MathCastle

Castle Cup

Individual Round

15 questions • 45 minutes

Name _____

School _____

INSTRUCTIONS

1. This round has 15 problems. You have 45 minutes.
2. No calculators. Write your final answer in the blank beside each problem.
3. Answers must be simplified, with fractions in lowest terms.

Total Correct	Scorer's Initials

DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.

1. Ten towers stand beside a long straight road, at the kilometre marks $1, 2, 3, \ldots, 10$. The king wants a well dug somewhere on the road, at kilometre mark x (any real number; the well may even lie beyond the last tower). He insists that the ten distances from the well to the ten towers add up to exactly 30 km. Find the sum of all possible values of x.

1. _____

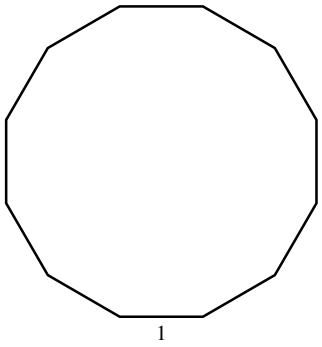
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2. Four fair six-sided dice, coloured red, white, blue and green, are rolled. What is the probability that the four dice can be split into two pairs whose sums are equal? (For example, red 1, white 4, blue 2, green 3 qualifies, since $1 + 4 = 2 + 3$; so does $5, 5, 2, 2$, since $5 + 2 = 5 + 2$.) Express your answer as a common fraction.

2. _____

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3. A rose window has the shape of a regular dodecagon (12-sided polygon) with side length 1. A glazier fills it completely, with no gaps and no overlaps, using two kinds of glass tiles: squares of side 1 and equilateral triangles of side 1. (This can be done, and in more than one way.) How many tiles does the glazier use in total?



3. _____

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4. A castle password is a three-digit number whose three digits are all different; the first digit is not 0. One such password is chosen at random, all of them equally likely. What is the probability that it is divisible by 3? Express your answer as a common fraction.

4. _____

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5. For how many pairs (a, b) of integers with $1 \le a < b \le 30$ does the fraction $\frac{a}{b}$ have a terminating decimal expansion? The fraction need not be in lowest terms: for instance $\frac{3}{6} = 0.5$ counts, and so does $\frac{6}{12}$ as a separate pair.

5. _____

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6. The numbers $1, 2, \ldots, 9$ are placed in the nine cells of a 3×3 grid, one number per cell, so that each of the three rows and each of the three columns has sum 15. The diagonals do not matter. How many such grids are there? (Grids that are rotations or reflections of each other count as different.)

6. _____

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7. A double-seven domino set has exactly one tile for every unordered pair of numbers from 0 to 7, doubles included, so 36 tiles in all (the tile 3-5 is the same as 5-3, and there is one tile 4-4). Tiles are laid end to end in a single straight line (a tile may be turned around), and wherever two tiles touch, the two touching halves must show the same number. Each tile can be used at most once. What is the largest number of tiles that can be placed in one such line?

7. _____

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8. Seven towers stand on seven different islands. The castle engineers will build some bridges, each bridge joining two of the towers, with at most one bridge between any pair of towers. The plan must give every tower exactly two bridges.

Two plans are different if some pair of towers is joined by a bridge in one plan but not in the other. How many different bridge plans are possible?

8. _____

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9. For a real number k, consider the equation $(x-1)(x-2)(x-3)(x-4) = k$.

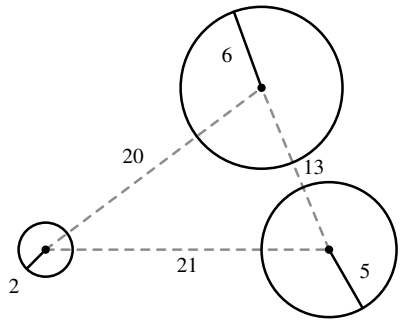
The set of all real numbers k for which this equation has four distinct real solutions x is an open interval (a, b). Find $b - a$. Express your answer as a common fraction.

9. _____

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10. Three round towers stand on flat ground. Seen from above, their walls are circles of radii 2, 5 and 6 metres. The distance between the centres of the radius-2 and radius-5 towers is 21 m, between the radius-2 and radius-6 towers 20 m, and between the radius-5 and radius-6 towers 13 m.

A herald stands at a point P on the ground outside the towers and stretches three taut ropes along the ground. Each rope runs from P to a point on the wall of one of the towers and is tangent to that wall there. There is exactly one point P for which the three ropes have equal lengths. How long is each rope, in metres?



10. _____

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11. A knight stands in a corner square of a 3×3 board. Each minute it makes one knight's move (two squares in one direction and one square in a perpendicular direction), choosing uniformly at random among the moves that stay on the board.

Counting its starting square as visited, what is the expected number of moves until the knight has visited every square of the board except the centre square?

11. _____

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12. A subset S of $\{1, 2, 3, \ldots, 12\}$ is called *peaceful* if there are no two different elements a and b of S such that a divides b.

How many peaceful subsets are there? (The empty set and all one-element sets count as peaceful.)

12. _____

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13. The numbers $1, 2, 3, \ldots, 100$ are written on a board. A *move* consists of choosing two numbers on the board, neither of which divides the other, erasing them, and writing their greatest common divisor and their least common multiple in their place (so there are always 100 numbers on the board, possibly with repeats).

Moves are made, in any order, until no move is possible. How many different values then appear on the board?

13. _____

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14. Find the volume of the solid consisting of all points (x, y, z) in space that satisfy $|x| + |y| + |z| + |x + y + z| \leq 2$. Express your answer as a common fraction.

14. _____

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15. A castle has 16 rooms arranged in a 4×4 grid; two rooms are *neighbours* when they share a wall, that is, when they are next to each other in the same row or in the same column (so a corner room has 2 neighbours, another edge room 3, and a middle room 4).

Each room is painted black or white so that *every* room, whatever its own colour, has exactly one black neighbour.

Two paintings are different if some room is black in one and white in the other. How many such paintings are there?

15. _____

Answer Key — Castle Cup Practice Contest 2

1. 11

2. $\frac{7}{27}$

3. 18

4. $\frac{19}{54}$

5. 113

6. 72

7. 33

8. 465

9. $\frac{25}{16}$

10. 10

11. 28

12. 253

13. 16

14. $\frac{10}{3}$

15. 4

Solutions — Castle Cup Practice Contest 2

1. Answer: 11**. The idea.** Let $f(x) = |x-1| + |x-2| + \cdots + |x-10|$. The towers sit symmetrically about 5.5, so $f(5.5+d) = f(5.5-d)$: whatever solutions $f(x) = 30$ has come in mirror pairs about 5.5. If we can show there are exactly two of them, their sum is $2 \cdot 5.5 = 11$ with no solving at all.

Work it out. Moving the well 1 km to the right changes each distance by ± 1: it grows for towers to its left and shrinks for towers to its right. So on the stretch between marks j and $j+1$ the graph of f is a straight segment of slope $j - (10-j) = 2j - 10$. The slope is negative left of 5, zero on $[5, 6]$ and positive right of 6, so f falls, stays flat at its minimum, then rises steadily. The minimum is $f(5) = 4+3+2+1+0+1+2+3+4+5 = 25 < 30$. Hence $f = 30$ happens exactly once to the right of 6 and, by symmetry, exactly once to the left of 5: two solutions, summing to 11.

Check. Find them anyway. $f(7) = 25 + 2 = 27$ and on $[7, 8]$ the slope is 4, so $27 + 4(x-7) = 30$ gives $x = 7.75$. The mirror image is $x = 3.25$, and $f(3.25) = 2.25 + 1.25 + 0.25 + 0.75 + 1.75 + 2.75 + 3.75 + 4.75 + 5.75 + 6.75 = 30$. Indeed $3.25 + 7.75 = 11$.

2. Answer: $\frac{7}{27}$**. The idea.** Call the four numbers a, b, c, d (red, white, blue, green). There are only three ways to split four dice into two pairs, so a roll is good when at least one of $a+b = c+d$, $a+c = b+d$, $a+d = b+c$ holds. Count each equation, then correct for rolls satisfying two or three of them (inclusion-exclusion). The overlaps turn out to be tiny, because two of the equations together force repeated numbers.

Work it out. There are $6^4 = 1296$ equally likely rolls. The number of ordered pairs of dice with sum $s = 2, 3, \ldots, 12$ is $1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1$, so the number of rolls with $a+b = c+d$ is $1^2 + 2^2 + \cdots + 6^2 + \cdots + 1^2 = 146$. The same holds for each of the three equations: $3 \cdot 146 = 438$.

If $a+b = c+d$ and $a+c = b+d$, subtracting gives $b - c = c - b$, so $b = c$ and then $a = d$; conversely any roll with $b = c$, $a = d$ satisfies both. That is $6 \cdot 6 = 36$ rolls, and likewise 36 for each of the other two pairs of equations: $3 \cdot 36 = 108$. All three equations together force $a = b = c = d$: 6 rolls.

Good rolls: $438 - 108 + 6 = 336$, and $\frac{336}{1296} = \frac{7}{27}$.

Check. Count by the pattern of values instead. All four equal: 6 rolls, all good. Two pairs x, x, y, y with $x \neq y$: always good ($x + y = x + y$), $\binom{6}{2} \cdot \frac{4!}{2!\,2!} = 90$ rolls. Three equal, x, x, x, y: never good. One pair x, x, y, z: good exactly when $2x = y + z$, i.e. x is the average of y and z; the sets $\{y, z\}$ are $\{1, 3\}$ for $x = 2$, $\{2, 4\}, \{1, 5\}$ for $x = 3$, $\{3, 5\}, \{2, 6\}$ for $x = 4$, $\{4, 6\}$ for $x = 5$: 6 sets, each giving 12 rolls, so 72. Four different values $p < q < r < s$: good exactly when $p + s = q + r$, which happens for the 7 sets $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$, $\{1, 2, 4, 5\}$, $\{2, 3, 5, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 4, 6\}$, each giving 24 rolls, so 168. Total $6 + 90 + 72 + 168 = 336$.

3. Answer: 18**. The idea.** Different tilings might look very different, but they all cover the same area. A square has area 1 and a triangle has area $\frac{\sqrt{3}}{4}$; since $\sqrt{3}$ is irrational, the area of the window pins down both counts.

Work it out. Find the area with one convenient tiling. Put a regular hexagon of side 1 in the middle, a unit square on each of its 6 sides, and fill each of the 6 gaps between neighbouring

squares with a unit equilateral triangle (the gap angle is $360° - 120° - 90° - 90° = 60°$). The outline is exactly the regular dodecagon of side 1. Its area is $\frac{3\sqrt{3}}{2} + 6 + 6 \cdot \frac{\sqrt{3}}{4} = 6 + 3\sqrt{3}$.

Now take any tiling with s squares and t triangles: $s + \frac{\sqrt{3}}{4}t = 6 + 3\sqrt{3}$, so $s - 6 = \left(3 - \frac{t}{4}\right)\sqrt{3}$. If $3 - \frac{t}{4} \neq 0$ then $\sqrt{3} = \frac{s-6}{3-t/4}$ would be rational, which it is not. So $t = 12$ and $s = 6$, for 18 tiles, whatever the tiling. (The hexagon in the middle, cut into 6 triangles, gives one such tiling: 6 squares and 12 triangles.)

Check. The standard area formula for a regular n-gon of side 1 is $\frac{n}{4}\cot\frac{180°}{n}$; for $n = 12$ this is $3\cot 15° = 3(2 + \sqrt{3}) = 6 + 3\sqrt{3}$, matching.

4. Answer: $\frac{19}{54}$. The idea. Divisibility by 3 depends only on the digit sum, and the digit sum mod 3 depends only on which residue class each digit lies in: $\{0, 3, 6, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$. Count digit *sets* first, then arrange them, paying attention to the digit 0, which cannot lead.

Work it out. Three residues sum to a multiple of 3 exactly when they are all equal or all different. So the digit set is three digits from one class, $\binom{4}{3} + \binom{3}{3} + \binom{3}{3} = 6$ sets, or one digit from each class, $4 \cdot 3 \cdot 3 = 36$ sets: 42 sets. Those containing 0: $\binom{3}{2} = 3$ from the first kind and $3 \cdot 3 = 9$ from the second, 12 in all; each gives $2 \cdot 2 \cdot 1 = 4$ passwords. The other 30 sets give $3! = 6$ each. Good passwords: $12 \cdot 4 + 30 \cdot 6 = 228$. All passwords: $9 \cdot 9 \cdot 8 = 648$. Probability $\frac{228}{648} = \frac{19}{54}$.

Check. The answer is a bit above $\frac{1}{3}$, which makes sense: over all 900 three-digit numbers exactly $\frac{1}{3}$ are multiples of 3, but requiring distinct digits favours the big class $\{0, 3, 6, 9\}$, which is the only class that can supply three different digits in 4 ways. Recount by strings: without the leading-zero rule there are $42 \cdot 6 = 252$ good strings out of 720; strings with a leading 0 number 72, of which the good ones are $12 \cdot 2 = 24$; so $\frac{252-24}{720-72} = \frac{228}{648}$.

5. Answer: 113**. The idea.** A fraction terminates exactly when its denominator in lowest terms has no prime factors other than 2 and 5. Write $b = t \cdot m$ where t is the largest divisor of b of the form $2^x 5^y$ and m is coprime to 10. Then $\frac{a}{b}$ terminates exactly when all of m cancels, i.e. when $m \mid a$. So we do not reduce fractions one by one; we just count multiples of m.

Work it out. The multiples of m below $b = tm$ are $m, 2m, \ldots, (t-1)m$: that is $t - 1$ good numerators for each b. The values of t for $b = 2, 3, \ldots, 30$ are 2, 1, 4, 5, 2, 1, 8, 1, 10, 1, 4, 1, 2, 5, 16, 1, 2, 1, 20, 1, 2, 1, 8, 25 which add up to 142. Subtracting 1 for each of the 29 denominators gives $142 - 29 = 113$.

Check. Try $b = 12 = 4 \cdot 3$: $t = 4$, so 3 numerators, namely 3, 6, 9 (0.25, 0.5, 0.75), and indeed $\frac{1}{12}, \frac{2}{12}, \frac{4}{12}, \ldots$ all keep a factor 3 downstairs. For $b = 25$, $t = 25$ and all 24 numerators work. A direct computer check of all 435 pairs also gives 113.

6. Answer: 72**. The idea.** Each row is a 3-element subset of $\{1, \ldots, 9\}$ with sum 15, and the three rows split $\{1, \ldots, 9\}$. There are very few such splittings, and the columns are forced to be a different one.

Work it out. The triples with sum 15 are $\{1, 5, 9\}$, $\{1, 6, 8\}$, $\{2, 4, 9\}$, $\{2, 5, 8\}$, $\{2, 6, 7\}$, $\{3, 4, 8\}$, $\{3, 5, 7\}$, $\{4, 5, 6\}$. Split $\{1, \ldots, 9\}$ into three of them by looking at the triple holding 5. With $\{1, 5, 9\}$ the rest is $\{2, 6, 7\}, \{3, 4, 8\}$. With $\{3, 5, 7\}$ the rest is $\{1, 6, 8\}, \{2, 4, 9\}$. With $\{2, 5, 8\}$ or $\{4, 5, 6\}$ the number 1 has no triple left. So there are exactly two splittings, $P = \{159, 267, 348\}$ and $Q = \{168, 249, 357\}$.

A row and a column share exactly one cell, so each column triple meets each row triple in exactly one number. The columns also form a splitting, and it cannot be the same one as the rows (a

column equal to a row would share three numbers with it). So if the rows use P the columns use Q, and vice versa; and each triple of P does meet each triple of Q in exactly one number (for example $\{1, 5, 9\}$ meets them in $1, 9, 5$). Once we choose which splitting gives the rows (2 ways), the order of the rows (3!) and the order of the columns (3!), every cell is forced: it holds the common number of its row and its column. Total $2 \cdot 6 \cdot 6 = 72$.

Check. The 8 ordinary magic squares are among these 72 (their rows are P or Q). A computer search over all 9! placements finds exactly 72 grids.

7. Answer: 33**. The idea.** Think of the numbers 0 to 7 as 8 points and each tile as a road between its two numbers (a double is a loop at one point). A domino line is a walk that uses each road at most once, and the number at each joint is where the walk passes through. Parity decides everything: a walk leaves every point it passes through as often as it enters, so every point except the two ends of the line is used an even number of times.

Work it out. Each number v appears on the 7 tiles v-w with $w \neq v$, plus the double, which counts twice: its degree is $7 + 2 = 9$, odd. In the tiles of a line, each number other than the two end numbers appears an even number of times (every appearance at a joint is paired with the touching half). So among the used tiles at most 2 numbers have odd degree, and at least 6 of the 8 odd numbers must be made even by leaving tiles out. Leaving out a double changes no parity; leaving out a non-double changes the parity of exactly 2 numbers. Hence at least 3 tiles are left out, and a line has at most 33 tiles.

Leaving out exactly 0-1, 2-3, 4-5 makes every degree even except at 6 and 7, and the remaining tiles still connect all eight numbers. By Euler's theorem there is a walk starting at 6, ending at 7, using every remaining road once, which is a line of $36 - 3 = 33$ tiles.

Check. The same reasoning for the double-three set (10 tiles, four numbers each of odd degree 5) says at most $10 - 1 = 9$ tiles, and an exhaustive search confirms that 9 is the longest line there. For the double-seven set, a computer builds an explicit 33-tile line and finds that no line uses 34 or more.

8. Answer: 465**. The idea.** Walk along the bridges. Leaving a tower by one bridge, you arrive at a tower that has exactly one other bridge, so the walk is forced and can only stop by returning to where it began. So a plan is a collection of disjoint closed loops that together use every tower, and each loop has at least 3 towers (two towers could only be joined twice, which is not allowed).

Work it out. The loop lengths are at least 3 and add up to 7, so they are either 7 or $3 + 4$.

One loop through all 7 towers: list the towers around the loop starting from tower 1; the other 6 can follow in $6! = 720$ orders, but each loop is listed twice (once in each direction), giving $720/2 = 360$ plans.

A 3-loop and a 4-loop: choose the 3 towers of the triangle in $\binom{7}{3} = 35$ ways; they form exactly one loop. The remaining 4 towers form a 4-loop in $3!/2 = 3$ ways (the tower opposite a fixed tower can be any of the other 3). That gives $35 \cdot 3 = 105$ plans.

Total: $360 + 105 = 465$.

Check. With 4 towers only a single 4-loop is possible, and $3!/2 = 3$ is right: a plan is fixed by which two pairs of towers are left unbridged, and 4 towers split into two pairs in 3 ways. For 6 towers the same reasoning gives $5!/2 + \frac{1}{2}\binom{6}{3} = 60 + 10 = 70$, the known count of 2-regular graphs on 6 labelled vertices.

9. Answer: $\frac{25}{16}$**. The idea.** Pair the outer factors and the inner factors: $(x-1)(x-4) = x^2-5x+4$ and $(x-2)(x-3) = x^2-5x+6$ differ by exactly 2. So with one new variable y the quartic becomes a quadratic, and the question becomes: how many x does each y come from?

Work it out. Let $y = x^2-5x+4 = \left(x-\frac{5}{2}\right)^2 - \frac{9}{4}$. The equation is $y(y+2) = k$, i.e. $y^2+2y-k = 0$, with roots $y = -1 \pm \sqrt{1+k}$.

Each value $y > -\frac{9}{4}$ comes from exactly two values of x (symmetric about $\frac{5}{2}$), the value $y = -\frac{9}{4}$ from only $x = \frac{5}{2}$, and $y < -\frac{9}{4}$ from none. Different y values come from different x values. So we get four distinct real x exactly when the quadratic in y has two distinct real roots, both greater than $-\frac{9}{4}$.

Two distinct real roots: $1+k > 0$, so $k > -1$. The smaller root must exceed $-\frac{9}{4}$: $-1-\sqrt{1+k} > -\frac{9}{4}$, i.e. $\sqrt{1+k} < \frac{5}{4}$, i.e. $k < \frac{25}{16} - 1 = \frac{9}{16}$. (The larger root is then automatically above $-\frac{9}{4}$.)

So the interval is $\left(-1, \frac{9}{16}\right)$ and $b-a = \frac{9}{16}+1 = \frac{25}{16}$.

Check. The graph of $p(x) = (x-1)(x-2)(x-3)(x-4)$ is W-shaped and symmetric about $x = \frac{5}{2}$. Its middle bump is $p\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2} = \frac{9}{16}$, and its two valleys are where $y = -1$, giving $p = (-1)(1) = -1$. A horizontal line $p = k$ cuts the W four times exactly when it lies strictly between the valleys and the bump: $-1 < k < \frac{9}{16}$.

10. Answer: 10**. The idea.** If a rope from P touches a circle with centre O and radius r at the point T, then $OT \perp PT$, so the rope has length t with $t^2 = PO^2 - r^2$. Equal ropes therefore mean $PA^2 - 4 = PB^2 - 25 = PC^2 - 36$, where A, B, C are the centres. Each equation compares two squared distances, and in coordinates the $x^2 + y^2$ terms cancel: each is the equation of a straight line. Two lines, one point.

Work it out. Put $A = (0,0)$ and $B = (21,0)$, and let $C = (u,v)$ with $v > 0$. From $u^2 + v^2 = 400$ and $(u-21)^2 + v^2 = 169$, subtracting gives $42u - 441 = 231$, so $u = 16$ and $v = 12$.

Let $P = (x,y)$. $PA^2 - PB^2 = 4 - 25 = -21$ becomes $x^2 - (x-21)^2 = 42x - 441 = -21$, so $x = 10$. $PA^2 - PC^2 = 4 - 36 = -32$ becomes $x^2 + y^2 - (x-16)^2 - (y-12)^2 = 32x + 24y - 400 = -32$, so $320 + 24y = 368$ and $y = 2$. Then $t^2 = PA^2 - 4 = 100 + 4 - 4 = 100$, so each rope is 10 m long.

Check. $PB^2 - 25 = (11^2 + 2^2) - 25 = 100$ and $PC^2 - 36 = (6^2 + 10^2) - 36 = 100$. The point $P = (10,2)$ is outside all three towers ($PA^2 = 104 > 4$, $PB^2 = 125 > 25$, $PC^2 = 136 > 36$), so the ropes exist, and the towers do not overlap ($2+5 < 21$, $2+6 < 20$, $5+6 < 13$). The point is unique because the two lines are not parallel: they are perpendicular to AB and to AC, and the centres are not in a line.

11. Answer: 28**. The idea.** No knight's move touches the centre, and each of the other eight squares has exactly two knight's moves. Following them, the eight squares form a single loop, so the knight is just taking a fair random walk around a cycle of 8 squares. On a cycle the visited squares always form one unbroken arc, and the knight always stands at an end of it right after reaching a new square.

Work it out. Label the squares by row and column from $(1,1)$. Starting at a corner, the knight's moves chain as $(1,1) \to (2,3) \to (3,1) \to (1,2) \to (3,3) \to (2,1) \to (1,3) \to (3,2) \to (1,1)$, one cycle of 8, each square with exactly two neighbours.

Suppose j squares have been visited, forming an arc, and the knight is at one end. Unroll the arc as positions $1, 2, \ldots, j$ on a line with the knight at position 1; the unvisited neighbours are

positions 0 and $j+1$ (for $j=7$ they are the same square, which does not matter). Let T_i be the expected time to reach 0 or $j+1$ from position i. Then $T_0 = T_{j+1} = 0$ and $T_i = 1 + \frac{1}{2}(T_{i-1} + T_{i+1})$, whose solution is $T_i = i(j+1-i)$ (the second differences are the constant -2). So $T_1 = j$.

Reaching the new square makes an arc of $j+1$ with the knight at its end again. Starting from $j=1$, the expected total is $1+2+3+4+5+6+7 = 28$.

Check. The first move always finds a new square ($j=1$ costs exactly 1). For $j=2$, the knight at an end of a 2-arc either steps out (probability $\frac{1}{2}$) or steps back to the other end, from where it again has probability $\frac{1}{2}$ of stepping out: a geometric wait with mean 2, matching $T_1 = 2$.

12. Answer: 253**. The idea.** The big numbers $7, 8, \ldots, 12$ never divide anything in the set (twice 7 is 14), and none of them divides another. So they only cause trouble as *multiples* of small elements. Choose the small part $A = S \cap \{1, \ldots, 6\}$ first; then any collection of the numbers from 7 to 12 that are not multiples of an element of A may be added freely.

Work it out. If $1 \in S$ then $S = \{1\}$: one subset. Otherwise A is a peaceful subset of $\{2,3,4,5,6\}$ (forbidden pairs: 2, 4; 2, 6; 3, 6). The multiples in 7..12 are: $2 \to 8, 10, 12$; $3 \to 9, 12$; $4 \to 8, 12$; $5 \to 10$; $6 \to 12$. If f numbers of 7..12 remain free, A contributes 2^f.

$A = \varnothing$: $f = 6$, giving 64.

One element: $\{2\}$: free 7, 9, 11, giving 8; $\{3\}$: free 7, 8, 10, 11, giving 16; $\{4\}$: free 7, 9, 10, 11, giving 16; $\{5\}$: free 7, 8, 9, 11, 12, giving 32; $\{6\}$: free 7, 8, 9, 10, 11, giving 32. Subtotal 104.

Two elements (allowed pairs $\{2,3\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{5,6\}$): free sets $\{7,11\}$, $\{7,9,11\}$, $\{7,10,11\}$, $\{7,8,11\}$, $\{7,9,11\}$, $\{7,9,10,11\}$, $\{7,8,9,11\}$, giving $4+8+8+8+8+16+16 = 68$.

Three elements: the only peaceful triples are $\{2,3,5\}$ (free 7, 11), $\{3,4,5\}$ (free 7, 11) and $\{4,5,6\}$ (free 7, 9, 11), giving $4+4+8 = 16$. No four of 2..6 are peaceful.

Total: $1+64+104+68+16 = 253$.

Check. The same method on $\{1, \ldots, 6\}$ alone: 4, 5, 6 are the free 'big' numbers, and $A \subseteq \{2,3\}$: $\varnothing \to 2^3$, $\{2\} \to 2^1$ (5), $\{3\} \to 2^2$ (4, 5), $\{2,3\} \to 2^1$, plus $\{1\}$: $8+2+4+2+1 = 17$, which a direct listing confirms.

13. Answer: 16**. The idea.** Look at one prime p at a time. If $a = p^s \cdots$ and $b = p^t \cdots$, then $\gcd(a,b)$ has p-exponent $\min(s,t)$ and $\operatorname{lcm}(a,b)$ has $\max(s,t)$: the two exponents are just handed back. So for every prime, the multiset of its exponents over the 100 numbers never changes. And when no move is possible, every two numbers are comparable under divisibility, so the board is a chain $N_1 \mid N_2 \mid \cdots \mid N_{100}$, which forces each prime's exponents to be sorted. The final board is therefore completely determined.

The game ends. If neither of a, b divides the other, then $g = \gcd(a,b) < \min(a,b)$ and $l = \operatorname{lcm}(a,b) > \max(a,b)$ with $gl = ab$; for a fixed product, numbers further apart have a larger sum, so $g + l > a + b$. The sum on the board strictly increases, yet every number stays a divisor of $\operatorname{lcm}(1, \ldots, 100)$, so only finitely many boards are possible and the moves must stop.

Work it out. Sort the final numbers $N_1 \le \cdots \le N_{100}$. The p-exponent of N_k is the k-th smallest p-exponent among $1, \ldots, 100$. Exactly $\lfloor 100/p^j \rfloor$ of the numbers 1..100 have exponent at least j, so N_k has exponent at least j exactly when $k > 100 - \lfloor 100/p^j \rfloor$. Hence $N_{k+1} \neq N_k$ exactly when $100 - k = \lfloor 100/p^j \rfloor$ for some prime power $p^j \le 100$.

The values of $\lfloor 100/p^j \rfloor$: powers of 2 give $50, 25, 12, 6, 3, 1$; powers of 3 give $33, 11, 3, 1$; $5, 25$ give $20, 4$; $7, 49$ give $14, 2$; the primes $11, 13, 17, 19, 23$ give $9, 7, 5, 5, 4$; primes 29 to 47 give 3 or 2; primes from 53 up give 1. The distinct values are $1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 20, 25, 33, 50$: fifteen jumps. So the chain takes $15 + 1 = 16$ different values.

Check. The smallest final value is 1, appearing 50 times (only $k \le 100 - 50$ has every exponent zero), then $N_{51} = 2$ (the jump at $\lfloor 100/2 \rfloor = 50$), and the largest is $\mathrm{lcm}(1, \ldots, 100)$, appearing once. Playing the game out by computer, in any order, ends with exactly 16 distinct values.

14. Answer: $\frac{10}{3}$. **The idea.** Put $w = -(x + y + z)$. Then $x + y + z + w = 0$, so the positive ones among x, y, z, w add up to the same amount P as the absolute values of the negative ones, and the left side is $2P$. The condition says $P \le 1$, i.e. *every* sub-collection of x, y, z, w has sum at most 1. Since the complement of a sub-collection has the opposite sum, this is: $|x|, |y|, |z|, |x+y+z| \le 1$ and $|x + y|, |y + z|, |z + x| \le 1$.

Work it out. The solid is the cube $[-1, 1]^3$ trimmed by the six planes $x + y = \pm 1$, $y + z = \pm 1$, $z + x = \pm 1$ and the two planes $x + y + z = \pm 1$. It is convex and symmetric about the origin, so we cut it into pyramids with apex at the origin, one over each face, each of volume $\frac{1}{3} \cdot (\text{height}) \cdot (\text{area})$.

Face on $x = 1$: then $|1 + y| \le 1$ and $|1 + z| \le 1$ force $y, z \le 0$, and $|1 + y + z| \le 1$ gives $y + z \ge -1$. It is the right triangle with legs 1, area $\frac{1}{2}$, height 1: pyramid $\frac{1}{6}$. Six such faces: 1.

Face on $x + y + z = 1$: then $|x + y| = |1 - z| \le 1$ gives $z \ge 0$, and likewise $x, y \ge 0$. It is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and the pyramid is the corner tetrahedron, volume $\frac{1}{6}$. Two such faces: $\frac{1}{3}$.

Face on $x + y = 1$: then $0 \le x \le 1$, $y = 1 - x$, and $|x + y + z| = |1 + z| \le 1$ gives $-1 \le z \le 0$; all other conditions then hold. It is the parallelogram with corner $(1, 0, 0)$ and edge vectors $(-1, 1, 0)$ and $(0, 0, -1)$, of area $\sqrt{2} \cdot 1 = \sqrt{2}$; the plane $x + y = 1$ is at distance $\frac{1}{\sqrt{2}}$ from the origin, so the pyramid has volume $\frac{1}{3} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{2} = \frac{1}{3}$. Six such faces: 2.

These 14 faces cover the whole boundary (the twelve vertices are $\pm(1, 0, 0)$, $\pm(0, 1, 0)$, $\pm(0, 0, 1)$, $\pm(1, -1, 0)$, $\pm(1, 0$ so the volume is $1 + \frac{1}{3} + 2 = \frac{10}{3}$.

Check. In four dimensions, the points with $x + y + z + w = 0$ and $|x| + |y| + |z| + |w| \le 2$ form a regular cuboctahedron with the 12 vertices obtained by permuting $(1, -1, 0, 0)$; its edge is $\sqrt{2}$, so its volume is $\frac{5\sqrt{2}}{3}(\sqrt{2})^3 = \frac{20}{3}$. Our solid is its shadow on the xyz-space; the hyperplane's unit normal $\frac{1}{2}(1, 1, 1, 1)$ makes $\cos\theta = \frac{1}{2}$ with the w-axis, so volumes shrink by $\frac{1}{2}$: $\frac{10}{3}$.

15. Answer: 4. **The idea.** Write $b(r, c) = 1$ if the room in row r, column c is black and 0 if white, and imagine an all-white row 0 above the grid and an all-white row 5 below it. The condition at room (r, c) reads $b(r - 1, c) + b(r, c - 1) + b(r, c + 1) + b(r + 1, c) = 1$ (terms outside the grid are 0). Only one term involves row $r + 1$, so once rows $r - 1$ and r are known, row $r + 1$ is *forced*: $b(r + 1, c) = 1 - b(r - 1, c) - b(r, c - 1) - b(r, c + 1)$. Hence the top row determines the whole painting, and a top row works exactly when every forced value is 0 or 1 and the forced row 5 comes out all white (that is the condition for the rooms of row 4). So there are at most 16 paintings, and we just test the 16 top rows.

Work it out. Write rows as strings of 0s and 1s. Row 2 is $b(2, c) = 1 - b(1, c - 1) - b(1, c + 1)$, so row 1 cannot have black rooms in both columns 1 and 3, nor in both 2 and 4. That leaves $3 \cdot 3 = 9$ top rows.

$0000 \to 1111$, then row 3 starts $1-0-0-1 = 0$ and next $1-1-1-1 = -1$: fails. $0001 \to 1101 \to$ row 3 has $b(3,3) = 1-0-1-1 = -1$: fails. Its mirror 1000 fails likewise. $0010 \to 1010 \to$ row 3 has $b(3,2) = 1-0-1-1 = -1$: fails; its mirror 0100 fails too.

The four top rows with two adjacent black rooms or two black end rooms all succeed: $0011 \to 1000 \to 1000 \to 0011 \to 0000$, $1100 \to 0001 \to 0001 \to 1100 \to 0000$, $0110 \to 0000 \to 1001 \to 1001 \to 0000$, $1001 \to 1001 \to 0000 \to 0110 \to 0000$. In each, all forced values are 0 or 1 and row 5 is white. So there are exactly 4 paintings.

Check. The four paintings are the four rotations of one picture: three black dominoes against the outer walls, two of them in corners and one in the middle of the opposite wall. Two facts confirm it. Every black room has exactly one black neighbour, so the black rooms pair up into dominoes; and counting pairs (room, black neighbour of it) gives 16 on one side and the sum of the numbers of neighbours of the black rooms on the other. For $0011, 1000, 1000, 0011$ the black rooms have $2+3+3+3+3+2 = 16$ neighbours in all, as required. A computer search of all 2^{16} paintings also finds exactly 4.