MathCastle

# Castle Cup

Individual Round

15 questions • 45 minutes

**Name** _____

**School** _____

**INSTRUCTIONS**

1. This round has 15 problems. You have 45 minutes.
2. No calculators. Write your final answer in the blank beside each problem.
3. Answers must be simplified, with fractions in lowest terms.

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| Total Correct | Scorer's Initials |

**DO NOT BEGIN UNTIL YOU ARE TOLD TO DO SO.**

**1.** A positive integer $N$ is written on the castle blackboard. Two players take turns; on a turn the player chooses one of the nonzero digits of the number currently on the board and subtracts it, replacing the number by the result. (From 2026 a player may move to 2024 or to 2020.) The player who makes the number exactly 0 wins. For how many starting values $N$ with $1 \le N \le 1000$ can the *second* player force a win, no matter how the first player moves?

1. _____

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**2.** A wizard has two jugs. Jug A holds 10 litres of wine and jug B holds 10 litres of water. One round of his spell goes as follows: he pours 2 litres from jug A into jug B and stirs B until it is perfectly mixed, then pours 2 litres from jug B back into jug A and stirs A until it is perfectly mixed. After 3 rounds, how many litres of wine are in jug A? Express your answer as a common fraction.

2. _____

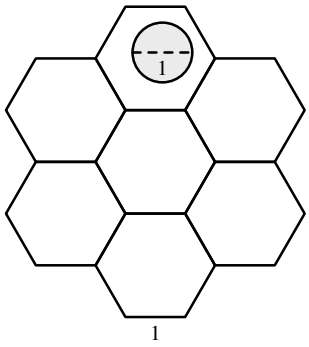
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**3.** A 24-hour digital clock shows the time as HH:MM, from 00:00 to 23:59. Read the display as an ordinary integer $100 \cdot \mathrm{HH} + \mathrm{MM}$, so that 07:15 reads as 715, 00:04 as 4 and 00:00 as 0. During how many of the 1440 minutes of a day is the integer on the display divisible by 11?

3. _____

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**4.** The floor of the great hall is tiled with regular hexagons, each with side length 1 metre, covering the floor with no gaps. A giant drops a coin of diameter 1 metre onto the floor, and the centre of the coin is equally likely to land at any point of the (very large) floor. What is the probability that the coin lies entirely inside a single tile? Express your answer in simplest radical form.




4. _____

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**5.** Each pane of enchanted glass reflects exactly $\frac{1}{3}$ of any light that hits it (from either side) and lets the other $\frac{2}{3}$ pass through; it absorbs none. Three such panes stand parallel to one another, one behind the other, and a beam of light is shone straight at the front pane. Light may bounce back and forth between the panes any number of times. What fraction of the light eventually comes out beyond the back pane? Express your answer as a common fraction.

5. _____

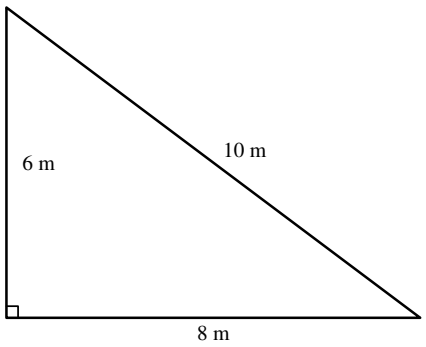
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**6.** Nine treasure cards numbered 1 through 9 are shuffled, with all orders equally likely, and placed face down in a pile. Ada turns the cards over one at a time and keeps a running total of the numbers she has turned over. She stops as soon as the total is greater than 12. What is the expected number of cards she turns over? Express your answer as a common fraction.

6. _____

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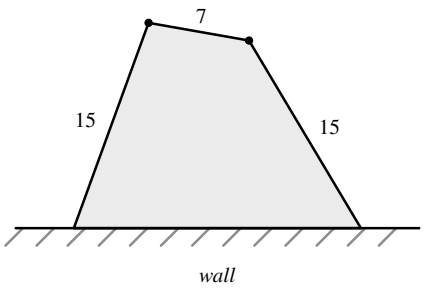
**7.** A castle lawn is shaped like a right triangle with legs 6 m and 8 m (so its hypotenuse is 10 m). It is watered by two identical sprinklers, each of which waters exactly the points within distance $r$ of the spot where it stands (a closed disc of radius $r$). The sprinklers may be placed anywhere. What is the least value of $r$ for which the two sprinklers together can water every point of the lawn, including its edges? Express your answer as a common fraction.




7. _____

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**8.** Three straight fence panels, of lengths 15, 7 and 15 metres, are hinged end to end in that order. The two free ends are set against the long, straight outer wall of a castle, so that the three panels and the stretch of wall between the two free ends enclose a region of flat ground on one side of the wall. The free ends may slide along the wall and the hinges may take any angles, as long as the panels stay on that side of the wall and do not cross each other. What is the largest possible area of the enclosed region, in square metres?




8. _____

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**9.** A castle vault opens with a positive integer that is a multiple of 99 and whose decimal digits add up to exactly 99. The vault-keeper always uses the smallest such integer. What is the vault-keeper's number?

9. _____

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**10.** Compute $\lfloor \log_2 2026 \rfloor + \lfloor \log_3 2026 \rfloor + \lfloor \log_4 2026 \rfloor + \cdots + \lfloor \log_{2026} 2026 \rfloor$, the sum of $\lfloor \log_k 2026 \rfloor$ over all integers $k$ from 2 to 2026. (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.)

10. _____

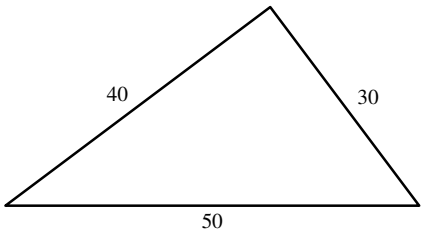
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**11.** Three pages each flip their own fair coin repeatedly until it first lands heads, and each records how many flips that took (so each number is a positive integer, equal to $n$ with probability $2^{-n}$). What is the probability that the three recorded numbers are the side lengths of a non-degenerate triangle? Express your answer as a common fraction.

11. _____

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**12.** A triangular field has sides of length 30, 40 and 50 metres. A single straight fence is built from one side of the field to another so that it divides the field into two parts with equal areas and also equal perimeters (the perimeter of each part includes the fence). How long is the fence, in metres? Express your answer in simplest radical form.




12. _____

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**13.** Ten stepping stones stand in a row across a moat, numbered 1 to 10 in order. A page must stand on every stone exactly once: he starts on any stone he likes, and each hop takes him to a stone at most two places away (from stone 5 he may hop to 3, 4, 6 or 7), never to a stone he has already visited. He may finish on any stone. How many different routes are there, where a route is the ordered list of the ten stones in the order he visits them?

13. _____

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**14.** For a positive integer $n$, the squares $1^2, 2^2, \ldots, n^2$ are split into two groups (each square going into exactly one group; a group may be empty). Let $D(n)$ be the smallest possible value of $|A - B|$, where $A$ and $B$ are the two group sums (so $D(n) = 0$ exactly when the sums can be made equal). Find $D(1) + D(2) + \cdots + D(100)$.

14. _____

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**15.** A sentry paces a long straight corridor. He takes 12 steps, each exactly one pace forward or one pace back, so there are $2^{12}$ possible step sequences. In how many of these sequences does the sentry stand on at most 4 different spots of the corridor during his walk, counting his starting spot? (Equivalently: his farthest-forward and farthest-back positions are at most 3 paces apart.)

15. _____

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# Answer Key — Castle Cup Practice Contest 1

**1.** 100

**2.** $\frac{175}{27}$

**3.** 131

**4.** $\frac{4-2\sqrt{3}}{3}$

**5.** $\frac{2}{5}$

**6.** $\frac{55}{18}$

**7.** $\frac{25}{8}$

**8.** 192

**9.** 1098999999999

**10.** 2094

**11.** $\frac{1}{3}$

**12.** $20\sqrt{3}$

**13.** 208

**14.** 60

**15.** 1028

# Solutions — Castle Cup Practice Contest 1

**1. Answer:** 100**. The idea.** Look at the units digit. A multiple of 10 has units digit 0, and any legal move subtracts something from 1 to 9, so it always leaves a number that is not a multiple of 10. From a number that is not a multiple of 10 the units digit is nonzero, so a player may subtract exactly that digit and land on a multiple of 10. So the player who faces a multiple of 10 is stuck on the losing side forever.

**Work it out.** Suppose the first player faces a multiple of 10. Whatever she subtracts, the second player answers by subtracting the new units digit, which returns the board to a (smaller) multiple of 10. The numbers keep shrinking, and 0 is itself a multiple of 10, so eventually the second player is the one who reaches 0 and wins. (A move never overshoots below 0: a digit of a number is never larger than the number.) If $N$ is not a multiple of 10, the first player uses the same strategy herself and wins. So the second player wins exactly when $N$ is a multiple of 10: $10, 20, \ldots, 1000$, which is 100 values.

**Check.** Small cases: from 1 through 9 the first player subtracts the whole number and wins at once. From 10 she must subtract 1, reaching 9, and her opponent subtracts 9 to win. From 11 she subtracts 1 to reach 10 and wins. The pattern holds, giving 100.

**2. Answer:** $\frac{175}{27}$**. The idea.** After every round each jug again holds exactly 10 litres, and the total amount of wine is always 10. So one number tells the whole story: the difference $d$ between the wine in A and the wine in B. One round turns out to multiply $d$ by a fixed factor.

**Work it out.** Say that at the start of a round A holds $a$ litres of wine and B holds $b = 10 - a$. Pouring 2 of A's 10 litres moves $\frac{a}{5}$ litres of wine, so B then holds 12 litres containing $b + \frac{a}{5}$ of wine. Pouring 2 of those 12 litres back moves $\frac{1}{6}\left(b + \frac{a}{5}\right)$ of wine. So the new amount in A is $a' = \frac{4a}{5} + \frac{b}{6} + \frac{a}{30} = \frac{5a}{6} + \frac{b}{6}$, and similarly $b' = 10 - a' = \frac{a}{6} + \frac{5b}{6}$. Subtracting, $a' - b' = \frac{4}{6}(a - b) = \frac{2}{3}(a - b)$.

At the start $d = 10 - 0 = 10$; after 3 rounds $d = 10 \cdot \left(\frac{2}{3}\right)^3 = \frac{80}{27}$. Since $a + b = 10$, jug A holds $\frac{1}{2}\left(10 + \frac{80}{27}\right) = \frac{175}{27}$ litres of wine.

**Check.** Round by round: $a$ goes $10 \to \frac{25}{3} \to \frac{65}{9} \to \frac{175}{27}$. For instance after round 1: $a = \frac{5 \cdot 10}{6} + \frac{0}{6} = \frac{25}{3}$ and $b = \frac{5}{3}$, and $d = \frac{25}{3} - \frac{5}{3} = \frac{20}{3} = \frac{2}{3} \cdot 10$. As the rounds go on, $a$ drifts down toward 5, as it should.

**3. Answer:** 131**. The idea.** Since $100 = 9 \cdot 11 + 1$, we have $100 \equiv 1 \pmod{11}$, so $100\,\mathrm{HH} + \mathrm{MM} \equiv \mathrm{HH} + \mathrm{MM} \pmod{11}$. The question becomes: for how many pairs (HH, MM) is HH + MM a multiple of 11?

**Work it out.** Among the minutes $0, 1, \ldots, 59$ (that is $5 \cdot 11 + 5$ numbers), the residues $0, 1, 2, 3, 4 \pmod{11}$ occur 6 times each and the residues $5, \ldots, 10$ occur 5 times each. For a given hour $h$ we need $\mathrm{MM} \equiv -h \pmod{11}$. Over any 11 consecutive hours, $-h$ runs through every residue once, giving $5 \cdot 6 + 6 \cdot 5 = 60$ good minutes. The hours 0 to 21 are two full cycles: 120. Hour 22 needs $\mathrm{MM} \equiv 0$: 6 minutes. Hour 23 needs $\mathrm{MM} \equiv -1 \equiv 10$: 5 minutes (10, 21, 32, 43, 54). Total $120 + 6 + 5 = 131$.

**Check.** Spot-check hour 22: the minutes 00, 11, 22, 33, 44, 55 give $2200 = 11 \cdot 200$, $2211 = 11 \cdot 201$, and so on up to $2255 = 11 \cdot 205$ — six times, as counted. And 23:10 gives $2310 = 11 \cdot 210$.

**4. Answer:** $\frac{4-2\sqrt{3}}{3}$. **The idea.** Every tile is the same, so it is enough to look at one tile: the probability is the fraction of a tile's area where the centre may land so that the coin stays inside. The coin (radius $\frac{1}{2}$) stays inside exactly when its centre is at distance at least $\frac{1}{2}$ from every side. Pushing each side of a regular hexagon inward by $\frac{1}{2}$ gives a smaller regular hexagon with the same centre, so the answer is a ratio of similar figures.

**Work it out.** Measure hexagons by their apothem (distance from the centre to a side). A regular hexagon of side 1 is six equilateral triangles of side 1, so its apothem is $\frac{\sqrt{3}}{2}$. The good region is the regular hexagon with apothem $\frac{\sqrt{3}}{2} - \frac{1}{2}$. The ratio of apothems is $\frac{\sqrt{3}-1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$, and areas scale by its square: $\left(1 - \frac{1}{\sqrt{3}}\right)^2 = 1 - \frac{2}{\sqrt{3}} + \frac{1}{3} = \frac{4}{3} - \frac{2\sqrt{3}}{3} = \frac{4-2\sqrt{3}}{3}$.

**Check.** Numerically this is about 0.179: a coin as wide as a tile's side usually lands across a grout line, which is plausible. Also, $\frac{4-2\sqrt{3}}{3} = \frac{(\sqrt{3}-1)^2}{3}$, which is exactly $\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)^2$.

**5. Answer:** $\frac{2}{5}$. **The idea.** Treat any stack of panes as a single "super-pane" with a transmission $T$ and a reflection $R$ (with $R + T = 1$, since nothing is absorbed). Two layers combine by summing a geometric series over the bounces between them, and we can add one pane at a time.

**Work it out.** Put a layer with $(R_1, T_1)$ in front of a layer with $(R_2, T_2)$, and assume (as holds here, by symmetry of each step) that the front layer reflects equally from both sides. Light gets through the first layer ($T_1$); then each time it reaches the second layer, the fraction $T_2$ escapes and $R_2$ comes back, of which $R_1$ is sent forward again. So the total transmission is $T_1T_2\left(1 + R_1R_2 + (R_1R_2)^2 + \cdots\right) = \frac{T_1T_2}{1-R_1R_2}$.

Two panes: $T = \frac{\frac{2}{3}\cdot\frac{2}{3}}{1-\frac{1}{9}} = \frac{4/9}{8/9} = \frac{1}{2}$, so $R = \frac{1}{2}$. (A two-pane stack is symmetric front to back, so its reflection is the same from either side.) Now add the third pane behind it: $T = \frac{\frac{1}{2}\cdot\frac{2}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \frac{1/3}{5/6} = \frac{2}{5}$.

**Check.** For a lossless layer the ratio $\frac{R}{T}$ simply adds when layers are stacked. One pane has $\frac{R}{T} = \frac{1}{2}$; three panes give $\frac{R}{T} = \frac{3}{2}$, so $\frac{1}{T} = 1 + \frac{R}{T} = \frac{5}{2}$ and $T = \frac{2}{5}$. Two panes give $\frac{R}{T} = 1$ and $T = \frac{1}{2}$, matching the step above.

**6. Answer:** $\frac{55}{18}$. **The idea.** For a count $T$ of cards turned, $E[T] = \sum_{k\geq 0} P(T > k)$. Ada is still going after $k$ cards exactly when the first $k$ cards total at most 12, and the first $k$ cards form a uniformly random $k$-element set. So each term is a count of small subsets of $\{1, \ldots, 9\}$ divided by $\binom{9}{k}$.

**Work it out.** $k = 0$ and $k = 1$: every single card is at most 9, so $P(T > 0) = P(T > 1) = 1$.

$k = 2$: of the $\binom{9}{2} = 36$ pairs, those with sum at least 13 are $\{4,9\}$, $\{5,8\}$, $\{5,9\}$, $\{6,7\}$, $\{6,8\}$, $\{6,9\}$, $\{7,8\}$, $\{7,9\}$, {
which is 9 pairs, so $P(T > 2) = \frac{27}{36} = \frac{3}{4}$.

$k = 3$: count sets of three different numbers with sum $6, 7, \ldots, 12$. They number $1, 1, 2, 3, 4, 5, 7$ respectively (for 12: $\{1,2,9\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,6\}$, $\{2,3,7\}$, $\{2,4,6\}$, $\{3,4,5\}$), a total of 23 out of $\binom{9}{3} = 84$.

$k = 4$: the smallest sum is 1+2+3+4 = 10; the sets with sum at most 12 are $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,3,6\}$, {1, 2,
so $P(T > 4) = \frac{4}{126} = \frac{2}{63}$.

$k \geq 5$: five different cards total at least 15, so the probability is 0.

Hence $E[T] = 1 + 1 + \frac{3}{4} + \frac{23}{84} + \frac{2}{63} = \frac{504+189+69+8}{252} = \frac{770}{252} = \frac{55}{18}$.

**Check.** Compute the distribution: $P(T=2) = 1 - \frac{3}{4} = \frac{1}{4}$, $P(T=3) = \frac{3}{4} - \frac{23}{84} = \frac{40}{84}$, $P(T=4) = \frac{23}{84} - \frac{2}{63} = \frac{61}{252}$, $P(T=5) = \frac{2}{63}$. These add to $\frac{63+120+61+8}{252} = 1$, and $2\cdot\frac{63}{252} + 3\cdot\frac{120}{252} + 4\cdot\frac{61}{252} + 5\cdot\frac{8}{252} = \frac{126+360+244+40}{252} = \frac{770}{252} = \frac{55}{18}$.

**7. Answer: $\frac{25}{8}$. The idea.** Cut the lawn into two pieces, one per sprinkler. The obvious cuts (along the altitude, or from the right angle to the midpoint of the hypotenuse) leave a piece whose longest side is 8, which needs $r = 4$. The better cut is the perpendicular bisector of the hypotenuse: it makes both pieces fit inside circles of the same size. To prove nothing smaller works, follow five special points around.

**Work it out.** Put the right angle at $C = (0,0)$, with $A = (8,0)$ and $B = (0,6)$. Let $M = (4,3)$ be the midpoint of $AB$. The perpendicular bisector of $AB$ meets $CA$ at a point $Q = (q,0)$ with $QA = QB$: $(8-q)^2 = q^2 + 36$, so $q = \frac{7}{4}$ and $QA = QB = \frac{25}{4}$.

*Upper bound:* the segment $MQ$ cuts the lawn into triangle $AMQ$ and quadrilateral $CQMB$. Triangle $AMQ$ has a right angle at $M$, so it lies inside the circle with diameter $AQ$, of radius $\frac{25}{8}$. Quadrilateral $CQMB$ has right angles at $C$ and at $M$, so all four vertices lie on the circle with diameter $BQ$, again of radius $\frac{25}{8}$, and the convex quadrilateral lies inside that circle. So $r = \frac{25}{8}$ works.

*Lower bound:* suppose $r < \frac{25}{8}$, which is less than 4. A disc of radius $r$ containing $A$ cannot also contain $C$ (distance 8) or $B$ (distance 10), so one sprinkler covers $A$ and the other covers both $B$ and $C$. Where is $M$? If the second disc also covers $M$, it contains $B$, $C$ and $M$. Triangle $BCM$ has sides $6, 5, 5$, so it is acute, and the smallest disc containing an acute triangle is its circumcircle, of radius $\frac{6\cdot5\cdot5}{4\cdot12} = \frac{25}{8}$ — too big. So $M$ is in the first disc, with $A$. Now $Q$: triangle $BCQ$ has a right angle at $C$ and triangle $AMQ$ a right angle at $M$, so a disc containing $B, C, Q$ needs radius at least $\frac{BQ}{2} = \frac{25}{8}$, and a disc containing $A, M, Q$ needs at least $\frac{AQ}{2} = \frac{25}{8}$. Either way $Q$ cannot be watered — a contradiction. So the least radius is $\frac{25}{8}$.

**Check.** $\frac{25}{8} = 3.125$, below the 4 of the obvious cuts and above the 3 forced by the two closest corners $B$ and $C$. The circumcircle of $B, C, M$ is the circle on diameter $BQ$ (angles $C$ and $M$ are right), which is why both bounds meet at the same point $Q$.

**8. Answer:** 192**. The idea.** Reflect the whole enclosure in the wall. The region and its mirror image form a hexagon with sides $15, 7, 15, 15, 7, 15$ whose area is twice the region's. Among all polygons with given side lengths the cyclic one has the largest area, and that cyclic hexagon is symmetric about a diameter, so the best enclosure has all four corners on a semicircle whose diameter lies along the wall. The lengths are chosen so that this diameter comes out to be 25.

**Work it out.** Put the four corners on a circle of radius $R$ with the two free ends at the ends of a diameter. A chord of length $\ell$ subtends a central angle $2\theta$ with $\sin\theta = \frac{\ell}{2R}$. With half-angles $\beta$ for the panels of length 15 and $\alpha$ for the panel of length 7, the three central angles fill the half-turn: $2\beta + 2\alpha + 2\beta = 180°$, i.e. $\alpha = 90° - 2\beta$, so $\sin\alpha = \cos 2\beta = 1 - 2\sin^2\beta$. Writing $d = 2R$, this says $\frac{7}{d} = 1 - \frac{2\cdot225}{d^2}$, i.e. $d^2 - 7d - 450 = 0 = (d-25)(d+18)$, so $d = 25$ and $R = \frac{25}{2}$. Then $\sin\beta = \frac{3}{5}$, $\cos\beta = \frac{4}{5}$ and $\sin\alpha = \frac{7}{25}$, $\cos\alpha = \frac{24}{25}$. Joining the centre to the corners splits the region into three isosceles triangles with apex angles $2\beta, 2\alpha, 2\beta$, so the area is $\frac{1}{2}R^2(2\sin 2\beta + \sin 2\alpha) = \frac{1}{2}\cdot\frac{625}{4}\left(2\cdot\frac{24}{25} + \frac{336}{625}\right) = \frac{625}{8}\cdot\frac{1536}{625} = 192$. Why this is the maximum: any admissible enclosure and its mirror image form a simple hexagon with sides $15, 7, 15, 15, 7, 15$ and twice the area; that area is at most the cyclic hexagon's, which is exactly the doubled semicircle configuration above. So no enclosure beats 192.

**Check.** Coordinates: the diameter runs from $(-\frac{25}{2}, 0)$ to $(\frac{25}{2}, 0)$. Each triangle on the centre has area $\frac{1}{2}\cdot$ base $\cdot$ distance from the centre to the chord: for a 15-chord the distance is $\sqrt{12.5^2 - 7.5^2} = 10$, giving 75; for the 7-chord it is $\sqrt{12.5^2 - 3.5^2} = 12$, giving 42. Total $75 + 42 + 75 = 192$.

**9. Answer:** 1098999999999**. The idea.** A digit sum of 99 already makes the number a multiple of 9, so the real condition is divisibility by 11: the sum of the digits in the odd places (units, hundreds, …) minus the sum in the even places must be a multiple of 11. That difference has the same parity as the total 99, so it is odd: $\pm 11$ or $\pm 33$ (or more). This forces the number of digits, and then we minimise greedily from the left.

**Work it out.** Let $O$ and $E$ be the digit sums in the odd and even places (counting from the units digit), so $O + E = 99$ and $O - E \in \{\pm 11, \pm 33, \dots\}$. Then $\{O, E\}$ is $\{55, 44\}$ or $\{66, 33\}$ or more lopsided, so one class of places has digit sum at least 55 and therefore at least $\lceil 55/9 \rceil = 7$ digits. With 12 or fewer digits each class has at most 6 digits (sum at most 54), so we need at least 13 digits. With 13 digits the odd places are the 7 places $1, 3, \dots, 13$ (including the leading digit) and the even places are the 6 places $2, \dots, 12$. The 6 even places sum to at most 54, so $E = 44$ and $O = 55$ (the choice $E = 33$, $O = 66$ would need $O \le 63$, impossible). To make the number small, make the leading digit (an odd place) as small as possible: the other six odd places hold at most 54, so the leading digit is at least $55 - 54 = 1$; take 1, and then those six odd places are all 9. The even places hold 44 in six digits; reading left to right, the first even place (the second digit of the number) is as small as possible: the other five hold at most 45, so it can be 0; then the next even place is at least $44 - 36 = 8$, and the remaining four are 9. The digits are $1, 0, 9, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9$.

**Check.** Digit sum $1 + 0 + 9 + 8 + 9 \cdot 9 = 99$. Counting places from the units digit, the odd places hold $9, 9, 9, 9, 9, 9, 1$ (sum 55) and the even places hold $9, 9, 9, 9, 8, 0$ (sum 44); the difference 11 confirms divisibility by 11, and indeed $1098999999999 = 99 \cdot 11101010101$.

**10. Answer:** 2094**. The idea.** For $k \ge 2$, $\lfloor \log_k 2026 \rfloor$ is the number of exponents $j \ge 1$ with $k^j \le 2026$. So the sum counts pairs $(k, j)$ with $k \ge 2$, $j \ge 1$, $k^j \le 2026$. Count them the other way round: for each exponent $j$, how many bases $k \ge 2$ have $k^j \le 2026$? That is $\lfloor 2026^{1/j} \rfloor - 1$, and only the first few $j$ matter.

**Work it out.** $j = 1$: $k = 2, \dots, 2026$, which is 2025 bases. $j = 2$: $45^2 = 2025 \le 2026 < 46^2$, so 44 bases. $j = 3$: $12^3 = 1728 \le 2026 < 2197 = 13^3$, so 11. $j = 4$: $6^4 = 1296 \le 2026 < 2401 = 7^4$, so 5. $j = 5$: $4^5 = 1024 \le 2026 < 3125$, so 3. $j = 6$: $3^6 = 729 \le 2026 < 4096 = 4^6$, so 2. $j = 7$: $2^7 = 128$ but $3^7 = 2187 > 2026$, so 1. $j = 8, 9, 10$: only $k = 2$ ($2^{10} = 1024$), so 1 each. $j \ge 11$: $2^{11} = 2048 > 2026$, so none. The total is $2025 + 44 + 11 + 5 + 3 + 2 + 1 + 1 + 1 + 1 = 2094$.

**Check.** Count by bases instead. Bases $k \ge 46$ contribute 1 each: $2026 - 45 = 1981$. Bases 13 to 45 contribute 2: $33 \cdot 2 = 66$. Bases 7 to 12 contribute 3: 18. Base 6 gives 4 (1296) and base 5 gives 4 (625, while $3125 > 2026$). Base 4 gives 5 (1024), base 3 gives 6 (729), base 2 gives 10 (1024). Sum: $1981 + 66 + 18 + 4 + 4 + 5 + 6 + 10 = 2094$.

**11. Answer:** $\frac{1}{3}$**. The idea.** Go through the complement. Three lengths fail to form a triangle exactly when one of them is at least the sum of the other two. The three failure events "$C \ge A + B$", "$B \ge A + C$", "$A \ge B + C$" cannot happen together (two of them would force a length to be $\le 0$), so their probabilities simply add, and by symmetry they are equal. The geometric tail $P(C \ge m) = 2^{-(m-1)}$ makes each one a clean double sum.

**Work it out.** For fixed $A = a$, $B = b$, $P(C \geq a + b) = 2^{-(a+b-1)}$. Hence $P(C \geq A + B) = \sum_{a,b\geq 1} 2^{-a}2^{-b} \cdot 2^{1-a-b} = 2\sum_{a\geq 1} 4^{-a} \sum_{b\geq 1} 4^{-b} = 2 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$. If both $C \geq A+B$ and $B \geq A+C$ held, adding gives $0 \geq 2A$, impossible; so the three events are disjoint and $P(\text{no triangle}) = 3 \cdot \frac{2}{9} = \frac{2}{3}$. The answer is $1 - \frac{2}{3} = \frac{1}{3}$.

**Check.** A partial sum agrees in size. The outcomes with all numbers at most 2 that form triangles are $(1, 1, 1)$, probability $\frac{1}{8}$, the three orders of $(2, 2, 1)$, $3 \cdot \frac{1}{32}$, and $(2, 2, 2)$, $\frac{1}{64}$: already $\frac{15}{64} \approx 0.23$, while the three orders of $(1, 1, 2)$ correctly fail (degenerate, $1+1 = 2$). Separately, $P(C \geq A+B) = \frac{2}{9}$ can be checked at its biggest term: $A = B = 1$, $C \geq 2$ has probability $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$, and the full sum $\frac{2}{9}$ is a bit more, as it should be.

**12. Answer:** $20\sqrt{3}$. **The idea.** A straight cut crossing two sides cuts off a small triangle at the vertex $V$ where those sides meet. If it cuts off $x$ and $y$ along the two sides (measured from $V$), equal perimeters say $x + y$ is half the perimeter of the field, and equal areas say $xy$ is half the product of those two sides (areas of triangles sharing the angle at $V$ scale like $xy$). So $x, y$ are roots of a quadratic, and we test the three vertices.

**Work it out.** The field is a right triangle; call the right-angle vertex $R$ (sides 30, 40), the vertex $P$ between the sides 30 and 50, and $Q$ between 40 and 50. Half the perimeter is 60, so $x + y = 60$ in every case. At $R$: $xy = \frac{1}{2} \cdot 30 \cdot 40 = 600$, so $x, y = 30 \pm \sqrt{300} \approx 47.3,\ 12.7$, but 47.3 is longer than both sides at $R$. At $Q$: $xy = \frac{1}{2} \cdot 40 \cdot 50 = 1000$, and $t^2 - 60t + 1000$ has discriminant $3600-4000 < 0$. At $P$: $xy = \frac{1}{2} \cdot 30 \cdot 50 = 750$, so $x, y = 30 \pm \sqrt{150} = 30 \pm 5\sqrt{6}$; put $30+5\sqrt{6} \approx 42.2$ on the side 50 and $30 - 5\sqrt{6} \approx 17.8$ on the side 30. A cut through a vertex would be a median (to halve the area), leaving the two parts' perimeters differing by the difference of the other two sides, never 0 here. So the fence is unique. At $P$, $\cos P = \frac{30}{50} = \frac{3}{5}$, and by the law of cosines $L^2 = x^2 + y^2 - 2xy \cdot \frac{3}{5} = (x + y)^2 - 2xy - \frac{6}{5}xy = 3600 - 1500 - 900 = 1200$, so $L = 20\sqrt{3}$.

**Check.** Coordinates: $R = (0, 0)$, $P = (0, 30)$, $Q = (40, 0)$. The point $30 - 5\sqrt{6}$ from $P$ toward $R$ is $(0,\ 5\sqrt{6})$; the point $30 + 5\sqrt{6}$ from $P$ toward $Q$ is $P + \frac{30+5\sqrt{6}}{50}(40, -30) = (24 + 4\sqrt{6},\ 12 - 3\sqrt{6})$. The squared distance is $(24 + 4\sqrt{6})^2 + (12 - 8\sqrt{6})^2 = (672 + 192\sqrt{6}) + (528 - 192\sqrt{6}) = 1200$.

**13. Answer:** 208. **The idea.** First count routes that start at an end stone, by looking at the first hop; they obey a Fibonacci-like recursion. Then handle an interior start: the only link between the stones to its left and the stones to its right (once the start is used) is the hop $k - 1 \leftrightarrow k + 1$, so the route crosses from one side to the other exactly once. The side visited first must be swept out and back in a forced zigzag.

**Work it out.** Let $g(m)$ be the number of routes on $m$ stones that start at stone 1. From 1 the page hops to 2 or 3. If to 2: what remains is a route on stones $2..m$ starting at its end, $g(m - 1)$ ways. If to 3, stone 2 (neighbours $1, 3, 4$) is still unvisited. Either he hops $3 \to 2 \to 4$ at once, leaving a route on $4..m$ from its end: $g(m - 3)$ ways; or stone 2 is entered later, from 4, and then it is a dead end, so it is the last stone. A route $1, 3, \ldots, 4, 2$ must go out to the far end and come back, and that is forced: $1, 3, 5, 7, \ldots$ up the odd stones and back down the even ones, exactly 1 way. So $g(m) = g(m - 1) + g(m - 3) + 1$ for $m \geq 4$, with $g(1) = 1$, $g(2) = 1$, $g(3) = 2$. This gives $g(4..10) = 4, 6, 9, 14, 21, 31, 46$.

Now a start at stone $k$ with $2 \leq k \leq 9$. The stones $< k$ and $> k$ are joined only by the hop $k - 1 \leftrightarrow k + 1$, so the route visits one whole side first and then crosses once. If it does the left side first, it must go from $k$ down, cover $1..k - 1$, and end that part at $k - 1$ to cross: this is again

a forced zigzag ($k \to k-2 \to k-4 \to \cdots$ and back up to $k-1$), 1 way; then it continues on $k+1..10$ from its end, $g(10-k)$ ways. Doing the right side first gives $g(k-1)$ ways. Summing over $k = 2..9$ gives $2\,(g(1)+\cdots+g(8)) = 2(1+1+2+4+6+9+14+21) = 116$. Starts at stone 1 or 10 give $2g(10) = 92$. Total $116+92 = 208$.

**Check.** The same formula $2g(n) + 2\sum_{j\le n-2} g(j)$ gives 6 routes for 3 stones (all 3! orders, since every hop is allowed) and 12 for 4 stones: of the 24 orders of $1, 2, 3, 4$, exactly those with 1 and 4 not adjacent in the list, $24 - 12 = 12$.

**14. Answer:** 60**. The idea.** Parity gives a lower bound: the difference is the total minus twice one group's sum, so $D(n)$ has the same parity as the total $S(n) = \frac{n(n+1)(2n+1)}{6}$. A block of eight consecutive squares splits evenly by itself, for any starting point: $(k+1)^2 + (k+4)^2 + (k+6)^2 + (k+7)^2 = (k+2)^2 + (k+3)^2 + (k+5)^2 + (k+8)^2$. Adding such a block never changes the best difference, so $D(n+8) \le D(n)$; once eight consecutive $n$ meet the parity bound, every larger $n$ does too. The small cases have to be done by hand, and that is where the trap lies.

**Work it out.** The block identity: both sides equal $4k^2 + 36k + 102$ (linear terms $1+4+6+7 = 18 = 2+3+5+8$, constants $1+16+36+49 = 102 = 4+9+25+64$). $S(n)$ is odd exactly when $4 \nmid n(n+1)$, i.e. $n \equiv 1, 2 \pmod 4$; then $D(n) \ge 1$. Base cases $n = 6, \ldots, 13$, one for each residue mod 8, all meet the parity bound: $n = 6$: $S = 91$, $3^2 + 6^2 = 45$ against 46, $D = 1$. $n = 7$: $3^2 + 5^2 + 6^2 = 70 = \frac{140}{2}$, $D = 0$. $n = 8$: $1^2 + 4^2 + 6^2 + 7^2 = 102 = \frac{204}{2}$, $D = 0$. $n = 9$: $S = 285$, $5^2 + 6^2 + 9^2 = 142$ against 143, $D = 1$. $n = 10$: $S = 385$, $1^2 + 5^2 + 6^2 + 7^2 + 9^2 = 192$ against 193, $D = 1$. $n = 11$: $2^2 + 6^2 + 7^2 + 8^2 + 10^2 = 253 = \frac{506}{2}$, $D = 0$. $n = 12$: $9^2 + 10^2 + 12^2 = 325 = \frac{650}{2}$, $D = 0$. $n = 13$: $S = 819$, $1^2 + 3^2 + 7^2 + 9^2 + 10^2 + 13^2 = 409$ against 410, $D = 1$. Adding blocks of 8, for every $n \ge 6$ we get $D(n) = 1$ if $n \equiv 1, 2 \pmod 4$ and $D(n) = 0$ otherwise. Small $n$: $D(1) = 1$ (1 against nothing). $D(2) = 3$ (4 against 1). $n = 3$: the subset sums of $\{1, 4, 9\}$ are $0, 1, 4, 5, 9, 10, 13, 14$; closest to 7 are 5 and 9, so $D(3) = 4$. $n = 4$: $S = 30$, but no group sums to 15 (without 16 the most is 14, with it at least 16), so $D(4) = 2$ (14 against 16). $n = 5$: $S = 55$; a group sum of 27 or 28 would need the group with 25 to add 2 or 3 from $\{1, 4, 9, 16\}$, impossible; $D(5)$ is odd, so $D(5) \ge 3$, and $25 + 1 = 26$ against 29 gives $D(5) = 3$. So $D(1) + \cdots + D(5) = 1 + 3 + 4 + 2 + 3 = 13$. For $6 \le n \le 100$ the sum counts the $n \equiv 1, 2 \pmod 4$ there: 50 such $n$ in 1..100, minus $1, 2, 5$, leaves 47. The answer is $13 + 47 = 60$.

**Check.** The block identity at $k = 8$: $9^2 + 12^2 + 14^2 + 15^2 = 81 + 144 + 196 + 225 = 646$ and $10^2 + 11^2 + 13^2 + 16^2 = 100 + 121 + 169 + 256 = 646$, so the split for $n = 7$ plus this block settles $n = 15$. The tempting shortcut "$D(n)$ is just the parity of $S(n)$" gives 50; it is wrong exactly at $n = 2, 3, 4, 5$, which add $2 + 4 + 2 + 2 = 10$ more.

**15. Answer:** 1028**. The idea.** Counting walks whose range is at most 3 directly is messy, because the range is only known at the end. Instead count pairs (window, walk inside the window). A walk of range $r \le 3$ fits inside exactly $4 - r$ windows of 4 consecutive spots (that contain the start) and inside exactly $3 - r$ windows of 3 consecutive spots. So (walks counted in all 4-windows) − (walks counted in all 3-windows) counts each walk of range at most 3 exactly once, and walks of range 4 or more zero times.

**Work it out.** Walks on 4 spots in a row: let $e(n)$ be the number of $n$-step walks from an end spot and $m(n)$ from an inner spot. From an end the first step is forced to the neighbouring inner spot, so $e(n) = m(n-1)$; from an inner spot one step goes to the end, one to the other inner spot, so $m(n) = e(n-1) + m(n-1) = m(n-2) + m(n-1)$. With $m(0) = 1$, $m(1) = 2$ this is Fibonacci: $m(n) = F_{n+2}$, $e(n) = F_{n+1}$. The start can sit at any of the 4 positions of the window:

$2e(12) + 2m(12) = 2(F_{13} + F_{14}) = 2(233 + 377) = 1220$. On 3 spots every other step is forced (an end must step to the middle), and from the middle there are 2 choices; in 12 steps there are 6 free choices from any start, so 3 positions give $3 \cdot 2^6 = 192$. The answer is $1220 - 192 = 1028$.

**Check.** Small case, 2 steps: all 4 walks have range at most 2. The formula gives $2(F_3+F_4)-3\cdot 2^1 = 2(2+3)-6 = 4$. With 4 steps only the two straight walks ++++, −−−− have range 4, so $16-2 = 14$; the formula gives $2(F_5 + F_6) - 3 \cdot 2^2 = 2(5 + 8) - 12 = 14$.