MathCastle

NATIONAL COMPETITION

Sprint Round

Problems 1–30

in the format of the MATHCOUNTS competition

Name _____

School _____

Chapter _____

INSTRUCTIONS

1. This round has 30 problems. You have 40 minutes to work on all of them.
2. No calculators, books or other aids. Work on scratch paper.
3. Write only your final answer in the blank in the right-hand column. Answers must be complete, legible and simplified, with fractions in lowest terms.
4. If you finish before time is called, use the remaining time to check your answers.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The nine integers $1, 2, 3, \ldots, 9$ are split into three groups of three numbers each, and the product of the numbers in each group is computed. What is the smallest possible value of the largest of the three products? 1. _____

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2. The seven-digit number 1,000,027 is not prime. What is the largest prime factor of 1,000,027? 2. _____

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3. The equation $x^2+bx+c = 0$ has roots r and s. The equation $x^2+cx+b = 0$, with the same two coefficients in swapped places, has roots $r + 1$ and $s + 1$. What is the value of $r^2 + s^2$? 3. _____

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4. A triangle is formed by choosing three of the twelve vertices of a regular 12-sided polygon. How many such triangles have no side in common with the polygon, that is, no two of the three chosen vertices are adjacent vertices of the polygon? 4. _____

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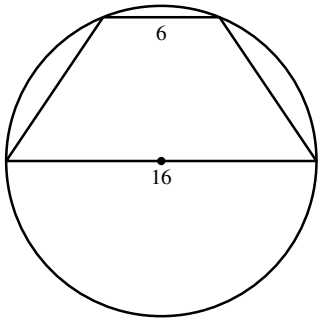
5. A hiking club has adult members and child members. The mean age of the adults is 40, the mean age of the children is 10, and the mean age of all the members is 22. Then five new adults join, and their mean age is 43. After they join, the mean age of all the members is 25. How many children are in the club? 5. _____

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6. Four fair six-sided dice, with faces numbered 1 through 6, are rolled. What is the probability that some two of the four dice show numbers whose sum is 7? Express your answer as a common fraction. 6. _____

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7. An isosceles trapezoid has parallel sides of lengths 16 and 6. A circle passes through all four of its vertices, and the center of that circle lies on the longer of the two parallel sides. What is the area of the trapezoid? Express your answer in simplest radical form.



7. _____

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

8. A jar holds only red and blue marbles, at least one of each, and the total number of marbles is between 10 and 100, inclusive. Two marbles are drawn at random without replacement, and the probability that they are the same color is exactly $\frac{1}{2}$. How many different numbers of red marbles are possible?

8. _____

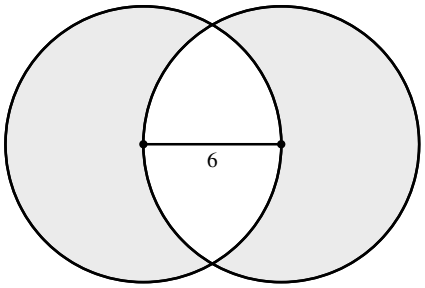
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9. The number $N = 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdots 30^{30}$ is the product of k^k for every integer k from 1 to 30. When N is written out in full, how many zeros does it end with?

9. _____

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10. Two circles, each of radius 6, are drawn so that each circle passes through the center of the other. What is the total area of the region consisting of all points that lie inside exactly one of the two circles? Express your answer in terms of π and in simplest radical form.



10. _____

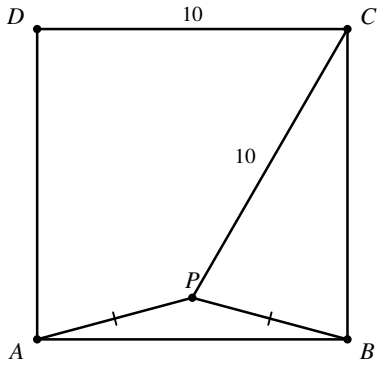
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11. How many ordered pairs (a, b) of positive integers satisfy $a+b \leq 100$ and have the property that $a+b$ is a divisor of ab? (For example, $(3, 6)$ works, since 9 divides 18.)

11. _____

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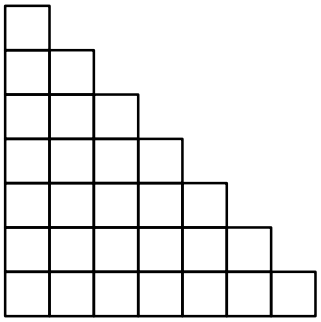
12. Square $ABCD$ has side length 10, with its vertices labeled in order around the square. Point P lies inside the square, $PA = PB$, and $PC = 10$. What is the length PA? Express your answer in simplest radical form.



12. _____

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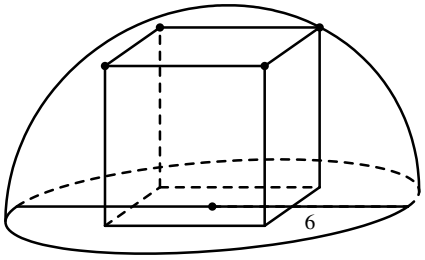
13. A staircase is built from 28 unit squares arranged in 7 left-aligned rows: the bottom row has 7 squares, the row above it has 6, and so on, up to a single square in the top row. How many rectangles (squares included) have all four sides along the grid lines of the staircase and lie completely inside it?



13. _____

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14. A solid hemisphere of radius 6 rests on its flat circular face. A cube is placed inside it: the bottom face of the cube lies on the flat face of the hemisphere, centered at the center of that circle, and the four top vertices of the cube lie on the curved surface of the hemisphere. What is the volume of the cube? Express your answer in simplest radical form.



14. _____

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15. A parabola $y = ax^2 + bx + c$, where $a \neq 0$, crosses the x-axis at $x = 1$ and $x = 5$, and it is tangent to the line $y = 2x - 1$ (it touches the line at exactly one point). There are two such parabolas. What is the sum of the y-coordinates of their vertices?

15. _____

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16. The ten letters of the word MATHCOUNTS are arranged in a row at random, with every distinguishable arrangement equally likely (the two T's are identical; the other eight letters are all different). What is the probability that the two T's are next to each other AND the vowels appear in the order A, then O, then U from left to right (not necessarily next to each other)? Express your answer as a common fraction.

16. _____

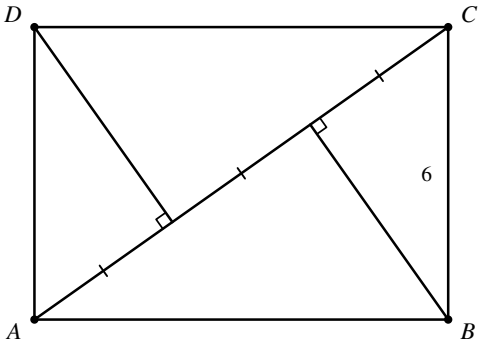
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17. How many four-digit positive integers are multiples of 9 and have only odd digits? (For example, $1179 = 9 \cdot 131$ is counted, but 3593 is not a multiple of 9, and 1233 is a multiple of 9 with an even digit.)

17. _____

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18. Rectangle $ABCD$ has its vertices labeled in order around it, and its shorter side has length 6. Perpendiculars are dropped from B and from D to the diagonal $\overline{AC}$. Their feet divide $\overline{AC}$ into three segments of equal length. What is the area of the rectangle? Express your answer in simplest radical form.



18. _____

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19. A three-digit positive integer N has a units digit that is not 0, and R is the three-digit number formed by writing the digits of N in reverse order. (For example, if $N = 172$ then $R = 271$.) For how many such integers N is $N + R$ a perfect square?

19. _____

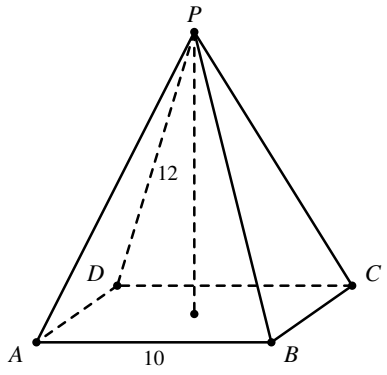
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20. A geometric progression is a list of numbers in which each term after the first equals the previous term multiplied by the same constant. How many three-element subsets of $\{1, 2, 3, \ldots, 25\}$ have the property that their three elements, listed in some order, form a geometric progression?

20. _____

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21. A pyramid has a square base $ABCD$ with side length 10, with its vertices labeled in order around the square, and its apex P is directly above the center of the square at a height of 12. What is the distance from vertex A to the plane that contains face PCD? Express your answer as a common fraction.



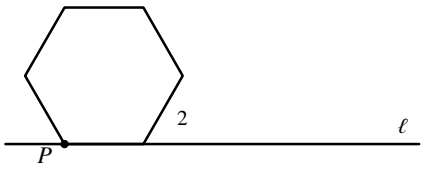
21. _____

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22. A number begins with 2026 if its leftmost four digits are 2, 0, 2, 6 in that order; for example, 2026, 20261 and 2026953 all begin with 2026. What is the smallest positive integer n such that the decimal digits of n^2 begin with 2026?

22. _____

23. A regular hexagon with side length 2 sits in a plane with one side lying on a straight line ℓ. Call the left endpoint of that bottom side P. The hexagon rolls to the right along ℓ without slipping: it rotates clockwise about its rightmost vertex on ℓ until the next side lands flat on ℓ, then does the same about the new rightmost vertex on ℓ, and so on. It stops the first time P is again the left endpoint of the side lying on ℓ. What is the total length of the path traced by P? Express your answer in terms of π and in simplest radical form.



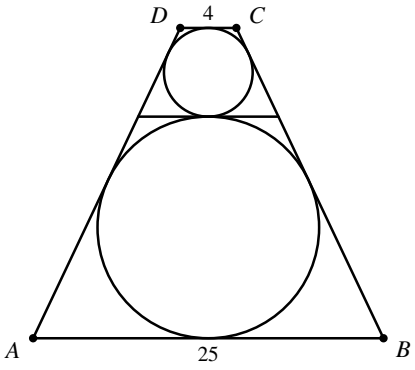
23. _____

24. The numbers $1, 2, 3, 4, 5, 6, 7$ are written down one at a time in a random order, with all 5040 orders equally likely. What is the probability that every number written after the first differs by exactly 1 from at least one of the numbers written before it? Express your answer as a common fraction.

24. _____

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25. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, with $AB = 25$ and $CD = 4$. A segment parallel to $\overline{AB}$, with one endpoint on leg $\overline{AD}$ and the other on leg $\overline{BC}$, cuts $ABCD$ into two smaller trapezoids. Each of the two smaller trapezoids has a circle inscribed in it, tangent to all four of its sides. What is the perimeter of $ABCD$?



25. _____

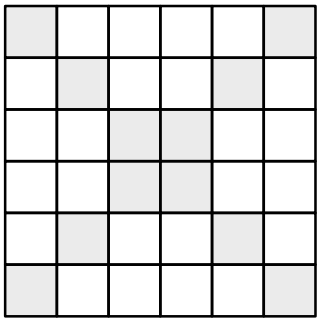
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26. A queen is placed on a square of a 12×12 board, but not on the bottom-left corner square. Two players take turns moving the queen, and each move takes it any positive number of squares to the left, or down, or diagonally down-and-left (staying on the board). The player who moves the queen onto the bottom-left corner square wins. For how many of the 143 possible starting squares does the second player (the one who does not move first) have a winning strategy?

26. _____

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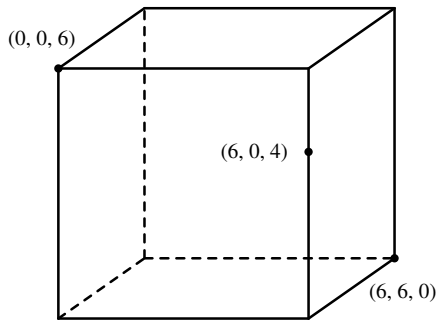
27. Six rooks are placed on the squares of a 6×6 chessboard so that no two are in the same row or the same column. None of the rooks may stand on either of the two long diagonals of the board (the 12 squares running corner to corner). In how many ways can the rooks be placed?



27. _____

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28. A $6 \times 6 \times 6$ cube is built from 216 unit cubes. Place it in space so that its corners are the eight points whose coordinates are each 0 or 6. A plane passes through the corner $(0, 0, 6)$, the corner $(6, 6, 0)$, and the point $(6, 0, 4)$ on one of the cube's edges. How many of the 216 unit cubes does the plane pass through the interior of?



28. _____

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29. For how many integers n with $2 \leq n \leq 100$ is $(n-1)! = 1 \cdot 2 \cdot 3 \cdots (n-1)$ divisible by n^2?

29. _____

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30. A round table has 10 seats, numbered 1 through 10 in order around it. The integers $1, 2, 3, \ldots, 10$ are placed in the seats, one per seat, so that no two numbers in neighbouring seats have a sum divisible by 3. (Seats 10 and 1 are neighbours. Placements are different if some seat holds a different number, so rotations count as different.) How many placements are possible?

30. _____

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Answer Key — MathCounts Nationals Sprint Practice Contest 1

1. 72

2. 103

3. 7

4. 112

5. 18

6. $\frac{139}{216}$

7. $11\sqrt{55}$

8. 8

9. 130

10. $24\pi + 36\sqrt{3}$

11. 105

12. $5\sqrt{6} - 5\sqrt{2}$

13. 210

14. $48\sqrt{6}$

15. 5

16. $\frac{1}{30}$

17. 81

18. $36\sqrt{2}$

19. 15

20. 16

21. $\frac{120}{13}$

22. 4502

23. $\frac{(8+4\sqrt{3})\,\pi}{3}$

24. $\frac{4}{315}$

25. 78

26. 8

27. 80

28. 60

29. 57

30. 43200

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Solutions — MathCounts Nationals Sprint Practice Contest 1

1. Answer: 72. **The idea.** The three products always multiply to the same number, $1 \cdot 2 \cdots 9 = 9! = 362880$, no matter how the groups are chosen. If all three products were at most M, their product would be at most M^3; so the largest product is at least the cube root of 9!. That gives a lower bound, and then we look for a split that meets it.

Work it out. $71^3 = 357911 < 362880$, so if every product were at most 71, the three would multiply to at most 357911, which is too small. Hence the largest product is at least 72. And 72 is achievable: $\{1, 8, 9\}$ gives 72, $\{3, 4, 6\}$ gives 72, and $\{2, 5, 7\}$ gives 70. The largest of these is 72, so the answer is 72.

Check. $72 \cdot 72 \cdot 70 = 5184 \cdot 70 = 362880 = 9!$, as it must be. The bound is nearly tight ($\sqrt[3]{9!} \approx 71.3$), so the groups must be very balanced; in particular 9 and 8 must go in different groups or with 1, since $8 \cdot 9 \cdot 2 = 144$. A computer search over all 280 splits confirms that none does better than 72.

2. Answer: 103. **The idea.** $1{,}000{,}027 = 1{,}000{,}000 + 27 = 100^3 + 3^3$, a sum of two cubes, and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. That splits the number into two much smaller factors which are easy to finish.

Work it out. With $a = 100$ and $b = 3$: $a+b = 103$ and $a^2 - ab + b^2 = 10000 - 300 + 9 = 9709$. So $1{,}000{,}027 = 103 \cdot 9709$. Now 103 is prime (it is not divisible by 2, 3, 5 or 7, and $11^2 > 103$). For 9709: $9709 = 7 \cdot 1387$, and $1387 = 19 \cdot 73$. Both 19 and 73 are prime. So $1{,}000{,}027 = 7 \cdot 19 \cdot 73 \cdot 103$, and the largest prime factor is 103.

Check. $7 \cdot 19 = 133$, $133 \cdot 73 = 9709$, and $9709 \cdot 103 = 970900 + 29127 = 1{,}000{,}027$.

3. Answer: 7. **The idea.** Do not try to find the roots first. Vieta's formulas turn each equation into two facts about the sum and product of its roots, and the roots of the second equation are the first roots shifted by 1, so its sum and product can be rewritten in terms of $r + s$ and rs. That gives two linear equations in b and c.

Work it out. From the first equation, $r+s = -b$ and $rs = c$. From the second, $(r+1)+(s+1) = -c$ and $(r + 1)(s + 1) = b$. The third fact says $-b + 2 = -c$, so $c = b - 2$. The fourth says $rs + (r + s) + 1 = b$, that is $c - b + 1 = b$, so $c = 2b - 1$. Then $b - 2 = 2b - 1$ gives $b = -1$ and $c = -3$. Finally $r^2 + s^2 = (r + s)^2 - 2rs = b^2 - 2c = 1 + 6 = 7$.

Check. The first equation is $x^2 - x - 3 = 0$: its roots have sum 1 and product -3, and they are real since $1 + 12 > 0$. The second is $x^2 - 3x - 1 = 0$: its roots should have sum $1 + 2 = 3$ and product $rs + (r + s) + 1 = -3 + 1 + 1 = -1$, which matches $x^2 - 3x - 1$. And $r^2 + s^2 = 1^2 - 2(-3) = 7$.

4. Answer: 112. **The idea.** Count all triangles, then remove the ones that use polygon sides, sorting them by how many sides they use. A triangle can use two sides (three consecutive vertices) or exactly one side, and these two groups are easy to count separately.

Work it out. There are $\binom{12}{3} = 220$ triangles in all. Two polygon sides: the three vertices are consecutive, and there are 12 runs of three consecutive vertices. Exactly one side: pick the side (12 ways); the third vertex must not be either endpoint or a neighbor of either endpoint, which rules out 4 vertices and leaves $12 - 4 = 8$. That gives $12 \cdot 8 = 96$. (A triangle cannot use all three sides, since the polygon has more than 3 vertices.) So $220 - 12 - 96 = 112$ triangles use no side.

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Check. Count directly. Going around the polygon, the three chosen vertices leave three gaps of unchosen vertices, each gap at least 1, totaling 9. Fix one chosen vertex V: the gaps after V and after the next two chosen vertices form an ordered triple of positive integers with sum 9, which can be chosen in $\binom{8}{2} = 28$ ways. That counts each triangle once per vertex taken as V, and V has 12 choices, so there are $\frac{12\cdot 28}{3} = 112$ triangles.

5. Answer: 18**. The idea.** Work with total ages, not means. The second piece of information fixes the size of the club; the first splits it into adults and children.

Work it out. Let the club have n members before the newcomers. Their ages total $22n$, and the newcomers add $5 \cdot 43 = 215$. So $22n + 215 = 25(n + 5) = 25n + 125$, which gives $3n = 90$ and $n = 30$. Now let there be a adults and c children: $40a + 10c = 22 \cdot 30 = 660$ and $a + c = 30$. Subtracting $10(a + c) = 300$ from the first equation, $30a = 360$, so $a = 12$ and $c = 18$.

Check. $40 \cdot 12 + 10 \cdot 18 = 480 + 180 = 660 = 22 \cdot 30$. After the newcomers, the total is $660 + 215 = 875$ over 35 members, a mean of 25. Also the mean 22 sits 12 above the children's 10 and 18 below the adults' 40, so children outnumber adults 18 : 12, as found.

6. Answer: $\frac{139}{216}$**. The idea.** The faces split into three "partner" pairs $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$, and two dice sum to 7 exactly when they show partners. Count the complement: rolls in which no number appears together with its partner. In such a roll, all the dice whose values lie in the same pair must show the *same* member of that pair.

Work it out. Build a bad roll by first deciding which pair each die lands in ($3^4 = 81$ ways), then for each pair that is actually used, choosing which of its two numbers every die in it shows (2 ways per used pair). Group by the number of pairs used. One pair: 3 assignments, times 2, giving 6. Two pairs: choose the pairs (3 ways) and split the four dice onto both of them ($2^4 - 2 = 14$ ways), times 2^2, giving $3 \cdot 14 \cdot 4 = 168$. All three pairs: $81 - 3 - 42 = 36$ assignments, times 2^3, giving 288. So $6 + 168 + 288 = 462$ of the $6^4 = 1296$ rolls have no pair summing to 7, and the probability we want is $1 - \frac{462}{1296} = \frac{834}{1296} = \frac{139}{216}$.

Check. Two dice: the method gives $3\cdot 2+6\cdot 4 = 30$ bad rolls out of 36, leaving 6 rolls summing to 7, which is right. Also $139/216 \approx 0.64$: with four dice there are six pairs of dice, each summing to 7 with probability $\frac{1}{6}$, so an answer somewhat below $1 - (\frac{5}{6})^6 \approx 0.67$ is reasonable.

7. Answer: $11\sqrt{55}$**. The idea.** The longer parallel side is a chord of the circle that passes through the center, so it is a diameter. That gives the radius at once, and the shorter side is a chord of known length, whose distance from the center is the height of the trapezoid.

Work it out. The side of length 16 is a diameter, so the radius is 8 and the center O is its midpoint. By symmetry the shorter side, of length 6, is centered directly above O at some height h, so each of its endpoints is 3 units sideways and h units up from O. Such an endpoint is on the circle, so $3^2 + h^2 = 8^2$, giving $h^2 = 55$ and $h = \sqrt{55}$. The area is $\frac{16+6}{2} \cdot \sqrt{55} = 11\sqrt{55}$.

Check. Put O at the origin: the vertices are $(\pm 8, 0)$ and $(\pm 3, \sqrt{55})$, and each is at distance 8 from O (for instance $9 + 55 = 64$), so the circle really passes through all four. The legs have length $\sqrt{5^2 + 55} = \sqrt{80}$, which is less than 16, so the shape is a genuine trapezoid, and $11\sqrt{55} \approx 81.6$ is less than half the circle's area, $32\pi \approx 100.5$, as it must be for a figure inside a half-disk.

8. Answer: 8**. The idea.** Write the condition "same color" = "different colors" (each has probability $\frac{1}{2}$) as a count of pairs, and it collapses to a perfect square: the difference of the two colors

squared equals the total.

Work it out. Let there be r red and b blue marbles, $n = r+b$. Same-color pairs: $\binom{r}{2}+\binom{b}{2}$; different-color pairs: rb. These are equal, so $r(r-1)+b(b-1) = 2rb$, i.e. $r^2 - 2rb + b^2 = r + b$, that is $(r-b)^2 = n$. So n must be a perfect square m^2 and $r - b = \pm m$, giving $r = \frac{m^2+m}{2}$ or $r = \frac{m^2-m}{2}$, two consecutive triangular numbers. With $10 \le n \le 100$ we need $m = 4, 5, \ldots, 10$, and the red counts are $\{6, 10\}, \{10, 15\}, \{15, 21\}, \{21, 28\}, \{28, 36\}, \{36, 45\}, \{45, 55\}$. The distinct values are 6, 10, 15, 21, 28, 36, 45, 55: there are 8 of them.

Check. Take 6 red and 10 blue: same color $\binom{6}{2} + \binom{10}{2} = 15 + 45 = 60$, different $6 \cdot 10 = 60$, out of $\binom{16}{2} = 120$ pairs, so each is $\frac{1}{2}$. Note the trap: there are 14 jars, but each triangular number except 6 and 55 appears for two different jars, so only 8 red counts.

9. Answer: 130**. The idea.** Each trailing zero is a factor $10 = 2 \cdot 5$, so count the factors of 5 and of 2 in N and take the smaller. The factor k^k contributes k times as many 5s as k itself has.

Work it out. Only multiples of 5 contain the prime 5. The numbers 5, 10, 15, 20, 30 each contain one 5, and $25 = 5^2$ contains two. So the exponent of 5 in N is $5 + 10 + 15 + 20 + 30 + 2 \cdot 25 = 80 + 50 = 130$. There are far more 2s: $16^{16} = 2^{64}$ and 24^{24} (which contains 2^{72}) already give $136 > 130$. So the 5s run out first, and N ends in exactly 130 zeros.

Check. Count the 2s properly: summing $k \cdot$ (exponent of 2 in k) over even $k \le 30$ gives $2 + 8 + 6 + 24 + 10 + 24 + 14 + 64 + 18 + 20 + 22 + 48 + 26 + 28 + 30 = 344 > 130$. So the 5s are the bottleneck.

10. Answer: $24\pi + 36\sqrt{3}$**. The idea.** Everything hinges on the overlap (the lens). The two centers and either crossing point of the circles are all 6 apart, so they form equilateral triangles; that pins the lens down as two circular segments with 120° arcs. Then "inside exactly one" is the total area of both disks with the lens removed from each.

Work it out. Let the centers be O_1, O_2 and the crossing points X, Y. Triangles O_1XO_2 and O_1YO_2 are equilateral, so $\angle XO_1Y = 120°$. The part of the lens on O_2's side of line XY is a segment of circle O_1: a 120° sector, $\frac{1}{3} \cdot 36\pi = 12\pi$, minus triangle O_1XY, whose area is $\frac{1}{2} \cdot 6 \cdot 6 \cdot \sin 120° = 9\sqrt{3}$. The lens is two such segments: $2(12\pi - 9\sqrt{3}) = 24\pi - 18\sqrt{3}$. The two disks have total area 72π, and the lens is counted in both, so the region inside exactly one has area $72\pi - 2(24\pi - 18\sqrt{3}) = 24\pi + 36\sqrt{3}$.

Check. Numerically, the lens is about $75.40 - 31.18 = 44.22$ and the answer about $75.40 + 62.35 = 137.75$. Adding back one copy of the lens gives the union, $137.75 + 44.22 = 181.97$, and the union plus the lens must equal the two disks, $181.97 + 44.22 = 226.19 \approx 72\pi$. A tempting slip is to subtract the lens only once, which gives the union, not "exactly one".

11. Answer: 105**. The idea.** Pull out the greatest common divisor. Write $a = dx$ and $b = dy$ with $d = \gcd(a, b)$ and $\gcd(x, y) = 1$. The condition becomes $d(x + y) \mid d^2xy$, that is, $x + y \mid d\,xy$. Since x and y share no factor, $x + y$ shares no factor with x (a common factor would divide y too), nor with y, so $\gcd(x + y, xy) = 1$. Therefore $x + y$ must divide d.

Work it out. Let $s = x + y \ge 2$ and $d = ms$. Then $a + b = ds = ms^2 \le 100$. For a fixed s, the coprime ordered pairs (x, y) of positive integers with $x + y = s$ number $\varphi(s)$ (choose x from 1 to $s - 1$ with $\gcd(x, s) = 1$), and m can be any of $1, \ldots, \lfloor 100/s^2 \rfloor$. Sum over $s = 2, \ldots, 10$:

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$s = 2$: $1 \cdot 25 = 25$; $s = 3$: $2 \cdot 11 = 22$; $s = 4$: $2 \cdot 6 = 12$; $s = 5$: $4 \cdot 4 = 16$; $s = 6$: $2 \cdot 2 = 4$; $s = 7$: $6 \cdot 2 = 12$; $s = 8$: $4 \cdot 1 = 4$; $s = 9$: $6 \cdot 1 = 6$; $s = 10$: $4 \cdot 1 = 4$.

The total is $25 + 22 + 12 + 16 + 4 + 12 + 4 + 6 + 4 = 105$.

Check. The example $(3, 6)$: $d = 3$, $(x, y) = (1, 2)$, $s = 3$, $m = 1$, and $a + b = 9 = 1 \cdot 3^2$. Every valid sum $a+b$ has the form ms^2, so the smallest sums are 4 (the pair $(2, 2)$) and 8 (the pair $(4, 4)$), then 9 (the pairs $(3, 6)$ and $(6, 3)$): a direct check of all sums below 10 finds exactly these.

12. Answer: $5\sqrt{6} - 5\sqrt{2}$. **The idea.** $PA = PB$ puts P on the perpendicular bisector of $\overline{AB}$, which is also the perpendicular bisector of $\overline{CD}$. So $PD = PC = 10 = CD$: triangle PCD is equilateral. That locates P exactly, and then PA is one Pythagorean computation, followed by a nested radical that needs untangling.

Work it out. Place $A = (0, 0)$, $B = (10, 0)$, $C = (10, 10)$, $D = (0, 10)$. Triangle PCD is equilateral with side 10 and P below $\overline{CD}$, so $P = (5, 10 - 5\sqrt{3})$. Then $PA^2 = 5^2 + (10 - 5\sqrt{3})^2 = 25 + 100 - 100\sqrt{3} + 75 = 200 - 100\sqrt{3}$, so $PA = 10\sqrt{2 - \sqrt{3}}$. To simplify, look for $\sqrt{2 - \sqrt{3}} = \sqrt{x} - \sqrt{y}$ with $x + y = 2$ and $2\sqrt{xy} = \sqrt{3}$, i.e. $xy = \frac{3}{4}$: $x = \frac{3}{2}$, $y = \frac{1}{2}$. So $\sqrt{2 - \sqrt{3}} = \frac{\sqrt{6} - \sqrt{2}}{2}$ and $PA = 5\sqrt{6} - 5\sqrt{2}$.

Check. $(5\sqrt{6} - 5\sqrt{2})^2 = 150 + 50 - 50\sqrt{12} = 200 - 100\sqrt{3}$. Numerically $PA \approx 12.25 - 7.07 = 5.18$, and $P = (5, 1.34)$ is indeed inside the square, just above $\overline{AB}$. In angle terms: $\angle PDA = 90^\circ - 60^\circ = 30^\circ$ with $DA = DP$, so $\angle DAP = 75^\circ$ and $\angle PAB = 15^\circ$; then $PA = \frac{5}{\cos 15^\circ} \approx 5.18$, agreeing.

13. Answer: 210. **The idea.** Classify each rectangle by its top-right unit square. The staircase is "closed downward and to the left": if a square is in the staircase, so is every square below it and every square to its left. So once the top-right square is fixed, *any* choice of a bottom row at or below it and a left column at or left of it gives a rectangle that fits.

Work it out. Number rows 1 to 7 from the bottom and columns 1 to 7 from the left; the square in row i and column j exists exactly when $i + j \le 8$. With top-right square (i, j) there are i choices for the bottom row and j for the left column, so ij rectangles. Group the squares by $k = i + j$, from 2 to 8: the total for a given k is $\sum_{i=1}^{k-1} i(k - i) = \binom{k+1}{3}$, giving $1, 4, 10, 20, 35, 56, 84$. Their sum is $1+4+10+20+35+56+84 = 210$. (By the hockey-stick identity, $\binom{3}{3} + \binom{4}{3} + \cdots + \binom{9}{3} = \binom{10}{4} = 210$.)

Check. Try a staircase with 2 rows (3 squares in an L): the formula gives $\binom{5}{4} = 5$, and indeed there are 3 single squares and 2 dominoes. With 3 rows (row lengths 3, 2, 1) it gives $\binom{6}{4} = 15$, and a direct count agrees: 6 unit squares, $2 + 1 = 3$ horizontal dominoes, $2 + 1 = 3$ vertical dominoes, one 1×3 strip, one 3×1 strip and one 2×2 square, for $6 + 3 + 3 + 1 + 1 + 1 = 15$. For comparison, a full 7×7 board has $\binom{8}{2}^2 = 784$ rectangles, so 210 is a plausible share for a shape with 28 of its 49 squares.

14. Answer: $48\sqrt{6}$. **The idea.** Every point of the curved surface is at distance 6 from the center O of the flat face. So each top vertex of the cube is 6 from O. Drop a perpendicular from a top vertex V to the base: its foot is a bottom vertex W, and OVW is a right triangle with a vertical leg $WV = s$ (the cube's edge) and a horizontal leg OW equal to half the diagonal of the bottom face.

Work it out. The bottom face is an $s \times s$ square centered at O, so $OW = \frac{s\sqrt{2}}{2}$ and $OW^2 = \frac{s^2}{2}$. Then $OV^2 = \frac{s^2}{2} + s^2 = \frac{3s^2}{2} = 36$, so $s^2 = 24$ and $s = 2\sqrt{6}$. The volume is $s^3 = 24 \cdot 2\sqrt{6} = 48\sqrt{6}$.

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Check. Reflect the cube in the flat face: together with its mirror image it forms a $2\sqrt{6}\times 2\sqrt{6}\times 4\sqrt{6}$ box inscribed in the full sphere of radius 6. Its space diagonal is $\sqrt{24+24+96} = \sqrt{144} = 12$, the sphere's diameter, as it should be.

15. Answer: 5. **The idea.** The roots fix the parabola up to the single number a: $y = a(x-1)(x-5)$. Tangency to a line means that setting the parabola equal to the line gives a quadratic with exactly one root, so its discriminant is 0. That is one equation for a.

Work it out. Set $a(x^2-6x+5) = 2x-1$, i.e. $ax^2-(6a+2)x+(5a+1) = 0$. Its discriminant is $(6a+2)^2-4a(5a+1) = 36a^2+24a+4-20a^2-4a = 16a^2+20a+4 = 4(4a+1)(a+1)$. So $a=-1$ or $a=-\frac{1}{4}$. The vertex is midway between the roots, at $x=3$, where $y = a(3-1)(3-5) = -4a$. The two vertices have $y=4$ and $y=1$, and the sum is 5.

Check. For $a=-1$: $-x^2+6x-5 = 2x-1$ becomes $x^2-4x+4=0$, a double root at $x=2$, touching at $(2,3)$. For $a=-\frac{1}{4}$: multiply by -4 to get $x^2-6x+5 = -8x+4$, i.e. $x^2+2x+1=0$, a double root at $x=-1$, touching at $(-1,-3)$. Both points lie on $y=2x-1$. (A parabola opening upward cannot work: with $a>0$ the discriminant $4(4a+1)(a+1)$ is positive.)

16. Answer: $\frac{1}{30}$. **The idea.** Deal with the two conditions separately. Where the T's sit says nothing about the order of A, O and U, so the two events are independent and their probabilities multiply.

Work it out. Think of an arrangement as first choosing the 2 positions (out of 10) for the T's, then filling the other 8 positions with the eight different letters. All $\binom{10}{2} = 45$ position pairs for the T's are equally likely, and 9 of them are adjacent pairs, so $P(\text{T's adjacent}) = \frac{9}{45} = \frac{1}{5}$.

Whatever positions the T's take, the other eight letters fill the other eight spots in a uniformly random order. So the three vowels are equally likely to appear in any of their $3! = 6$ relative orders, and $P(\text{A before O before U}) = \frac{1}{6}$, no matter where the T's are.

So the probability is $\frac{1}{5}\cdot\frac{1}{6} = \frac{1}{30}$.

Check. Count directly: 9 adjacent T-pairs, then $\binom{8}{3} = 56$ ways to choose the vowel spots (with the order forced), then $5! = 120$ ways for M, H, C, N, S. That gives $9\cdot 56\cdot 120 = 60480$ arrangements, out of $\frac{10!}{2} = 1814400$ in all, and $\frac{60480}{1814400} = \frac{1}{30}$.

17. Answer: 81. **The idea.** A number is a multiple of 9 exactly when its digit sum is. Four odd digits add to an even number between 4 and 36, so the digit sum must be 18 or 36. Then count with stars and bars.

Work it out. The sum of four odd numbers is even, and it lies between $1\cdot 4 = 4$ and $9\cdot 4 = 36$. The even multiples of 9 in that range are 18 and 36.

Sum 36: every digit is 9, so the only number is 9999.

Sum 18: write each digit as $2e+1$ with $e\in\{0,1,2,3,4\}$. Then $2(e_1+e_2+e_3+e_4)+4 = 18$, so $e_1+e_2+e_3+e_4 = 7$. With no upper limit there are $\binom{7+3}{3} = 120$ solutions. Remove those where some $e_i \geq 5$: put $e_i = 5+e_i'$, leaving a sum of 2, which has $\binom{5}{3} = 10$ solutions, for each of the 4 choices of i (two of them can't both be ≥ 5). That leaves $120-40 = 80$.

Total: $80+1 = 81$.

Check. A few examples with sum 18: 1179, 1359, 3555, 9711, each divisible by 9 (for instance $1179 = 9\cdot 131$). The count 80 also matches a brute-force search of 1000 to 9999.

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18. Answer: $36\sqrt{2}$. **The idea.** In right triangle ABC (right angle at B), the foot of the altitude from B cuts the hypotenuse AC into pieces, and the piece next to A has length $\frac{AB^2}{AC}$ (from the similar triangles, $AB^2 = AC \cdot AH$). So the positions of the two feet tell us AB^2 and AD^2 as fractions of AC^2.

Work it out. Let H be the foot from B and K the foot from D. Then $AH = \frac{AB^2}{AC}$. Triangle ADC has its right angle at D, so $AK = \frac{AD^2}{AC}$. Also $AB^2 + AD^2 = AC^2$, so $AH + AK = AC$: the two feet are placed symmetrically about the middle of AC.

For the feet to trisect AC they must be at $\frac{1}{3}AC$ and $\frac{2}{3}AC$ from A. So one of AB^2, AD^2 is $\frac{1}{3}AC^2$ and the other is $\frac{2}{3}AC^2$. The longer side squared is twice the shorter side squared, so the longer side is $6\sqrt{2}$.

The area is $6 \cdot 6\sqrt{2} = 36\sqrt{2}$.

Check. With $AB = 6$, $AD = 6\sqrt{2}$: $AC^2 = 36 + 72 = 108$, $AC = 6\sqrt{3}$. Then $AH = \frac{36}{6\sqrt{3}} = 2\sqrt{3}$, which is exactly $\frac{1}{3}$ of $6\sqrt{3}$, and $AK = \frac{72}{6\sqrt{3}} = 4\sqrt{3} = \frac{2}{3}AC$. So the three pieces are each $2\sqrt{3}$.

19. Answer: 15. **The idea.** Write $N = 100a + 10b + c$. Then $N + R = 101(a + c) + 20b$, which depends only on $s = a + c$ and b. Find which squares have this form, then count the digit choices.

Work it out. Here $a, c \geq 1$, so $2 \leq s \leq 18$, and $0 \leq b \leq 9$. Then $N + R$ is between 202 and 1998, so it is one of the squares $15^2, 16^2, \ldots, 44^2$.

For a square q we need $q - 101s = 20b$ with $0 \leq 20b \leq 180$. Mod 20, $101s \equiv s$, so s is the remainder of q mod 20 (it can't be that plus 20, since $s \leq 18$). That fixes s from q; then check whether $0 \leq q - 101s \leq 180$. Going through the squares from 225 to 1936, only three pass:

$484 = 101 \cdot 4 + 80$: $s = 4$, $b = 4$.

$625 = 101 \cdot 5 + 120$: $s = 5$, $b = 6$.

$1089 = 101 \cdot 9 + 180$: $s = 9$, $b = 9$.

(For example 1024 would need $s \equiv 4$, but $1024 - 404 = 620$ is too big for $20b$, and $s = 24$ is too big for s.)

Now count digits $a, c \geq 1$ with $a + c = s$: there are $s - 1$ choices. That gives $3 + 4 + 8 = 15$.

Check. The fifteen numbers are $143, 242, 341$; $164, 263, 362, 461$; $198, 297, \ldots, 891$. For instance $164 + 461 = 625 = 25^2$ and $396 + 693 = 1089 = 33^2$.

20. Answer: 16. **The idea.** The ratio need not be a whole number. List the elements in increasing order as $a < b < c$ with $b^2 = ac$. The ratio is a fraction $\frac{p}{q} > 1$ in lowest terms, and then the triple must be $(q^2k,\ pqk,\ p^2k)$ for a positive integer k.

Work it out. If $b = a \cdot \frac{p}{q}$ and $c = a \cdot \frac{p^2}{q^2}$ are integers with $\gcd(p, q) = 1$, then q^2 divides a. Writing $a = q^2k$ gives the triple (q^2k, pqk, p^2k). The only condition is $p^2k \leq 25$, so $p \leq 5$.

$p = 2$, $q = 1$: $4k \leq 25$, $k \leq 6$: 6 triples.

$p = 3$: $q = 1$ gives $k \leq 2$ (2 triples); $q = 2$ gives $9k \leq 25$, again 2 triples: $(4, 6, 9)$ and $(8, 12, 18)$.

$p = 4$: $q = 1$ or $q = 3$, with $16k \leq 25$ so $k = 1$: $(1, 4, 16)$ and $(9, 12, 16)$: 2 triples.

$p = 5$: $q = 1, 2, 3, 4$ with $k = 1$: $(1, 5, 25)$, $(4, 10, 25)$, $(9, 15, 25)$, $(16, 20, 25)$: 4 triples.

Total: $6 + 4 + 2 + 4 = 16$.

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Check. Each set is counted once because the increasing order of a set is unique. Forgetting fractional ratios would give only $6 + 2 + 1 + 1 = 10$. A computer search over all $\binom{25}{3} = 2300$ subsets finds exactly 16.

21. Answer: $\frac{120}{13}$**. The idea.** Edge $\overline{AB}$ is parallel to $\overline{CD}$, which lies in the plane PCD, so every point of $\overline{AB}$ is the same distance from that plane. Use the midpoint M of $\overline{AB}$ instead of A. Then everything happens in one vertical cross-section.

Work it out. Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$, and O the center of the base. The plane through P, M, N is vertical and perpendicular to $\overline{CD}$. So it is perpendicular to plane PCD, and the perpendicular from M to plane PCD lies in it. The distance we want is the distance from M to line PN in triangle PMN.

In triangle PMN: $MN = 10$, the height from P is $PO = 12$, and $PN = \sqrt{5^2 + 12^2} = 13$. Its area is $\frac{1}{2} \cdot 10 \cdot 12 = 60$. With PN as the base, the height from M is $\frac{2 \cdot 60}{13} = \frac{120}{13}$.

Check. With volumes: tetrahedron $APCD$ is half the pyramid, $\frac{1}{2} \cdot \frac{1}{3} \cdot 100 \cdot 12 = 200$. Face PCD has base 10 and slant height 13, so its area is 65. Distance $= \frac{3 \cdot 200}{65} = \frac{120}{13}$. It is also less than $MN = 10$, as it must be.

22. Answer: 4502**. The idea.** If n^2 has 2026 followed by k more digits, then $2026 \cdot 10^k \le n^2 < 2027 \cdot 10^k$. So we want a whole number between $\sqrt{2026 \cdot 10^k}$ and $\sqrt{2027 \cdot 10^k}$. Try $k = 0, 1, 2, \ldots$ in turn; the first k that works gives the smallest n, because a smaller k means a smaller square.

Work it out. The gap between the two square roots is about $\frac{10^k}{2\sqrt{2026 \cdot 10^k}}$, which starts tiny and grows, so check each k with the next integer above $\sqrt{2026 \cdot 10^k}$.

$k = 0$: $45^2 = 2025$ is just below 2026 and $46^2 = 2116$ is too big. No.

$k = 1$: $142^2 = 20164 < 20260$ and $143^2 = 20449 \ge 20270$. No.

$k = 2$: $450^2 = 202500 < 202600$ and $451^2 = 203401 \ge 202700$. No.

$k = 3$: $1423^2 = 2024929 < 2026000$ and $1424^2 = 2027776 \ge 2027000$. No.

$k = 4$: $4501^2 = 20259001 < 20260000$ and $4502^2 = 20268004$, which is below 20270000. Yes!

So $n = 4502$.

Check. $4502^2 = (4500 + 2)^2 = 20250000 + 18000 + 4 = 20268004$, which does begin with 2026. Every smaller n has a square with at most 8 digits, and the cases above show none of those squares starts with 2026.

23. Answer: $\frac{(8+4\sqrt{3})\pi}{3}$**. The idea.** Each tip is a 60° turn about a vertex (the exterior angle of a regular hexagon). During that turn P moves along a circular arc whose radius is its distance to the pivot. It takes 6 tips to return P to its starting role, and the pivots are the six vertices in turn, ending with P itself.

Work it out. Label the vertices $P = V_0, V_1, \ldots, V_5$ going counterclockwise from P, so V_1 is the right end of the bottom side. The first tip is about V_1, the next about V_2, then V_3, V_4, V_5, and finally about $V_0 = P$. After that P is the left end of the bottom side again.

The distances from P to the vertices of a regular hexagon with side 2 are: to a neighbour, 2; to a vertex two steps away, $2\sqrt{3}$ (a short diagonal); to the opposite vertex, 4. So the radii are $2, 2\sqrt{3}, 4, 2\sqrt{3}, 2, 0$.

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Each arc is $\frac{60}{360} = \frac{1}{6}$ of a full circle, so its length is $\frac{\pi}{3} \cdot r$. The total is $\frac{\pi}{3}(2+2\sqrt{3}+4+2\sqrt{3}+2+0) = \frac{\pi}{3}(8+4\sqrt{3}) = \frac{(8+4\sqrt{3})\,\pi}{3}$.

Check. During the six tips the hexagon moves forward by its perimeter, 12, so P ends 12 to the right of where it started. The path length is about $\frac{3.14 \cdot 14.93}{3} \approx 15.6$, which is more than 12, as a curved path must be. A square of side 1 rolled the same way gives $\frac{\pi}{2}(1+\sqrt{2}+1) = \frac{(2+\sqrt{2})\,\pi}{2}$, which is the standard answer for a square.

24. Answer: $\frac{4}{315}$**. The idea.** If the condition holds, the numbers written so far always form a block of consecutive integers. Each new number extends the block by one, either at the low end or at the high end. So a good order is just a starting number followed by a sequence of "left" and "right" steps.

Work it out. Suppose the numbers written so far form a block $\{m, m+1, \ldots, M\}$. A new number differs by 1 from something in the block exactly when it is $m-1$ or $M+1$. Either choice again leaves a block. So by induction the written numbers always form a block.

If the first number is k, the block must grow down to 1, which takes $k-1$ "left" steps, and up to 7, which takes $7-k$ "right" steps. These 6 steps can come in any order, so there are $\binom{6}{k-1}$ good orders starting with k. Summing over k: $\sum_{k=1}^{7} \binom{6}{k-1} = 2^6 = 64$.

Another way to see it: each of the 6 later steps is freely "left" or "right", and the starting number is then forced (it is 1+ the number of left steps).

Probability: $\frac{64}{5040} = \frac{4}{315}$.

Check. With just 1, 2, 3, the rule gives $2^2 = 4$ good orders: 123, 213, 231, 321. The two bad ones, 132 and 312, both put 1 and 3 first, and they don't touch. A computer check of all 5040 orders for $n = 7$ finds exactly 64.

25. Answer: 78**. The idea.** Extend the legs until they meet at a point O. Both circles sit inside the angle at O, touching both of its sides, so a scaling (a dilation) centred at O carries the small circle onto the big one. That scaling carries each tangent line of the small circle to the matching tangent line of the big one: the near line $\overline{CD}$ goes to the cut, and the cut goes to $\overline{AB}$. So the two pieces are similar, and the cut is the geometric mean of the bases. Once the cut is known, a trapezoid with an inscribed circle has leg sum equal to base sum.

Work it out. Let the cut have length x. The dilation that sends the upper circle to the lower one multiplies lengths by $\frac{x}{4}$ (it sends $\overline{CD}$ onto the cut) and also by $\frac{25}{x}$ (it sends the cut onto $\overline{AB}$). So $\frac{x}{4} = \frac{25}{x}$, $x^2 = 100$, $x = 10$.

In a quadrilateral with an inscribed circle, the two tangent segments from each corner are equal, so opposite sides have equal sums. For the upper piece, its two leg pieces add up to $4 + 10 = 14$. For the lower piece, its two leg pieces add up to $10 + 25 = 35$. Together these make up both full legs, so $AD + BC = 14 + 35 = 49$.

Perimeter: $AB + CD + AD + BC = 25 + 4 + 49 = 78$.

Check. Such a trapezoid really exists: take the isosceles one. The upper piece has bases 4 and 10 and legs 7 each (sum 14), so it overhangs by 3 on each side and its height is $\sqrt{49-9} = 2\sqrt{10}$. The lower piece is that one scaled by $\frac{10}{4} = \frac{5}{2}$: bases 10 and 25, legs $\frac{35}{2}$, overhang $\frac{15}{2}$. The overhangs match the legs' slope, $\frac{3}{2\sqrt{10}} = \frac{15/2}{5\sqrt{10}}$, so the legs are straight. The answer does not depend on the

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trapezoid being isosceles: a computer solving for the two circles in several lopsided trapezoids with bases 25 and 4 gets perimeter 78 every time.

26. Answer: 8**. The idea.** Label squares by (x, y), the number of columns and rows from the corner, so the corner is $(0, 0)$. Call a square losing if the player about to move from it loses with best play. A square is losing exactly when every move from it goes to a winning square. Two losing squares can never share a row, a column, or a down-left diagonal (the same value of $x - y$), since then one could move to the other. Build the losing squares from the corner outward.

Work it out. $(0, 0)$ counts as losing: the player who has to "move" there has just watched the opponent win. So every square in row 0, column 0, or on the main diagonal is winning (move straight to the corner).

Build them outward. Each new losing square must use a column, a row and a difference $|x - y|$ that no earlier losing square uses, and the first free choice always works: the k-th losing pair is $(a, a + k)$ where a is the smallest number not yet used as a coordinate. Using the symmetry $(x, y) \leftrightarrow (y, x)$, the losing squares come in mirror pairs:

$(1, 2)$, $(2, 1)$ (difference 1); $(3, 5)$, $(5, 3)$ (difference 2); $(4, 7)$, $(7, 4)$ (difference 3); $(6, 10)$, $(10, 6)$ (difference 4). The next pair would be $(8, 13)$, off the board (coordinates go only up to 11).

Each of these is losing: no row, column or diagonal holds two of them, so every move from one leads to a non-losing square. Every other square has a move to one of them (or to the corner). Together with the corner, these squares use every column and every row except 8, 9 and 11, so any other square in a used row or column can move sideways or down onto a losing square. That leaves the squares with both coordinates in $\{8, 9, 11\}$; their differences $x - y$ are between -3 and 3, and each of those differences is used by the corner or a losing square, so a diagonal move works.

So the second player wins from exactly 8 starting squares.

Check. From $(1, 2)$ the moves go to $(0, 2)$, $(1, 1)$, $(1, 0)$ or $(0, 1)$. From each of these the next player moves straight to the corner, so $(1, 2)$ is losing, as claimed. From $(3, 5)$ every move lands on a winning square: for example $(3, 4) \to (1, 2)$, and $(2, 4) \to (2, 1)$. A full computer table of the board agrees.

27. Answer: 80**. The idea.** Number the rows and columns 1 to 6. Row i must avoid columns i and $7 - i$, and row $7 - i$ must avoid the same two columns. So the 12 forbidden squares form three separate 2×2 blocks: rows $\{1, 6\}$ with columns $\{1, 6\}$, rows $\{2, 5\}$ with columns $\{2, 5\}$, and rows $\{3, 4\}$ with columns $\{3, 4\}$. Use inclusion–exclusion on how many rooks sit on forbidden squares.

Work it out. Let r_k be the number of ways to put k non-attacking rooks on forbidden squares. For one 2×2 block: 1 way with 0 rooks, 4 with 1, 2 with 2. Since the blocks use different rows and columns, we multiply: the r_k are the coefficients of $(1 + 4x + 2x^2)^3$. First $(1 + 4x + 2x^2)^2 = 1 + 8x + 20x^2 + 16x^3 + 4x^4$, and multiplying by $1 + 4x + 2x^2$ gives $1 + 12x + 54x^2 + 112x^3 + 108x^4 + 48x^5 + 8x^6$.

Each choice of k forbidden rooks extends in $(6 - k)!$ ways to a full placement. By inclusion–exclusion, the good placements number $720 - 12 \cdot 120 + 54 \cdot 24 - 112 \cdot 6 + 108 \cdot 2 - 48 \cdot 1 + 8 = 720 - 1440 + 1296 - 672 + 216 - 48 + 8 = 80$.

Check. Try the same method on a 4×4 board (two blocks): $(1 + 4x + 2x^2)^2$ gives $24 - 8 \cdot 6 + 20 \cdot 2 - 16 + 4 = 4$. By hand: row 1 must use column 2 or 3, and so must row 4, so rows 2 and 3 get columns 1 and 4. That is $2 \cdot 2 = 4$ placements, which agrees. A computer search of all 720 placements on the 6×6 board finds 80.

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28. Answer: 60**. The idea.** Find the plane's equation. A unit cube is cut exactly when the plane's expression is below the target value at one corner of the cube and above it at another. For a plane $x + 2y + 3z = 18$, that expression is smallest at a cube's lowest corner and largest at its highest corner, which turns the geometry into counting integer triples.

Work it out. Try $x + 2y + 3z = 18$: it holds at $(0, 0, 6)$ (18), at $(6, 6, 0)$ ($6 + 12 = 18$) and at $(6, 0, 4)$ ($6 + 12 = 18$). The three points are not on one line, so this is the plane.

The unit cube with lowest corner (a, b, c), where $a, b, c \in \{0, 1, \ldots, 5\}$, has $x + 2y + 3z$ ranging from $t = a + 2b + 3c$ at that corner up to $t + 6$ at the opposite corner, taking every value in between inside the cube. The plane passes through the interior exactly when $t < 18 < t + 6$, i.e. $13 \le t \le 17$. (If $t + 6 = 18$ or $t = 18$, the plane only touches a corner.)

Count by c. We need $a + 2b$ between $13 - 3c$ and $17 - 3c$, with $0 \le a, b \le 5$:

$c = 0$: $a + 2b \in [13, 17]$: 4 pairs. $c = 1$: $[10, 14]$: 11 pairs. $c = 2$: $[7, 11]$: 15 pairs. $c = 3$: $[4, 8]$: 15 pairs. $c = 4$: $[1, 5]$: 11 pairs. $c = 5$: $[-2, 2]$, so $a + 2b \le 2$: 4 pairs.

(For example, for $c = 0$ the pairs are $(a, b) = (3, 5), (4, 5), (5, 5), (5, 4)$.) Total: $4 + 11 + 15 + 15 + 11 + 4 = 60$.

Check. The count is symmetric under $(a, b, c) \to (5 - a, 5 - b, 5 - c)$, which sends t to $30 - t$ and the range 13–17 to itself. That matches the symmetric list 4, 11, 15, 15, 11, 4: the plane passes through the center $(3, 3, 3)$ of the big cube. A computer test of the signs at all 8 corners of every unit cube also gives 60.

29. Answer: 57**. The idea.** For most composite n we can find four different numbers less than n whose product is a multiple of n^2. So divisibility holds for most n, and the work is to find the exceptions: primes, twice primes, and two tiny cases.

Work it out. *Case 1: $n = ab$ with $3 \le a < b$.* If $b \ne 2a$, the four numbers $a, b, 2a, 2b$ are different and all less than n ($2b < ab$ because $a \ge 3$). Their product $4a^2b^2 = 4n^2$ divides $(n-1)!$. If $b = 2a$, then $n = 2a^2$. The numbers $a, 2a, 3a, 4a$ are different and less than $2a^2$ (since $a \ge 3$), and their product is $24a^4 = 6n^2$. Either way, it works.

Case 2: n has no such factorization. If n is prime, it fails (see below). Otherwise let p be the smallest prime factor of n and $m = \frac{n}{p} \ge p$. If $p \ge 3$ and $m > p$, then (p, m) is a Case 1 pair, so $m = p$ and $n = p^2$. If $p = 2$: either m is prime, so $n = 2p'$ for a prime p', or $m = uv$ with $2 \le u \le v$. In the second case $n = (2u) \cdot v$ with $2u \ge 4$, which is a Case 1 pair unless $v = 2$ (then $n = 8$) or $v = 2u$. When $v = 2u$, $n = u \cdot 4u$, a Case 1 pair unless $u = 2$ (then $n = 16$). So Case 2 leaves only: n prime, $n = 2p$ with p prime, $n = p^2$, $n = 8$, $n = 16$.

Primes fail: p does not divide $(p - 1)!$ at all. $n = 2p$ fails: the only multiple of p below $2p$ is p itself, so $p^2 \nmid (2p - 1)!$. (For $p = 2$, $n = 4$: $16 \nmid 6$.) $n = p^2$ with $p \ge 5$ works: $p, 2p, 3p, 4p$ are all less than p^2, giving p^4. $n = 9$ fails: $8!$ has only 3^2. $n = 8$ fails: $7!$ has only 2^4, not 2^6. $n = 16$ works: $15!$ contains $2^{7+3+1} = 2^{11}$, and $11 \ge 8$.

Count the failures from 2 to 100: 25 primes; $2p$ for the 15 primes $p \le 50$; and 8, 9. That is 42 failures among 99 numbers, so $99 - 42 = 57$ values work.

Check. Small cases: $n = 6$: $5! = 120$, $36 \nmid 120$ (twice a prime, fails). $n = 12$: $11! = 39916800 = 144 \cdot 277200$ (works). $n = 25$: $24!$ contains 5, 10, 15, 20, so 5^4 (works). A direct computer check over $2 \le n \le 100$ gives 57.

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30. Answer: 43200**. The idea.** Only remainders mod 3 matter. There are three multiples of 3 (remainder 0), four numbers with remainder 1 $(1, 4, 7, 10)$ and three with remainder 2 $(2, 5, 8)$. A neighbour sum is divisible by 3 exactly for the pairs $(0, 0)$ and $(1, 2)$. So the multiples of 3 are pairwise apart, and they split the circle into three stretches. Each stretch must be all 1s or all 2s. Count the remainder patterns first, then fill in the actual numbers.

Work it out. *Remainder patterns.* The three 0s sit in three non-adjacent seats, leaving three nonempty gaps whose sizes add up to 7. Inside a gap no 1 may touch a 2, so each gap is a single residue. The 1-gaps must add up to 4 and the 2-gaps to 3.

Go around the table from a chosen 0: the gap sizes form an ordered triple (g_1, g_2, g_3) of positive integers adding up to 7, and there are $\binom{6}{2} = 15$ such triples. For each triple, count the ways to choose which gaps get 1s:

$\{1, 1, 5\}$ (3 orderings): no set of gaps adds up to 4, so 0 ways.

$\{1, 2, 4\}$ (6 orderings): the 4-gap only, so 1 way.

$\{1, 3, 3\}$ (3 orderings): $1 + 3$ with either 3, so 2 ways.

$\{2, 2, 3\}$ (3 orderings): $2 + 2$, so 1 way.

Weighted total: $0 + 6 + 6 + 3 = 15$ (triple, colouring) pairs. A pattern on the numbered seats is fixed by choosing the seat of the "first" 0 (10 ways) and a (triple, colouring) pair. Each pattern is counted 3 times, once for each of its 0s as the starting one. So there are $\frac{10 \cdot 15}{3} = 50$ remainder patterns.

Filling in numbers. In each pattern the three multiples of 3 can be placed in 3! ways, the four remainder-1 numbers in 4! ways and the three remainder-2 numbers in 3! ways: $6 \cdot 24 \cdot 6 = 864$.

Total: $50 \cdot 864 = 43200$.

Check. The 0s alone can be seated with no two adjacent in $\frac{10}{7}\binom{7}{3} = 50$ ways, and on average each such seating allows exactly one valid colouring of its gaps (the weighted count 15 equals the number of triples). A backtracking search that places the actual numbers 1 to 10 seat by seat finds 43200 placements.

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