kind needs at least one ticket. $T = 38$ (so $M = 30$, $y_0 = 0$) has three: $(11, 1)$, $(6, 4)$, $(1, 7)$, so it is correctly left out. For $T \ge 46$ we have $M \ge 38$, so $M - 5y_0 \ge 30$ in every case (check $M = 40$, the worst one), giving at least three ways; so no larger value sneaks in.

19. Answer: $\frac{19}{55}$**. The idea.** Only the remainders mod 3 matter. The numbers 1 to 12 split evenly into three classes of four: remainder 0 $\{3, 6, 9, 12\}$, remainder 1 $\{1, 4, 7, 10\}$ and remainder 2 $\{2, 5, 8, 11\}$. A sum of three remainders is a multiple of 3 exactly when the remainders are all equal or all different.

Work it out. Remainders $r_1 + r_2 + r_3$ can be $0 + 0 + 0$, $1 + 1 + 1$, $2 + 2 + 2$ or $0 + 1 + 2$. The other patterns, two equal and one different, give $0 + 0 + 1$, $0 + 0 + 2$, $1 + 1 + 0$, $1 + 1 + 2$, $2 + 2 + 0$, $2 + 2 + 1$, none of which is a multiple of 3. All three from one class: $3 \cdot \binom{4}{3} = 12$ ways. One from each class: $4^3 = 64$ ways. That is 76 favourable sets out of $\binom{12}{3} = 220$, and $\frac{76}{220} = \frac{19}{55}$.

Check. Count the bad sets directly: two from one class and one from another is $3 \cdot 2 \cdot \binom{4}{2} \cdot 4 = 144$, and $144 + 76 = 220$.

20. Answer: 12**. The idea.** Split off the power of 2: write $n = 2^a m$ with m odd. The odd divisors of n are exactly the divisors of m, and every divisor of n is a power 2^i $(0 \le i \le a)$ times a divisor of m. So the two counts pin down a and the divisor count of m separately.

Work it out. Let $d(m)$ be the number of divisors of m. The odd divisors number $d(m) = 6$, and the even ones use $i = 1, \ldots, a$, so there are $a \cdot d(m) = 6a$ of them. Then $6a = 12$ gives $a = 2$, so $n = 4m$ with m odd, $d(m) = 6$ and $m < 250$. An integer with 6 divisors has the form p^5 or $p^2 q$ for distinct primes p, q, and here they must be odd. p^5: only $3^5 = 243$. $p = 3$, $9q < 250$: $q = 5, 7, 11, 13, 17, 19, 23$, giving 7 values. $p = 5$, $25q < 250$: $q = 3, 7$, giving 2. $p = 7$, $49q < 250$: $q = 3, 5$, giving 2. $p \ge 11$: $121 \cdot 3 = 363$ is too big. That is $1 + 7 + 2 + 2 = 12$ values of m, so 12 values of n.

Check. The largest is $n = 4 \cdot 245 = 980 < 1000$, and the next candidate $4 \cdot 9 \cdot 29 = 1044$ is too big. For $n = 180 = 2^2 \cdot 45$: 45 has 6 divisors, and 180 has $3 \cdot 6 = 18$ divisors in all, so $18 - 6 = 12$ are even, as required. A common slip is to take $a = 1$ (that gives only 6 even divisors) or to forget the form p^5.

21. Answer: 11**. The idea.** Work backwards from 0. The equation $f(y) = c$ means $y^2 = c + 1$: it has the two real solutions $\pm\sqrt{c+1}$ when $c > -1$, the single solution 0 when $c = -1$, and none when $c < -1$. So any value below -1 is a dead end, and most branches of the backward tree die. The survivors are described by the chain $a_0 = 0$, $a_1 = 1$, $a_2 = \sqrt{2}$, and in general $a_{k+1} = \sqrt{1 + a_k}$, which increases, so $a_k > 1$ for $k \ge 2$.

Work it out. Let S_n be the set of real x that give 0 after n applications. Going back one step: the preimages of a_k are $\pm a_{k+1}$; the preimage of $-a_1 = -1$ is $a_0 = 0$ only; and $-a_k$ for $k \ge 2$ is below -1, so it has no preimage. Starting from $S_0 = \{a_0\}$, this gives $S_1 = \{\pm a_1\}$, $S_2 = \{a_0, \pm a_2\}$, $S_3 = \{\pm a_1, \pm a_3\}$, $S_4 = \{a_0, \pm a_2, \pm a_4\}$, and in general S_n is made of $\pm a_k$ for all $k \le n$ with the same parity as n (where $\pm a_0$ is just 0). The a_k are all different, so for even n there are $1 + 2 \cdot \frac{n}{2} = n + 1$ solutions. For ten applications, the answer is 11.

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Check. Level 3 by hand: S_3 comes from $f(x) \in \{0, \sqrt{2}, -\sqrt{2}\}$, giving $x = \pm 1$ and $x = \pm\sqrt{1+\sqrt{2}}$, while $-\sqrt{2} < -1$ gives nothing: $4 = 3 + 1$ numbers, as the formula says. Note the answer is tiny compared with the degree: the equation is a polynomial of degree $2^{10} = 1024$, yet it has only 11 distinct real solutions (and $x = 0$ is a repeated root, since $f(x)$ has a turning point there).

22. Answer: $\sqrt{26}$. **The idea.** There is no need to find the legs, which turn out to be messy. Each median from an acute vertex is the hypotenuse of a smaller right triangle, and adding the two Pythagorean equations gives the sum of the squares of the legs. After that, the median to the hypotenuse is simply half the hypotenuse.

Work it out. Let $a = BC$ and $b = CA$. The median from A runs to a point $\frac{a}{2}$ from C along $\overline{CB}$, so $b^2 + \frac{a^2}{4} = 49$. Likewise $a^2 + \frac{b^2}{4} = 81$. Adding gives $\frac{5}{4}(a^2 + b^2) = 130$, so $AB^2 = a^2 + b^2 = 104$. The midpoint of the hypotenuse is the centre of the circle through A, B and C, so it is equally far from all three vertices. The median from C is therefore $\frac{AB}{2} = \frac{\sqrt{104}}{2} = \sqrt{26}$.

Check. Such a triangle really exists: solving the two equations gives $a^2 = \frac{220}{3}$ and $b^2 = \frac{92}{3}$, both positive, and these add to $\frac{312}{3} = 104$.

23. Answer: 108. **The idea.** Each formation is a choice of the number of rows r, which can be any divisor of n, so we are told the numbers of divisors of n, $2n$ and $3n$. Doubling only changes the exponent of 2 and tripling only changes the exponent of 3, so comparing the divisor counts as ratios pins down both exponents.

Work it out. Write $n = 2^a 3^b m$ with m not divisible by 2 or 3, and let $d(m)$ be the number of divisors of m. Then n has $(a+1)(b+1)d(m) = 12$ divisors, $2n$ has $(a+2)(b+1)d(m) = 16$ and $3n$ has $(a+1)(b+2)d(m) = 15$. Dividing the second count by the first, $\frac{a+2}{a+1} = \frac{16}{12} = \frac{4}{3}$, so $3a + 6 = 4a + 4$ and $a = 2$. Dividing the third by the first, $\frac{b+2}{b+1} = \frac{15}{12} = \frac{5}{4}$, so $b = 3$. Now $3 \cdot 4 \cdot d(m) = 12$ forces $d(m) = 1$, so $m = 1$ and $n = 2^2 \cdot 3^3 = 108$.

Check. $108 = 2^2 \cdot 3^3$ has $3 \cdot 4 = 12$ divisors, $216 = 2^3 \cdot 3^3$ has $4 \cdot 4 = 16$ and $324 = 2^2 \cdot 3^4$ has $3 \cdot 5 = 15$. The argument above left no choices, so 108 is the only possible band size.

24. Answer: 232. **The idea.** Build the strip from left to right and keep track of the colour of the last tile. Swapping red and blue is a symmetry, so the number of coverings ending in red equals the number ending in blue, and only two sequences are needed.

Work it out. For a $1 \times n$ strip let x_n be the number of coverings ending with a red tile (the same as ending with blue), and g_n the number ending with a green tile. A covering ending in red is a covering of length $n - 1$ ending in blue or green, followed by the red tile: $x_n = x_{n-1} + g_{n-1}$. A covering ending in green is a covering of length $n - 2$ ending in red or blue, plus the green tile: $g_n = 2x_{n-2}$ (and $g_2 = 1$, the lone green tile). Starting from $x_1 = 1$, $g_1 = 0$: x_n: 1, 1, 2, 4, 6, 10, 18, 30, 50, 86; g_n: 0, 1, 2, 2, 4, 8, 12, 20, 36, 60. The total for $n = 10$ is $2x_{10} + g_{10} = 172 + 60 = 232$.

Check. For $n = 3$ the total is $2 \cdot 2 + 2 = 6$: RBR, BRB, RG, BG, GR, GB, matching the example. For $n = 4$ it is $2 \cdot 4 + 2 = 10$: RBRB, BRBR, RBG, BRG, RGR, RGB, BGR, BGB, GRB, GBR. The totals $2, 3, 6, 10, 16, 28, 48, 80, 136, 232$ also obey $T_n = T_{n-1} + 2T_{n-3}$, for example $232 = 136 + 2 \cdot 48$.

25. Answer: $\frac{29}{90}$. **The idea.** Let $c(n, k)$ be the number of orders of n cards with exactly k new highs.

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Look at the last card. It is a new high only when it is the largest card. Otherwise it adds nothing, and the first $n-1$ cards behave like a shuffled deck of $n-1$ cards. This gives a Pascal-like recursion.

Work it out. If the last card is the largest, the first $n-1$ cards need $k-1$ new highs. Otherwise the last card is one of $n-1$ smaller values and the rest need k new highs. So $c(n,k) = c(n-1,k-1) + (n-1)\,c(n-1,k)$. Building the table row by row (listing $k = 1, 2, \ldots$): $n = 3$: $2, 3, 1$; $n = 4$: $6, 11, 6, 1$; $n = 5$: $24, 50, 35, 10, 1$; $n = 6$: $120, 274, 225, 85, 15, 1$. Then $c(7,3) = c(6,2) + 6\,c(6,3) = 274 + 1350 = 1624$, and the probability is $\frac{1624}{5040} = \frac{29}{90}$.

Check. The full row for $n = 7$ is $720, 1764, 1624, 735, 175, 21, 1$, which adds to $5040 = 7!$. As another check, the i-th card is a new high with probability $\frac{1}{i}$, independently of the others, so the numbers $c(7,k)$ are the coefficients of $x(x+1)(x+2)\cdots(x+6)$. The coefficient of x^3 there is also 1624.

26. Answer: 10**. The idea.** Line AI bisects angle A, so it meets the circumcircle of ABC again at the midpoint M of arc BC. This point is equally far from B, C and I, so M is the center of the circle through B, C, I. Then I and X are the two ends of a diameter along line AM, and $AX = AI + 2 \cdot MI$.

Work it out. Write α, β for the angles at A, B. Since M lies on the circumcircle, $\angle MBC = \angle MAC = \frac{\alpha}{2}$, so $\angle MBI = \frac{\alpha}{2} + \frac{\beta}{2}$. The angle $\angle MIB$ is an exterior angle of triangle ABI, so it is also $\frac{\alpha}{2} + \frac{\beta}{2}$. Hence $MB = MI$, and likewise $MC = MI$. Now the numbers. $\cos\alpha = \frac{4^2+5^2-6^2}{2\cdot4\cdot5} = \frac{1}{8}$, so $\cos\frac{\alpha}{2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ and $\sin\frac{\alpha}{2} = \frac{\sqrt{7}}{4}$. The angle $\angle BIC = 180^\circ - \frac{\beta}{2} - \frac{\gamma}{2} = 90^\circ + \frac{\alpha}{2}$, so by the law of sines in triangle BIC the radius is $MI = \frac{BC}{2\sin\angle BIC} = \frac{6}{2\cdot\frac{3}{4}} = 4$. The semiperimeter is $s = \frac{15}{2}$, and the incircle touches $\overline{AB}$ at distance $s - BC = \frac{3}{2}$ from A, so $AI = \frac{3/2}{\cos(\alpha/2)} = 2$. Therefore $AX = 2 + 2 \cdot 4 = 10$.

Check. The point X is the excenter opposite A (the incenter–excenter lemma), and the excircle touches line AB at distance $s = \frac{15}{2}$ from A. So $AX = \frac{15/2}{3/4} = 10$ again. Also $AI \cdot AX = 20 = AB \cdot AC$, a known identity for the incenter and this excenter.

27. Answer: 120**. The idea.** Do not count by position; count by shape. In any $a \times b$ rectangle on a checkerboard the colors are balanced as well as they can be: if ab is even, exactly half the squares are black; if ab is odd, the corner color has one more square than the other, $\frac{ab+1}{2}$ against $\frac{ab-1}{2}$. So exactly 4 black squares forces the area to be 8 (any position), or 7 with black corners, or 9 with white corners.

Work it out. Area 8: shapes 1×8 and 8×1 fit in 8 positions each, and 2×4 and 4×2 fit in $7 \cdot 5 = 35$ positions each, for 86. Area 7: the only shapes are 1×7 and 7×1. A 1×7 strip fits in 2 positions per row, and these start on squares of opposite colors, so exactly one per row has black corners: 8 strips, and 8 more vertical ones. Area 9: 1×9 does not fit, so the shape is 3×3, with $6 \cdot 6 = 36$ positions. Moving one step shifts the corner color, so exactly half, 18, have white corners (and 4 black squares). The total is $86 + 16 + 18 = 120$.

Check. No other area works: area 6 or less has at most 3 black squares when even and at most 3 when odd ($\frac{5+1}{2} = 3$), and area 10 or more has at least 5. A computer count over all $36^2 = 1296$ rectangles of the board also finds 120. The answer does not depend on which corner of the board is black, since swapping the colors just swaps which 1×7 strips and 3×3 squares qualify, not how many.

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28. Answer: $\frac{672}{65}$**. The idea.** The angles $\angle AXP$ and $\angle AYP$ are right angles, so X and Y lie on the circle with diameter $\overline{AP}$. In that circle the chord XY subtends the angle A of the triangle, so $XY = AP \cdot \sin A$. The angle A is fixed, so XY is smallest when AP is smallest, that is, when P is the foot of the altitude from A.

Work it out. By the extended law of sines in the circle of diameter AP, a chord seen from A at angle $\angle XAY = \angle BAC$ has length $AP \sin \angle BAC$. Heron's formula gives the area of the 13-14-15 triangle as $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. So the altitude from A is $\frac{2 \cdot 84}{14} = 12$, and it lands inside $\overline{BC}$ because angles B and C are acute. Also $\sin A = \frac{2 \cdot 84}{13 \cdot 15} = \frac{56}{65}$. The minimum is $12 \cdot \frac{56}{65} = \frac{672}{65}$.

Check. Coordinates: $B = (0, 0)$, $C = (14, 0)$, $A = (5, 12)$ and $P = (5, 0)$. Projecting P onto the two lines gives $X = \left(\frac{125}{169}, \frac{300}{169}\right)$ and $Y = \left(\frac{269}{25}, \frac{108}{25}\right)$. The differences are $\frac{42336}{4225} = \frac{672 \cdot 63}{4225}$ and $\frac{10752}{4225} = \frac{672 \cdot 16}{4225}$. Since $63^2 + 16^2 = 65^2$, $XY = \frac{672 \cdot 65}{4225} = \frac{672}{65}$. At the endpoints, $P = B$ gives $XY = BY = 14 \sin C = \frac{56}{5} > \frac{672}{65}$.

29. Answer: 1253**. The idea.** We need $n^2 \equiv 9 \pmod{10000}$, that is, $10000 = 16 \cdot 625$ divides $(n-3)(n+3)$. Handle 625 and 16 separately, then combine with the Chinese Remainder Theorem. The factors differ by 6, so 5 cannot divide both, but 2 divides both, and that is where the extra solutions come from.

Work it out. Mod 625: since 5 divides at most one of $n - 3$ and $n + 3$, one of them is a multiple of 625, so $n \equiv \pm 3 \pmod{625}$. Mod 16: n is odd, so both factors are even, and since they differ by $6 \equiv 2 \pmod 4$ one of them is $\equiv 2 \pmod 4$ and contributes just one factor 2. The other must supply 2^3: $n \equiv 3$ or $-3 \pmod 8$, that is, $n \equiv 3, 5, 11, 13 \pmod{16}$. Now combine, using $625 \equiv 1 \pmod{16}$. If $n = 625k + 3$, then $n \equiv k + 3 \pmod{16}$, which needs $k \equiv 0, 2, 8, 10$; the first case after $k = 0$ is $k = 2$, $n = 1253$. If $n = 625k - 3$, then $n \equiv k - 3$, which needs $k \equiv 6, 8, 14, 0$; the smallest is $n = 625 \cdot 6 - 3 = 3747$. The answer is 1253.

Check. $1253^2 = 1570009$. In all there are 8 solutions below 10000: 3, 1253, 3747, 4997, 5003, 6253, 8747, 9997, pairing up as n and $10000 - n$. Someone who assumed only $n \equiv \pm 3 \pmod{10000}$ would miss the four square roots of 9 mod 16 and answer 9997.

30. Answer: 646**. The idea.** Use inclusion–exclusion over the six forbidden pairs $\{1, 2\}, \{2, 3\}, \ldots, \{6, 7\}$. If we force a chosen set of j pairs to be adjacent, the pairs glue together into chains. A chain such as 3-4-5 must appear in increasing or decreasing order, so each chain contributes a factor 2, and the blocks can then be arranged freely. What matters is how many separate runs r the j chosen pairs form.

Work it out. Forcing j pairs that form r runs leaves $7 - j$ blocks, giving $(7 - j)! \cdot 2^r$ orders. The j-subsets of the six pairs (which sit in a row like the edges of a path) with exactly r runs number $\binom{j-1}{r-1}\binom{7-j}{r}$: the j chosen pairs are split into r runs, and the $6 - j$ unchosen pairs leave $7 - j$ gaps for them. The terms are: $j = 0$: 5040. $j = 1$: $6 \cdot 720 \cdot 2 = 8640$. $j = 2$: $5 \cdot 120 \cdot 2 + 10 \cdot 120 \cdot 4 = 6000$. $j = 3$: $4 \cdot 24 \cdot 2 + 12 \cdot 24 \cdot 4 + 4 \cdot 24 \cdot 8 = 2112$. $j = 4$: $3 \cdot 6 \cdot 2 + 9 \cdot 6 \cdot 4 + 3 \cdot 6 \cdot 8 = 396$. $j = 5$: $2 \cdot 2 \cdot 2 + 4 \cdot 2 \cdot 4 = 40$. $j = 6$: $1 \cdot 1 \cdot 2 = 2$. With alternating signs, $5040 - 8640 + 6000 - 2112 + 396 - 40 + 2 = 646$.

Check. The subset counts add up correctly: $5 + 10 = 15 = \binom{6}{2}$, $4 + 12 + 4 = 20 = \binom{6}{3}$, $3 + 9 + 3 = 15$ and $2 + 4 = 6$. The same method for four runners gives $24 - 36 + 16 - 2 = 2$, and indeed only 2413 and 3142 work. For six runners it gives 90, and the sequence 2, 14, 90, 646 for 4, 5, 6, 7 runners matches a direct listing.

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