MathCastle

Number Theory Open

Individual Round

10 questions • 30 minutes

Name _____

School _____

INSTRUCTIONS

1. This round has 10 problems. You have 30 minutes.
2. No calculators. Write your final answer in the blank beside each problem.
3. Answers must be simplified, with fractions in lowest terms.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

An original MathCastle competition.

1. For a positive integer n, let $S(n) = 1^n + 2^n + 3^n + \cdots + n^n$. For example, $S(3) = 1+8+27 = 36$, which is divisible by 3, while $S(2) = 1+4 = 5$ is not divisible by 2. For how many integers n with $1 \le n \le 1000$ is $S(n)$ divisible by n?

1. _____

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2. A *perfect power* is an integer of the form m^e with integers $m \ge 2$ and $e \ge 2$. For which positive integers n is $n! + 280$ a perfect power? Find the sum of all the perfect powers that arise this way, that is, the sum of $n! + 280$ over all such n.

2. _____

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3. A point (x, y) on the unit circle $x^2 + y^2 = 1$ is called *rational* if both x and y are rational numbers. Write each coordinate as a fraction in lowest terms with a positive denominator (so $0 = \frac{0}{1}$ and $-1 = \frac{-1}{1}$). How many rational points on the unit circle have both coordinates with denominator at most 100?

3. _____

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4. For positive integers x and y, the sum of squares $x^2 + y^2$ usually does not divide the sum of cubes $x^3 + y^3$: for $(x, y) = (1, 2)$, 5 does not divide 9. Sometimes it does: for $(x, y) = (5, 10)$, 125 divides $1125 = 9 \cdot 125$. How many ordered pairs (x, y) of positive integers with $x + y \le 100$ have the property that $x^2 + y^2$ divides $x^3 + y^3$?

4. _____

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5. Start with an integer x in $\{0, 1, 2, \ldots, 9999\}$ and repeatedly replace the current number by the remainder when its square is divided by 10000 (that is, by the last four digits of its square). Call x *recurrent* if, after at least one step, this process returns to x itself. For example, 0 and 1 are recurrent, but 2 is not. How many recurrent x are there in $\{0, 1, \ldots, 9999\}$?

5. _____

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6. The number $30^{10} = 2^{10} \cdot 3^{10} \cdot 5^{10}$ has $11^3 = 1331$ positive divisors. Some of them, such as 1, 8 and 15, leave remainder 1 when divided by 7. How many of the 1331 positive divisors of 30^{10} leave remainder 1 when divided by 7?

6. _____

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7. The numbers $2^k + 1$ for $k = 1, 2, 3, \ldots$ are $3, 5, 9, 17, 33, 65, 129, 257, \ldots$. Some primes divide at least one of them: for example, 3 divides 9 and 11 divides $2^5 + 1 = 33$. Other primes divide none of them: the powers of 2 leave only the remainders 1, 2, 4 on division by 7, so $2^k + 1$ is never a multiple of 7. Find the sum of all primes $p < 100$ that divide $2^k + 1$ for at least one positive integer k.

7. _____

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8. A six-digit positive integer N (so its first digit is not 0) is *rotation-proof* if each of the six numbers obtained by cyclically rotating its digits is divisible by 28. Rotating means moving some number of digits from the front to the back, keeping their order; N itself counts as one of the six rotations, and a rotation that begins with 0 is read as the smaller number it represents. For example, the rotations of 480480 are 480480, 804804, 48048, 480480, 804804, 48048. How many six-digit rotation-proof integers are there?

8. _____

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9. Find the sum of all positive integers n for which $\gcd(1,n)+\gcd(2,n)+\gcd(3,n)+\cdots+\gcd(n,n)=3n$.

9. _____

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10. For a positive integer N, consider the fifteen remainders obtained when N is divided by $2,3,4,\ldots,16$. Find the sum of all positive integers $N < 720720$ for which these fifteen remainders are pairwise different. (Note that 720720 is the least common multiple of $1,2,\ldots,16$.)

10. _____

Answer Key — Number Theory Open Practice Contest 1

1. 500
2. 1400
3. 132
4. 78
5. 252
6. 224
7. 638
8. 70
9. 58
10. 2278074

Solutions — Number Theory Open Practice Contest 1

1. Answer: 500**. The idea.** For odd n the terms cancel in pairs modulo n. For even n, look modulo the full power of 2 dividing n: there every odd term is 1 and every even term is 0, and the count of odd terms is exactly too small by one factor of 2.

Work it out. *Odd n.* Pair k with $n-k$ for $1 \le k \le n-1$. Since n is odd, $(n-k)^n \equiv (-k)^n = -k^n \pmod{n}$, so each pair contributes 0. The last term n^n is also $\equiv 0$. (For $n = 1$, $S(1) = 1$ is divisible by 1.) So every odd n works.

Even n. Write $n = 2^a m$ with $a \ge 1$ and m odd, and work modulo 2^a. If k is even, k^n is divisible by 2^n, and $n \ge 2^a > a$, so $k^n \equiv 0 \pmod{2^a}$. If k is odd, k is a unit modulo 2^a; the unit group modulo 2^a has 2^{a-1} elements, so $k^{2^{a-1}} \equiv 1$, and since $2^{a-1} \mid n$ we get $k^n \equiv 1 \pmod{2^a}$. Among $1, \ldots, n$ exactly $n/2 = 2^{a-1}m$ numbers are odd, so $S(n) \equiv 2^{a-1}m \pmod{2^a}$. As m is odd this is not 0 modulo 2^a, so $2^a \nmid S(n)$ and certainly $n \nmid S(n)$. No even n works.

Hence the answer is the number of odd n in $[1, 1000]$, which is **500**.

Check. $S(2) = 5$, $S(4) = 1 + 16 + 81 + 256 = 354 \equiv 2 \pmod 4$, matching $2^{a-1}m = 2$; $S(5) = 1 + 32 + 243 + 1024 + 3125 = 4425 = 5 \cdot 885$.

2. Answer: 1400**. The idea.** Look at the exponent of 5. Since $280 = 2^3 \cdot 5 \cdot 7$ contains exactly one factor of 5, as soon as $n!$ contains at least two factors of 5 the sum $n! + 280$ contains exactly one. A perfect power m^e has every prime appearing to an exponent divisible by $e \ge 2$, so an exponent of exactly 1 rules it out. That leaves only $n \le 9$ to check by hand.

Work it out. If m^e is a perfect power and a prime p divides it, then p appears to the power $e \cdot v_p(m)$, a multiple of e and hence at least 2. For $n \ge 10$, both 5 and 10 are factors of $n!$, so $25 \mid n!$, while $280 = 5 \cdot 56$ with $5 \nmid 56$. Hence $n! + 280 = 5\,(n!/5 + 56)$ with $n!/5 + 56 \equiv 56 \not\equiv 0 \pmod 5$: exactly one factor of 5, so $n! + 280$ is not a perfect power.

For $5 \le n \le 9$, $n!$ has exactly one factor of 5, so $n! + 280 = 5\,(n!/5 + 56)$ and we need the second factor. $n = 5$: $120 + 280 = 400 = 20^2$. Yes. $n = 6$: $720 + 280 = 1000 = 10^3$. Yes. $n = 7$: $5040 + 280 = 5320 = 5 \cdot 1064$, and $5 \nmid 1064$: exactly one 5, no. $n = 8$: $40320 + 280 = 40600 = 5^2 \cdot 1624$, and $5 \nmid 1624$: the exponent of 5 is 2, so it could only be a square; but $40600 = 100 \cdot 406$ and 406 is not a square ($20^2 < 406 < 21^2$). No. $n = 9$: $362880 + 280 = 363160 = 5 \cdot 72632$, and $5 \nmid 72632$: no. For $n \le 4$: 281 is prime; $282 = 2 \cdot 141$ and $286 = 2 \cdot 143$ with $141, 143$ odd, so the prime 2 appears once; $304 = 2^4 \cdot 19$ has 19 once. None is a perfect power.

So exactly $n = 5$ and $n = 6$ work, and the sum is $400 + 1000 =$ **1400**.

Check. $5! = 120$ and $120 + 280 = 400 = 20^2$; $6! = 720$ and $720 + 280 = 1000 = 10^3$. For $n = 10$: $3628800 + 280 = 3629080 = 5 \cdot 725816$, and 725816 ends in 6, so indeed only one factor of 5.

3. Answer: 132**. The idea.** On the unit circle the two coordinates of a rational point always have the *same* denominator c, and then the numerators with c form a primitive Pythagorean triple. So we are counting primitive triples with hypotenuse $c \le 100$.

Work it out. Let $x = p/q$ and $y = r/s$ in lowest terms. Then $p^2s^2 + r^2q^2 = q^2s^2$, so $q^2 \mid p^2s^2$; since $\gcd(p, q) = 1$, $q^2 \mid s^2$ and $q \mid s$. By symmetry $s \mid q$, so $q = s = c$ and $p^2 + r^2 = c^2$ with $\gcd(p, c) = 1$ (hence also $\gcd(r, c) = 1$). If p or r is 0 we get the four points $(\pm 1, 0)$,

$(0, \pm 1)$. Otherwise $(|p|, |r|, c)$ is a primitive Pythagorean triple, and conversely every primitive triple $a < b < c$ gives the 8 points $(\pm a/c, \pm b/c)$, $(\pm b/c, \pm a/c)$, all with denominator c.

Primitive triples are $c = m^2 + n^2$ with $m > n \geq 1$, $\gcd(m, n) = 1$, $m - n$ odd. With $c \leq 100$: $(2,1) \to 5$, $(3,2) \to 13$, $(4,1) \to 17$, $(4,3) \to 25$, $(5,2) \to 29$, $(6,1) \to 37$, $(5,4) \to 41$, $(7,2) \to 53$, $(6,5) \to 61$, $(7,4) \to 65$, $(8,1) \to 65$, $(8,3) \to 73$, $(7,6) \to 85$, $(9,2) \to 85$, $(8,5) \to 89$, $(9,4) \to 97$. That is 16 triples (note 65 and 85 each occur twice: $16^2 + 63^2 = 33^2 + 56^2 = 65^2$ and $13^2 + 84^2 = 36^2 + 77^2 = 85^2$).

Total: $16 \cdot 8 + 4 =$ **132**.

Check. A prime c is a primitive hypotenuse exactly when $c \equiv 1 \pmod 4$: the primes 5, 13, 17, 29, 37, 41, 53, 61, 7 give 11 triples; the composites 25 (one), 65 (two), 85 (two) give 5 more, for 16.

4. Answer: 78**. The idea.** Pull out $d = \gcd(x, y)$. For coprime a, b, the number $a^2 + b^2$ shares almost nothing with $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, so nearly all of $a^2 + b^2$ has to be supplied by d, which makes d large and leaves very few shapes (a, b).

Work it out. Write $x = da$, $y = db$ with $\gcd(a, b) = 1$. The condition $d^2(a^2 + b^2) \mid d^3(a + b)(a^2 - ab + b^2)$ becomes $a^2 + b^2 \mid d\,(a + b)(a^2 - ab + b^2)$. Now $\gcd(a^2 + b^2,\ a^2 - ab + b^2) = \gcd(a^2 + b^2,\ ab) = 1$, and $\gcd(a^2 + b^2,\ a + b) = \gcd(2b^2,\ a + b)$, which is 2 if a, b are both odd and 1 otherwise. So with $g \in \{1, 2\}$ as described, the condition is simply $\dfrac{a^2 + b^2}{g} \,\Big|\, d$, together with $d(a + b) \leq 100$.

Go through the shapes (take $a \leq b$, then double for order when $a \neq b$): $(1,1)$: need $1 \mid d$, $d \leq 50$: 50 pairs. $(1,2)$: $5 \mid d$, $d \leq 33$: 6 values, 12 pairs. $(1,3)$: $5 \mid d$, $d \leq 25$: 5 values, 10 pairs. $(2,3)$: $13 \mid d$, $d \leq 20$: 1 value, 2 pairs. $(1,4)$: $17 \mid d$, $d \leq 20$: 2 pairs. $(1,5)$: $13 \mid d$, $d \leq 16$: 2 pairs. Every other coprime shape has $(a^2 + b^2)/g > 100/(a + b)$ (for example $(3,4)$ needs $25 \mid d \leq 14$, $(1,7)$ needs $25 \mid d \leq 12$, $(3,5)$ needs $17 \mid d \leq 12$), and the gap only widens as a, b grow.

Total: $50 + 12 + 10 + 2 + 2 + 2 =$ **78**.

Check. $(x, y) = (13, 65)$: $x^2 + y^2 = 4394$ and $x^3 + y^3 = 276822 = 63 \cdot 4394$. $(26, 39)$: $2197 \mid 17576 + 59319 = 76895 = 35 \cdot 2197$.

5. Answer: 252**. The idea.** By the Chinese Remainder Theorem, squaring modulo $10000 = 16 \cdot 625$ is squaring modulo 16 and modulo 625 run side by side, and x is recurrent exactly when both of its components are recurrent (if they return after s and t steps, x returns after $\operatorname{lcm}(s, t)$ steps). So count the recurrent residues modulo 16 and modulo 625 and multiply.

Work it out. *Modulo* 16. An even x becomes divisible by 4, then by 16: it reaches 0 and stays there, so among evens only 0 is recurrent. An odd x has $x^2 \equiv 1$ or 9, and $9^2 \equiv 1$, so it reaches 1; among odds only 1 recurs. Two recurrent residues.

Modulo 625. A nonzero multiple of 5 gains factors of 5 with each squaring and reaches 0, so it is not recurrent, while 0 is. The units form a cyclic group of order $\varphi(625) = 500 = 4 \cdot 125$. Squaring a unit multiplies its discrete logarithm by 2 modulo 500. On the subgroup H of 125 elements of odd order, squaring is a bijection (since 2 is invertible modulo 125), so every element of H lies on a cycle. A unit outside H has order divisible by 2; each squaring halves the 2-part of its order, so after at most two steps it lands in H and can never come back out. Recurrent residues: 0 and the 125 elements of H, so 126.

Answer: $2 \cdot 126 =$ **252**.

Check. $2 \to 4 \to 16 \to 256 \to 5536 \to \cdots$ all stay even and gain powers of 2, so 2 never returns, as the statement says. And $x = 625$ is recurrent: it is odd, and $\equiv 0 \pmod{625}$; indeed $625^2 = 390625 \to 625$.

6. Answer: 224**. The idea.** Modulo 7 everything is a power of the primitive root 3: $3^1 = 3$, $3^2 \equiv 2$, $3^5 \equiv 5$, $3^6 \equiv 1$. So $d = 2^a 3^b 5^c \equiv 3^{2a+b+5c}$, and the condition $d \equiv 1$ becomes the linear congruence $2a + b + 5c \equiv 0 \pmod 6$ on the exponents.

Work it out. Here $30^{10} = 2^{10}3^{10}5^{10}$, so $a, b, c \in \{0, 1, \ldots, 10\}$. For each of the 121 choices of (a, c) the congruence fixes $b \equiv c - 2a \pmod 6$. In $\{0, \ldots, 10\}$ the residues $0, 1, 2, 3, 4$ modulo 6 each occur twice $(0, 6; 1, 7; 2, 8; 3, 9; 4, 10)$, but residue 5 occurs once (just 5). So the count is $2 \cdot 121$ minus the number of pairs (a, c) that force $b \equiv 5$, i.e. with $c \equiv 2a - 1 \pmod 6$.

If $a \equiv 0 \pmod 3$ $(a = 0, 3, 6, 9)$ then $c \equiv 5$: only $c = 5$, giving 4 pairs. If $a \equiv 1$ $(a = 1, 4, 7, 10)$ then $c \equiv 1$: $c = 1, 7$, giving 8 pairs. If $a \equiv 2$ $(a = 2, 5, 8)$ then $c \equiv 3$: $c = 3, 9$, giving 6 pairs. That is 18 pairs.

Answer: $242 - 18 =$ **224**.

Check. Heuristically about $1/6$ of the 1331 divisors should be $\equiv 1$; $1331/6 \approx 222$, very close. Small examples: $8 = 2^3$ has $2a + b + 5c = 6$, and $8 \equiv 1$; $15 = 3 \cdot 5$ has 6, and $15 \equiv 1$.

7. Answer: 638**. The idea.** The prime 2 never works, since every $2^k + 1$ is odd. For an odd prime p, some $2^k + 1$ is a multiple of p exactly when -1 is a power of 2 modulo p, and that happens exactly when the order of 2 modulo p (the least $d \geq 1$ with $2^d \equiv 1$) is *even*. Whether the order is even can be read off from 2^m, where m is the odd part of $p - 1$, and for most primes the residue class of p modulo 8 settles it at once.

Work it out. Let d be the order of 2 modulo the odd prime p. If $d = 2e$ is even, then $x = 2^e$ satisfies $x^2 \equiv 1$ but $x \not\equiv 1$; modulo a prime, $x^2 \equiv 1$ forces $x \equiv \pm 1$, so $2^e \equiv -1$ and $p \mid 2^e + 1$. Conversely, if $2^k \equiv -1$ then $2^{2k} \equiv 1$, so $d \mid 2k$ but $d \nmid k$; an odd d dividing $2k$ would divide k, so d is even.

Write $p - 1 = 2^s m$ with m odd. Since $d \mid p - 1$, d is odd exactly when $d \mid m$, that is, when $2^m \equiv 1 \pmod p$. So p works if and only if $2^m \not\equiv 1 \pmod p$.

Recall Euler's criterion $2^{(p-1)/2} \equiv \pm 1$, with $+1$ exactly when 2 is a square modulo p, and the fact that 2 is a square modulo p exactly when $p \equiv \pm 1 \pmod 8$.

$p \equiv 3$ *or* $5 \pmod 8$. Then $2^{(p-1)/2} \equiv -1$ directly (take $k = (p - 1)/2$), so p works. These are $3, 11, 19, 43, 59, 67, 83$ (sum 285) and $5, 13, 29, 37, 53, 61$ (sum 198).

$p \equiv 7 \pmod 8$. Here $m = (p - 1)/2$ and $2^m \equiv +1$, so d is odd and p fails: $7, 23, 31, 47, 71, 79$.

$p \equiv 1 \pmod 8$. Compute 2^m for the odd part m of $p - 1$. $p = 17$: $m = 1$, $2 \not\equiv 1$, works. $p = 41$: $m = 5$, $2^5 = 32 \not\equiv 1$, works. $p = 97$: $m = 3$, $2^3 = 8 \not\equiv 1$, works. $p = 73$: $m = 9$, $2^9 = 512 = 7 \cdot 73 + 1 \equiv 1$, fails. $p = 89$: $m = 11$, $2^{11} = 2048 = 23 \cdot 89 + 1 \equiv 1$, fails.

The sum is $285 + 198 + (17 + 41 + 97) = 285 + 198 + 155 =$ **638**.

Check. Some explicit witnesses: $2^1 + 1 = 3$, $2^2 + 1 = 5$, $2^4 + 1 = 17$, $2^5 + 1 = 33 = 3 \cdot 11$, $2^6 + 1 = 65 = 5 \cdot 13$, $2^9 + 1 = 513 = 27 \cdot 19$, $2^{10} + 1 = 1025 = 25 \cdot 41$, $2^7 + 1 = 129 = 3 \cdot 43$, and modulo 97: $2^8 = 256 \equiv -35$, $2^{16} \equiv 1225 \equiv 61$, $2^{24} \equiv 61 \cdot (-35) = -2135 = -22 \cdot 97 - 1 \equiv -1$. For the $7 \pmod 8$ primes one can see the square roots of 2 directly: $3^2 \equiv 2 \pmod 7$, $5^2 \equiv 2 \pmod{23}$, $8^2 \equiv 2 \pmod{31}$, $7^2 \equiv 2 \pmod{47}$, $12^2 \equiv 2 \pmod{71}$, $9^2 \equiv 2 \pmod{79}$, so $2^{(p-1)/2} \equiv x^{p-1} \equiv 1$ there. Of the 24 odd primes below 100, 8 fail and 16 work.

8. Answer: 70**. The idea.** Split $28 = 4 \cdot 7$. Divisibility by 4 sees only the last two digits, and every cyclically adjacent pair of digits becomes the last two digits of some rotation, which pins the digits down completely. Divisibility by 7 is automatically inherited by rotations because $7 \mid 999999$.

Work it out. *Modulo 4.* Every cyclically adjacent pair (u, v) (including the last digit followed by the first) must satisfy $4 \mid 10u + v$. Each digit is the v of some pair, so every digit is even; then $10u \equiv 2u \equiv 0 \pmod 4$, so $4 \mid v$ for every digit. Hence all digits lie in $\{0, 4, 8\}$, and conversely such digits make every rotation divisible by 4.

Modulo 7. If N has first digit d, the rotation moving it to the back equals $10N - d(10^6 - 1)$. Since $10^6 - 1 = 999999 = 7 \cdot 142857$, if $7 \mid N$ then 7 divides that rotation too, and repeating, every rotation. So N is rotation-proof iff its digits are in $\{0, 4, 8\}$, the first is nonzero, and $7 \mid N$.

Counting. Such N are $N = 4M$ where M has digits in $\{0, 1, 2\}$ with first digit 1 or 2, and $7 \mid N \iff 7 \mid M$. Write $M = 1000A + B$ with A, B three-digit strings over $\{0, 1, 2\}$ (A starting with 1 or 2). Since $1001 = 7 \cdot 143$, $1000 \equiv -1$, so the condition is $A \equiv B \pmod 7$. Tallying the 27 strings B by residue modulo 7: every residue occurs 4 times except residue 6, which occurs 3 times (020, 111, 202). The 18 strings A include exactly 2 with residue 6 (111 and 202). So the count is $18 \cdot 4 - 2 =$ **70**.

Check. $480480 = 28 \cdot 17160$, $804804 = 28 \cdot 28743$ and $48048 = 28 \cdot 1716$, so the example is rotation-proof; it is $4 \cdot 120120$ with $A = B = 120$.

9. Answer: 58**. The idea.** Group the terms by the value of the gcd. The sum is multiplicative in n, and dividing by n turns it into a product of simple factors, one per prime power, each at least $\frac{3}{2}$. A product equal to 3 can then have only one or two factors, and each case is a small Diophantine equation.

Work it out. The number of $k \in \{1, \ldots, n\}$ with $\gcd(k, n) = d$ equals the number of $j \le n/d$ with $\gcd(j, n/d) = 1$, namely $\varphi(n/d)$. So the sum is $P(n) = \sum_{d \mid n} d\,\varphi(n/d)$, a Dirichlet convolution of multiplicative functions, hence multiplicative. For a prime power, $P(p^e) = \sum_{i=0}^{e} p^i \varphi(p^{e-i}) = p^e + e(p^e - p^{e-1})$, so $\dfrac{P(p^e)}{p^e} = 1 + e\left(1 - \dfrac{1}{p}\right)$. Thus we need $\prod_{p^e \| n} \left(1 + e\left(1 - \frac{1}{p}\right)\right) = 3$. Each factor is at least $1 + \frac{1}{2} = \frac{3}{2}$, and $(\frac{3}{2})^3 > 3$, so n has one or two prime factors ($n = 1$ gives $1 \neq 3$).

One prime. $e(p - 1) = 2p$, so $p - 1 \mid 2p$, and as $\gcd(p - 1, p) = 1$, $p - 1 \mid 2$. $p = 2$: $e = 4$, $n = 16$. $p = 3$: $2e = 6$, $e = 3$, $n = 27$.

Two primes $p < q$. If some exponent is at least 2, that factor is at least $1 + 2 \cdot \frac{1}{2} = 2$, so the other factor is at most $\frac{3}{2}$, forcing it to be exactly $\frac{3}{2}$ (prime 2, exponent 1) and the first to be exactly 2, i.e. $e(p - 1) = p$ for an odd prime p, impossible since $p - 1 \nmid p$ for $p > 2$. So both exponents are 1: $(2 - \frac{1}{p})(2 - \frac{1}{q}) = 3$, i.e. $(2p - 1)(2q - 1) = 3pq$, which rearranges to $pq - 2p - 2q + 1 = 0$, i.e. $(p - 2)(q - 2) = 3$. So $p = 3$, $q = 5$, $n = 15$.

The answer is $15 + 16 + 27 =$ **58**.

Check. For $n = 15$: the gcds are 5 (twice: $k = 5, 10$), 3 (four times), 15 (once) and 1 (eight times): $10 + 12 + 15 + 8 = 45 = 3 \cdot 15$. For $n = 16$: 8 odd k give 1, four give 2, two give 4, one gives 8, one gives 16: $8 + 8 + 8 + 8 + 16 = 48$.

10. Answer: 2278074**. The idea.** Every even modulus sees the parity of N, so the eight remainders modulo $2, 4, \ldots, 16$ all have the same parity as N. There are only eight numbers of that parity in $\{0, \ldots, 15\}$, and the size limits force which remainder takes which value. The odd moduli are

then tied to the even ones ($3 \mid 6$, $5 \mid 10$, $7 \mid 14$, and $9, 15$ via 3 and 5), which leaves freedom only at 11 and 13.

Work it out. Write $r_k = N \bmod k$ and $p = N \bmod 2$. For even k, $r_k \equiv N \equiv p \pmod 2$, and $r_{2j} \le 2j-1$. The values available to $r_2, r_4, \ldots, r_{16}$ are the eight numbers of parity p in $\{0, \ldots, 15\}$. Inductively $r_2 = p$, then $r_4 \in \{p, p+2\}$ forces $r_4 = p+2$, and so on: $r_{2j} = 2j-2+p$. So all eight values of parity p are used up.

Case $p = 0$. $r_{2j} = 2j-2$, i.e. $N \equiv -2$ modulo $2, 4, \ldots, 16$. Then $r_3 \equiv r_6 = 4$ gives $r_3 = 1$; $r_5 \equiv r_{10} = 8$ gives $r_5 = 3$; $r_7 \equiv r_{14} = 12$ gives $r_7 = 5$; $r_9 \in \{1, 4, 7\}$ ($\equiv r_3$) must avoid the used 1 and 4, so $r_9 = 7$; $r_{15} \equiv 1 \pmod 3$ and $\equiv 3 \pmod 5$ gives $r_{15} = 13$. Now every even value and $1, 3, 5, 7, 13$ are used, so $r_{11} \le 10$ must be 9 and $r_{13} \le 12$ must be 11. Every remainder equals $k-2$, so $N \equiv -2 \pmod{720720}$: $N = 720718$.

Case $p = 1$. $r_{2j} = 2j-1$. Similarly $r_3 = 2$ (from $r_6 = 5$), $r_5 = 4$ (from $r_{10} = 9$), $r_7 = 6$ (from $r_{14} = 13$), $r_9 \in \{2, 5, 8\}$ gives $r_9 = 8$, and $r_{15} = 14$. So far $N \equiv -1$ modulo $16, 9, 5, 7$, i.e. $N \equiv -1 \pmod{5040}$, and all odd values plus $2, 4, 6, 8, 14$ are used. The unused values are $0, 10, 12$, so (r_{11}, r_{13}) is one of $(10, 12)$, $(10, 0)$, $(0, 10)$, $(0, 12)$. Each choice gives exactly one N modulo $720720 = 5040 \cdot 143$ by CRT: $(10, 12)$: $N \equiv -1$ everywhere, $N = 720719$. $(10, 0)$: $N = 5040t - 1$ with $13 \mid N$: since $5040 \equiv 9 \pmod{13}$, $9t \equiv 1$, $t \equiv 3 \pmod{13}$, and $N \equiv -1 \pmod{11}$ means $11 \mid t$: $t = 55$, $N = 277199$. $(0, 10)$: $11 \mid N$ means $2t \equiv 1 \pmod{11}$ ($5040 \equiv 2$), so $t \equiv 6 \pmod{11}$, and $r_{13} = 10$ means $9t \equiv 11 \pmod{13}$, so $t \equiv 7 \pmod{13}$: $t = 72$, $N = 362879$. $(0, 12)$: $t \equiv 6 \pmod{11}$ and $13 \mid t$: $t = 39$, $N = 196559$.

Sum: $720718 + 720719 + 277199 + 196559 + 362879 =$ **2278074**.

Check. $362879 = 9! - 1$. Since $9!$ is divisible by $2, 3, \ldots, 10$ and by $12, 14, 15, 16$, the remainders for those moduli are $k-1$; also $362879 = 11 \cdot 32989$ and $362880 = 13 \cdot 27913 + 11$, so $r_{11} = 0$ and $r_{13} = 10$. The fifteen remainders are $1, 2, \ldots, 9, 0, 11, 10, 13, 14, 15$, all different.