MathCastle

AMC 8

Practice Contest

25 questions • 40 minutes

Name _____

School _____

Grade _____

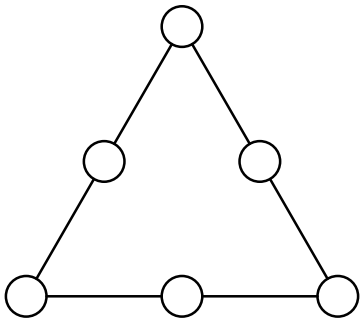
INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 40 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The numbers $1, 2, 3, 4, 5, 6$ are placed on a triangle, one number on each of its three corners and one on the midpoint of each of its three sides, using every number exactly once. Each side of the triangle then carries three numbers (two corners and one midpoint), and the placement must make all three sides have the same sum. What is the largest possible value of that common side sum?



(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

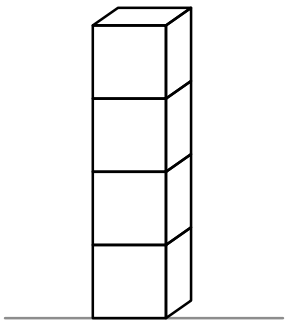
2. Theo has 20 coins, and every one of them is a nickel (worth 5 cents) or a dime (worth 10 cents). If each of his nickels were a dime and each of his dimes were a nickel, he would have 50 cents more than he has now. How many cents does Theo have now?

(A) 125 **(B)** 150 **(C)** 175 **(D)** 200 **(E)** 225

3. Three painters can paint 2 identical fences in 4 hours. A crew of three painters starts on a job of 5 such fences. After 2 hours, three more painters join them, and all six keep working until the job is done. Every painter works at the same constant rate the whole time. How many hours after the start is the job finished?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

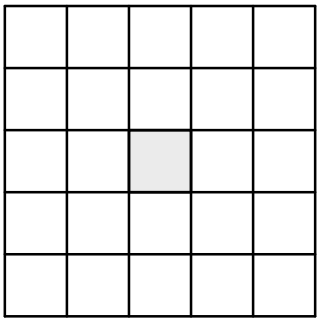
4. Four standard dice (faces 1 to 6, with opposite faces adding to 7) are stacked into a vertical tower on a table, each die resting squarely on the one below. Walking all the way around the tower you can see the four side faces of every die and the top face of the top die; the faces touching the table or another die are hidden. What is the largest possible total number of pips you can see?



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(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 66

5. A 5×5 board is divided into 25 unit squares. Consider all the squares, of every size from 1×1 up to 5×5, whose sides lie along the grid lines of the board. How many of these squares completely cover the unit square in the very center of the board?



(A) 15 **(B)** 17 **(C)** 19 **(D)** 25 **(E)** 55

6. A two-digit number with no zero digit is exactly 4 times the sum of its digits. When its two digits are swapped, the new number turns out to be exactly k times the same digit sum, where k is a whole number that is the same for every such number. What is k?

(A) 3 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

7. A four-digit lock code uses each of the digits $1, 2, 3, 4$ exactly once. The code must not contain the digit 1 immediately followed by the digit 2, and it must not contain the digit 3 immediately followed by the digit 4. (For example, 2143 is allowed, but 3412 and 4123 are not.) How many codes are allowed?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 22 **(E)** 24

8. A three-digit number is written with ordinary decimal digits, and the product of its three digits is exactly 24. For example, 146 is such a number ($1 \cdot 4 \cdot 6 = 24$), but 238 is not. How many three-digit numbers have a digit product equal to 24?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 20 **(E)** 21

9. Two positive numbers are chosen. Their sum, their difference (larger minus smaller) and their product are in the ratio $5 : 1 : 18$, in that order. What is the product of the two numbers?

(A) 18 **(B)** 24 **(C)** 54 **(D)** 96 **(E)** 150

10. Ten cards are numbered 1 through 10. Two different cards are drawn at random, all pairs being equally likely. What is the probability that the larger of the two numbers is a multiple of the smaller one? (For example, drawing 3 and 9 counts, but 4 and 6 does not.)

(A) $\frac{1}{9}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{16}{45}$ **(D)** $\frac{17}{45}$ **(E)** $\frac{2}{5}$

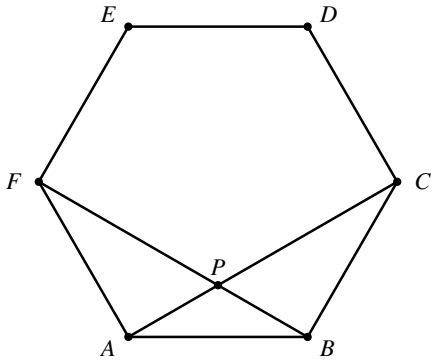
11. When 2026 is divided by a positive integer n, the remainder is 26. (The remainder is always a whole number from 0 up to $n - 1$.) How many different positive integers n have this property?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 19 **(E)** 20

12. Some whole numbers have exactly four positive divisors; for example, 10 has the divisors $1, 2, 5, 10$, and 8 has $1, 2, 4, 8$. How many of the integers $1, 2, 3, \ldots, 100$ have exactly four positive divisors?

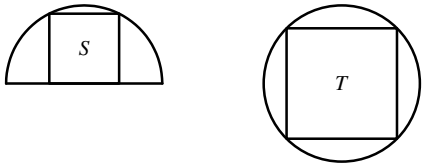
(A) 32 **(B)** 33 **(C)** 34 **(D)** 36 **(E)** 38

13. Regular hexagon $ABCDEF$ has its vertices labelled in order around the hexagon. The diagonal $\overline{AC}$ and the diagonal $\overline{BF}$ cross at the point P. What fraction of the area of the hexagon lies inside $\triangle ABP$?



(A) $\frac{1}{24}$ **(B)** $\frac{1}{18}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{9}$

14. A circle and a semicircle have the same radius. Square S is drawn inside the semicircle with one side lying along the semicircle's diameter and its other two corners on the arc. Square T is drawn inside the full circle with all four corners on the circle. What is the ratio of the area of S to the area of T?



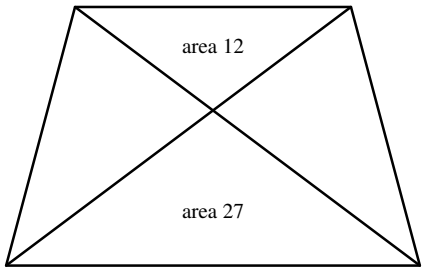
(A) $\frac{1}{3}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{4}{5}$

15. A robot starts at the origin of a flat field, facing north. It then takes 100 steps: on step k (for $k = 1, 2, \ldots, 100$) it rolls k metres straight ahead and then turns 90° to its right. So it rolls 1 metre north, turns, rolls 2 metres east, turns, rolls 3 metres south, and so on. How many metres is the robot from the origin after step 100 (measured in a straight line)?

(A) $25\sqrt{2}$ **(B)** 50 **(C)** $50\sqrt{2}$ **(D)** 100 **(E)** $100\sqrt{2}$

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16. The two diagonals of a trapezoid cross, cutting the trapezoid into four triangles. The triangle whose side is the shorter of the two parallel sides has area 12, and the triangle whose side is the longer parallel side has area 27. What is the area of the whole trapezoid?

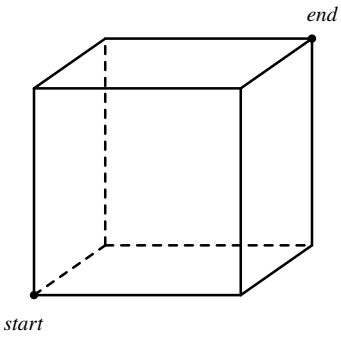


(A) 57 **(B)** 72 **(C)** 75 **(D)** 78 **(E)** 81

17. Ana and Bo start from the same point of a circular track at the same moment, and each runs at a constant speed, Ana faster than Bo. If they run in opposite directions, they meet for the first time after 48 seconds. If instead they both run in the same direction, Ana first catches up with Bo (having run exactly one lap more than him) after 240 seconds. How many seconds does Bo take to run one lap?

(A) 48 **(B)** 60 **(C)** 80 **(D)** 96 **(E)** 120

18. An ant sits at one corner of a wire-frame cube. Every minute it walks along one edge of the cube to a neighbouring corner; it may walk back along an edge it has already used. After exactly 5 minutes the ant is at the corner diametrically opposite its starting corner. How many different 5-edge routes could the ant have taken? (Two routes are different if the sequences of corners visited are different.)



(A) 21 **(B)** 36 **(C)** 42 **(D)** 54 **(E)** 60

19. A round table has 10 chairs, equally spaced and numbered 1 to 10 in order around the table, so chair 10 is next to chair 1. Ann, Ben and Cal each sit in one of the chairs, and no two of them may sit in neighbouring chairs. Two seatings count as different if at least one person is in a different numbered chair. How many seatings are possible?

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(A) 280 **(B)** 300 **(C)** 336 **(D)** 360 **(E)** 720

20. A right triangle has three whole-number side lengths, and the number of square units in its area equals the number of units in its perimeter. There are only finitely many such triangles. What is the sum of the perimeters of all of them? (Triangles that are congruent count only once.)

(A) 54 **(B)** 60 **(C)** 72 **(D)** 78 **(E)** 84

21. A list of 10 positive integers (not necessarily different) has mean 10 and median 8. The number 5 appears in the list more often than any other number, and no other number appears as often as 5 does. What is the largest possible value of the biggest number in the list?

(A) 36 **(B)** 38 **(C)** 39 **(D)** 40 **(E)** 42

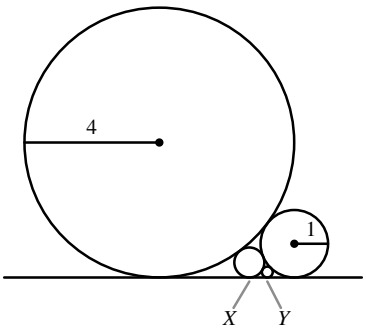
22. Eight identical counters are placed on a 4×4 board, at most one counter per square, so that every row contains exactly 2 counters and every column contains exactly 2 counters. In how many ways can this be done? (Placements that differ by a rotation or reflection of the board count as different.)

(A) 15 **(B)** 24 **(C)** 36 **(D)** 72 **(E)** 90

23. Some whole numbers have squares that end in the number itself; for example $25^2 = 625$ ends in the digits 25. Find the sum of all three-digit numbers n (that is, $100 \le n \le 999$) such that the last three digits of n^2 are the three digits of n, in the same order.

(A) 625 **(B)** 1001 **(C)** 1251 **(D)** 1501 **(E)** 1877

24. Two circles, of radius 4 and radius 1, rest on a straight line (each touching the line) on the same side of it, and they touch each other. Circle X lies in the curved gap enclosed by the line and these two circles, touching the line and both circles. Circle Y lies in the smaller gap enclosed by the line, circle X and the circle of radius 1, touching all three. What is the radius of circle Y?



(A) $\frac{1}{9}$ **(B)** $\frac{4}{25}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{2}{9}$ **(E)** $\frac{4}{9}$

25. Three pegs stand in a row: A on the left, B in the middle, C on the right. Six discs of different sizes are stacked on peg A, largest at the bottom. A move takes the top disc of one peg and places it on another peg, and a disc may never be placed on top of a smaller disc. There is one extra rule: a disc may only move between neighbouring pegs, so every move is between A and B or between B and C (never directly between A and C). What is the fewest number of moves needed to move all six discs to peg C?

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(A) 63 **(B)** 242 **(C)** 364 **(D)** 728 **(E)** 729

Answer Key — AMC 8 Practice Contest 4

1. (E) 12

2. (A) 125

3. (C) 6

4. (D) 62

5. (C) 19

6. (D) 7

7. (A) 14

8. (E) 21

9. (C) 54

10. (D) $\frac{17}{45}$

11. (A) 11

12. (A) 32

13. (B) $\frac{1}{18}$

14. (B) $\frac{2}{5}$

15. (C) $50\sqrt{2}$

16. (C) 75

17. (E) 120

18. (E) 60

19. (B) 300

20. (A) 54

21. (D) 40

22. (E) 90

23. (B) 1001

24. (B) $\frac{4}{25}$

25. (D) 728

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Solutions — AMC 8 Practice Contest 4

1. Answer: (E) 12**. The idea.** Add up the three side sums. Every number is counted once, except the corner numbers, which lie on two sides and are counted twice. So the side sum is controlled by the corner total.

Work it out. If each side sums to S, then $3S = (1+2+\cdots+6)+(\text{corner sum}) = 21+(\text{corner sum})$. The corners are three different numbers from 1 to 6, so their sum is at most $4+5+6 = 15$, giving $3S \le 36$ and $S \le 12$. To reach 12, put 4, 5, 6 on the corners. The side from 4 to 5 needs midpoint 3; the side from 5 to 6 needs 1; the side from 6 to 4 needs 2. The numbers $1, 2, 3$ are each used once, so $S = 12$ really happens.

Check. The sides are $4+3+5$, $5+1+6$, $6+2+4$, each 12. Forgetting that the corners are counted twice gives $3S = 21$ and $S = 7$; and 9 (corners $1, 2, 3$) is the smallest possible side sum, not the largest.

2. Answer: (A) 125**. The idea.** Look at what the swap does coin by coin: each nickel that becomes a dime gains 5 cents, and each dime that becomes a nickel loses 5 cents. So the swap gains 5 cents times (nickels minus dimes).

Work it out. Let Theo have n nickels and d dimes. The gain is $5(n - d) = 50$, so $n - d = 10$. Together with $n + d = 20$ this gives $n = 15$ and $d = 5$. Theo has $15 \cdot 5 + 5 \cdot 10 = 75 + 50 = 125$ cents.

Check. After the swap he would have 15 dimes and 5 nickels, which is $150 + 25 = 175$ cents, exactly 50 more than 125. The choice 175 is the amount after the swap, not now.

3. Answer: (C) 6**. The idea.** Measure work in painter-hours. Once you know how many painter-hours one fence takes, the question is just about adding up painter-hours.

Work it out. Three painters working 4 hours put in $3 \cdot 4 = 12$ painter-hours and finish 2 fences, so one fence takes 6 painter-hours, and 5 fences take 30. In the first 2 hours the three painters supply $3 \cdot 2 = 6$ painter-hours, which leaves 24. From then on six painters supply 6 painter-hours every hour, so they need $24 \div 6 = 4$ more hours. The job ends $2 + 4 = 6$ hours after the start.

Check. One painter paints $\frac{1}{6}$ of a fence per hour. The fences painted are $3 \cdot 2 \cdot \frac{1}{6} + 6 \cdot 4 \cdot \frac{1}{6} = 1+4 = 5$. The answer 4 forgets the first two hours, and 10 ignores the painters who joined.

4. Answer: (D) 62**. The idea.** The four side faces of a die are two pairs of opposite faces, and each opposite pair adds to 7. So no matter how a die is turned, its side faces always total 14.

Work it out. The faces of a die add to 21. The top and bottom are opposite, so they add to 7, and the four side faces add to $21 - 7 = 14$. Four dice therefore always show $4 \cdot 14 = 56$ pips on their sides. The only thing we control is the top face of the top die, which is at most 6. The largest visible total is $56 + 6 = 62$.

Check. Every orientation gives each die sides of 14, so the visible total is $56+(\text{top face})$, ranging from 57 to 62. The distractor 56 forgets the top face; 66 would need a top face of 10.

5. Answer: (C) 19**. The idea.** Sort the squares by size. For a square of side k, count the positions of its top-left corner that keep the middle cell inside it, one direction at a time.

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Work it out. Number the rows and columns 1 to 5, so the center cell is in row 3, column 3. A $k \times k$ square covering rows r to $r + k - 1$ contains row 3 exactly when $r \le 3 \le r + k - 1$, with $1 \le r \le 6 - k$. The number of good starting rows is $1, 2, 3, 2, 1$ for $k = 1, 2, 3, 4, 5$ (for $k = 4$, only $r = 1, 2$ fit, and both work). Columns behave the same way, so the counts are $1^2, 2^2, 3^2, 2^2, 1^2$, that is $1 + 4 + 9 + 4 + 1 = 19$.

Check. The board has $25 + 16 + 9 + 4 + 1 = 55$ squares in all; every 3×3, 4×4 and 5×5 square covers the center ($9 + 4 + 1 = 14$), and of the 1×1 and 2×2 squares only 1 and 4 do, giving 19 again.

6. Answer: (D) 7**. The idea.** Write the number as $10a + b$ and turn the condition into a relation between the digits. That relation makes both the number and its reversal simple multiples of a.

Work it out. $10a + b = 4(a + b)$ gives $6a = 3b$, so $b = 2a$. The numbers are $12, 24, 36, 48$ and the digit sum is $a + 2a = 3a$. The reversed number is $10b + a = 20a + a = 21a = 7 \cdot 3a$, which is 7 times the digit sum. So $k = 7$.

Check. $21 = 7 \cdot 3$, $42 = 7 \cdot 6$, $63 = 7 \cdot 9$, $84 = 7 \cdot 12$. A quick sanity check: the number plus its reversal is $11(a + b)$, so $4 + k = 11$ and $k = 7$ without any casework.

7. Answer: (A) 14**. The idea.** Count the bad codes and subtract them from all $4! = 24$ codes. The two kinds of bad codes overlap, so use inclusion-exclusion.

Work it out. Codes containing "12": glue 12 into one block; arranging the block, 3 and 4 gives $3! = 6$ codes. Likewise 6 codes contain "34". Codes containing both: arrange the two blocks 12 and 34, which gives $2! = 2$ codes (1234 and 3412). So $6 + 6 - 2 = 10$ codes are bad, and $24 - 10 = 14$ are allowed.

Check. Ruling out only one of the two forbidden pairs gives $24 - 6 = 18$, a common slip. Listing by first digit also works: codes starting with 2 are 2134, 2143, 2314, 2341, 2413, 2431 minus those with 34 (2134, 2341), leaving 4; codes starting with 1, 3, 4 similarly number 3, 3, 4, and $4 + 3 + 3 + 4 = 14$.

8. Answer: (E) 21**. The idea.** First find the possible sets of digits (ignoring order), then count the arrangements of each set, being careful with repeated digits.

Work it out. No digit can be 0, and each digit divides 24, so the digits come from $1, 2, 3, 4, 6, 8$. Go by the smallest digit. Smallest 1: the other two multiply to 24: $3 \cdot 8$ or $4 \cdot 6$. Smallest 2: the other two multiply to 12 and are at least 2: $2 \cdot 6$ or $3 \cdot 4$. Smallest 3 or more: $3 \cdot 3 \cdot 3 = 27 > 24$, impossible. So the digit sets are $\{1, 3, 8\}$, $\{1, 4, 6\}$, $\{2, 3, 4\}$ with $3! = 6$ orders each, and $\{2, 2, 6\}$ with only 3 orders (226, 262, 622). Total $6 + 6 + 6 + 3 = 21$.

Check. Giving $\{2, 2, 6\}$ only 2 orders gives 20; forgetting it gives 18. Both are traps, and 21 counts each number once.

9. Answer: (C) 54**. The idea.** The first two parts of the ratio pin down the numbers up to a scale factor; the product part then fixes the scale.

Work it out. Write the sum as $5k$ and the difference as k. Adding and subtracting, the larger number is $\frac{5k+k}{2} = 3k$ and the smaller is $2k$. Their product is $6k^2$, and it must be 18 times the difference, $18k$. So $6k^2 = 18k$, and since $k > 0$, $k = 3$. The numbers are 9 and 6, and their product is 54.

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Check. $9+6=15$, $9-6=3$, $9\cdot 6=54$, and $15:3:54=5:1:18$. The distractor 18 comes from reading the ratio as actual values, which no pair of numbers satisfies.

10. Answer: (D) $\frac{17}{45}$**. The idea.** Organise the favourable pairs by their smaller card a: the larger card must be one of $2a, 3a, \ldots$ that is at most 10.

Work it out. There are $\binom{10}{2}=45$ equally likely pairs. For $a=1$, any larger card works: 9 pairs. For $a=2$: $4,6,8,10$, which is 4. For $a=3$: $6,9$, which is 2. For $a=4$: 8. For $a=5$: 10. For $a\geq 6$, $2a>10$, so nothing. That is $9+4+2+1+1=17$ pairs, and the probability is $\frac{17}{45}$.

Check. In general the count is $\sum_a (\lfloor 10/a\rfloor - 1) = 9+4+2+1+1+0+0+0+0+0 = 17$. Counting only "exactly twice" gives $\frac{5}{45}=\frac{1}{9}$, and forgetting the pair $(1,10)$ or $(5,10)$ gives $\frac{16}{45}$.

11. Answer: (A) 11**. The idea.** A remainder of 26 means n divides $2026-26=2000$ exactly, but also that 26 is a legal remainder, which needs $n>26$.

Work it out. $2000=2^4\cdot 5^3$ has $(4+1)(3+1)=20$ divisors. Those that are 26 or less are $1,2,4,5,8,10,16,20,25$: that is 9 of them (the next divisor is 40). So $20-9=11$ values of n work: $40,50,80,100,125,200,250,400,500,1000,2000$.

Check. Try $n=40$: $2026=50\cdot 40+26$. Try $n=25$: $2026=81\cdot 25+1$, remainder 1, so small divisors really fail. Forgetting the condition $n>26$ gives 20.

12. Answer: (A) 32**. The idea.** If $n=p_1^{e_1}p_2^{e_2}\cdots$, the number of divisors is $(e_1+1)(e_2+1)\cdots$. The product 4 is either 4 or $2\cdot 2$, so $n=p^3$ or $n=pq$ with $p<q$ distinct primes.

Work it out. Cubes of primes up to 100: 8 and 27, so 2 numbers. Products $pq\leq 100$ with $p<q$, by the smaller prime: $p=2$, $q\leq 50$: $q=3,5,7,\ldots,47$, which is 14 primes. $p=3$, $5\leq q\leq 33$: $5,7,11,13,17,19,23,29,31$, which is 9. $p=5$, $7\leq q\leq 20$: $7,11,13,17,19$, which is 5. $p=7$, $11\leq q\leq 14$: $11,13$, which is 2. $p=11$ would need $q\geq 13$, but $11\cdot 13>100$. So there are $14+9+5+2=30$ products and 32 numbers in all.

Check. Counting $64=4^3$ as a cube gives 33, a likely slip, but $64=2^6$ has seven divisors. A spot check: $91=7\cdot 13$ has divisors $1,7,13,91$, and $98=2\cdot 7^2$ has six divisors, so it is correctly excluded.

13. Answer: (B) $\frac{1}{18}$**. The idea.** Diagonal BF is perpendicular to the line through A and the centre (the hexagon is symmetric in that line), so put the hexagon on coordinates with that line as the x-axis. Then BF is a vertical line and $\triangle ABP$ has a vertical base BP and an easy height.

Work it out. Take the centre at the origin and side 6: $A=(6,0)$, $B=(3,3\sqrt{3})$, $C=(-3,3\sqrt{3})$, $F=(3,-3\sqrt{3})$. Diagonal BF is the line $x=3$. On AC, moving from A to C changes x by -9 and y by $3\sqrt{3}$; reaching $x=3$ takes a third of the way, so $P=(3,\sqrt{3})$. In $\triangle ABP$ the base BP lies on $x=3$ with length $3\sqrt{3}-\sqrt{3}=2\sqrt{3}$, and the height is the distance from A to that line, $6-3=3$. So $[ABP]=\frac{1}{2}\cdot 2\sqrt{3}\cdot 3=3\sqrt{3}$. The hexagon is six equilateral triangles of side 6, with area $6\cdot\frac{\sqrt{3}}{4}\cdot 36=54\sqrt{3}$. The fraction is $\frac{3\sqrt{3}}{54\sqrt{3}}=\frac{1}{18}$.

Check. $\triangle ABF$ has base $BF=6\sqrt{3}$ and height 3, area $9\sqrt{3}$, which is $\frac{1}{6}$ of the hexagon. P divides BF with $BP:BF=2\sqrt{3}:6\sqrt{3}=1:3$, so $[ABP]=\frac{1}{3}[ABF]=\frac{1}{18}$ of the hexagon.

14. Answer: (B) $\frac{2}{5}$**. The idea.** By symmetry square S sits centred on the diameter, so the centre of

the semicircle is the midpoint of S's bottom side. A radius to a top corner is then the hypotenuse of a right triangle with legs $\frac{s}{2}$ and s.

Work it out. Let the radius be r and the side of S be s. A top corner is $\frac{s}{2}$ across and s up from the centre, so $\left(\frac{s}{2}\right)^2 + s^2 = r^2$, i.e. $\frac{5}{4}s^2 = r^2$ and $[S] = s^2 = \frac{4}{5}r^2$. Square T has diagonal $2r$, so $[T] = \frac{1}{2}(2r)^2 = 2r^2$. The ratio is $\frac{4/5}{2} = \frac{2}{5}$.

Check. Reflect the semicircle and S across the diameter: you get a full circle containing a $2s \times s$ rectangle with all four corners on the circle, so its diagonal is $2r$: $(2s)^2 + s^2 = 4r^2$, giving $s^2 = \frac{4}{5}r^2$ again.

15. Answer: (C) $50\sqrt{2}$**. The idea.** The directions repeat north, east, south, west every four steps. Group the steps in fours and look at what each group does to the robot's position.

Work it out. In steps $4m+1, 4m+2, 4m+3, 4m+4$ the robot moves $4m+1$ north, $4m+2$ east, $4m+3$ south and $4m+4$ west. North minus south is $(4m+1) - (4m+3) = -2$, east minus west is $(4m+2) - (4m+4) = -2$. So every group shifts the robot 2 m south and 2 m west, no matter which group it is. The 100 steps form 25 complete groups ($m = 0, \ldots, 24$), so the robot ends 50 m south and 50 m west of the origin. Its distance is $\sqrt{50^2 + 50^2} = 50\sqrt{2}$ metres.

Check. After the first group: $(0,0) \to (0,1) \to (2,1) \to (2,-2) \to (-2,-2)$, which is 2 south and 2 west, as claimed. The second group goes $(-2,-2) \to (-2,3) \to (4,3) \to (4,-4) \to (-4,-4)$, again 2 south and 2 west, so the pattern holds.

16. Answer: (C) 75**. The idea.** The two triangles on the parallel sides are similar, which fixes the ratio in which the diagonals cut each other. Then each of the two "side" triangles is tied to the known ones by shared heights, and they turn out to be equal with area $\sqrt{12 \cdot 27}$.

Work it out. Call the trapezoid $ABCD$ with $AB \parallel CD$, AB the longer side, and let the diagonals meet at O. Triangles OCD and OAB are similar (alternate angles), and their areas are in ratio $12 : 27 = 4 : 9$, so their sides are in ratio $2 : 3$. In particular $OC : OA = 2 : 3$. Now $\triangle OBC$ and $\triangle OBA$ share the vertex B and have bases OC and OA on the same line, so $[OBC] : [OBA] = 2 : 3$ and $[OBC] = \frac{2}{3} \cdot 27 = 18$. In the same way $[OAD] : [OAB] = OD : OB = 2 : 3$, so $[OAD] = 18$. The trapezoid's area is $12 + 27 + 18 + 18 = 75$.

Check. The two side triangles must be equal: $[ABD] = [ABC]$ (same base AB, same height), and removing $\triangle OAB$ from both leaves $[OAD] = [OBC]$. Also $18 = \sqrt{12 \cdot 27}$, and $12 \cdot 27 = 18 \cdot 18$ matches $[OCD] \cdot [OAB] = [OBC] \cdot [OAD]$.

17. Answer: (E) 120**. The idea.** The length of the track is never given, and it is not needed: measure speeds in laps per second. Running towards each other, the two together cover one lap before they meet; running the same way, Ana gains one full lap on Bo. So the sum and the difference of their speeds are known, and half of (sum − difference) is Bo's speed.

Work it out. Let Ana run a laps per second and Bo b laps per second. In opposite directions they meet when their distances add up to one lap: $48(a+b) = 1$, so $a + b = \frac{1}{48}$. In the same direction Ana catches Bo when she is one lap ahead: $240(a-b) = 1$, so $a - b = \frac{1}{240}$. Subtracting, $2b = \frac{1}{48} - \frac{1}{240} = \frac{5-1}{240} = \frac{1}{60}$, so $b = \frac{1}{120}$. Bo runs one lap in 120 seconds.

Check. Ana's speed is $a = \frac{1}{48} - \frac{1}{120} = \frac{5-2}{240} = \frac{1}{80}$, one lap in 80 seconds. In 48 seconds they cover $\frac{48}{80} + \frac{48}{120} = \frac{3}{5} + \frac{2}{5} = 1$ lap together, and in 240 seconds Ana runs 3 laps while Bo runs 2, exactly one lap more. (Forgetting to halve gives 60; answering with Ana's time gives 80.)

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18. Answer: (E) 60**. The idea.** Sort the corners by how many edges they are from the start: distance 0 (1 corner), 1 (3 corners), 2 (3 corners), 3 (1 corner, the goal). Every step changes the distance by exactly 1, so it is enough to track how many routes are at each distance after each minute.

Work it out. From a distance-0 corner, all 3 edges go to distance 1. From distance 1: 1 edge back to distance 0, 2 edges on to distance 2. From distance 2: 2 edges back to distance 1, 1 edge to distance 3. From distance 3: all 3 edges to distance 2. Write the counts as (dist 0, dist 1, dist 2, dist 3). Start: $(1, 0, 0, 0)$. After 1 minute: $(0, 3, 0, 0)$. After 2: $(3, 0, 6, 0)$. After 3: $(0,\ 3\cdot 3+6\cdot 2,\ 0,\ 6) = (0, 21, 0, 6)$. After 4: $(21,\ 0,\ 21\cdot 2+6\cdot 3,\ 0) = (21, 0, 60, 0)$. After 5: the routes at distance 3 are the 60 routes at distance 2, each with exactly 1 edge to the goal, so 60.

Check. After 3 minutes the table gives 6 routes to the far corner, and indeed the shortest routes are the $3! = 6$ orders of the three directions. Also the total after 5 minutes is $183 + 60 = 243 = 3^5$, every possible 5-step walk.

19. Answer: (B) 300**. The idea.** A circle has no ends, which makes gap-counting awkward, so break the circle: once Ann has chosen a chair, she and her two neighbours are out of play, and the 7 remaining chairs form an ordinary straight row.

Work it out. Ann has 10 choices. Her own chair and the two next to it are unavailable, and the other 7 chairs form a row (the two ends of this row are not next to each other, since Ann's three chairs lie between them). Ben and Cal need two non-neighbouring chairs in this row, in order. There are $7 \cdot 6 = 42$ ordered pairs of different chairs, and 6 neighbouring pairs in a row of 7, which give $2 \cdot 6 = 12$ ordered pairs. That leaves $42 - 12 = 30$. In total $10 \cdot 30 = 300$.

Check. Count the sets of three chairs first. Pick one of the three chairs (10 ways) and, going clockwise from it, the sizes of the three gaps of empty chairs: whole numbers at least 1 with sum 7, which is $\binom{6}{2} = 15$ ways. Each set is counted 3 times (once from each of its chairs), so there are $\frac{10\cdot 15}{3} = 50$ sets, and $50 \cdot 3! = 300$ seatings. (Forgetting that chairs 10 and 1 touch gives the straight-row answer 336.)

20. Answer: (A) 54**. The idea.** Get rid of the square root in the hypotenuse, then factor. With legs $a \le b$ and hypotenuse c, the condition $\frac{1}{2}ab = a + b + c$ becomes a product of two small whole numbers equal to 8.

Work it out. From $\frac{1}{2}ab = a + b + c$ we get $c = \frac{1}{2}ab - a - b$. Square it and use $c^2 = a^2 + b^2$: $\frac{1}{4}a^2b^2 + a^2 + b^2 - a^2b - ab^2 + 2ab = a^2 + b^2$, so $\frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab = 0$. Divide by $\frac{1}{4}ab$: $ab - 4a - 4b + 8 = 0$, that is $(a-4)(b-4) = 8$. The factors cannot both be negative: then each is between -3 and -1 (as $a, b \ge 1$), and no such pair multiplies to 8. So $(a-4, b-4) = (1, 8)$ or $(2, 4)$: the triangles 5-12-13 (area 30, perimeter 30) and 6-8-10 (area 24, perimeter 24). The sum of perimeters is $30 + 24 = 54$.

Check. Another route: a right triangle's inradius is $r = \frac{a+b-c}{2}$, and area $= r \cdot \frac{\text{perimeter}}{2}$. Area = perimeter forces $r = 2$. Both triangles have $r = \frac{5+12-13}{2} = 2$ and $\frac{6+8-10}{2} = 2$.

21. Answer: (D) 40**. The idea.** The list's sum is fixed at 100, so to make the biggest number large, make the other nine as small as possible. The median pins the middle pair, and the unique-mode rule forces some numbers above the middle to be different from each other.

Work it out. Sort the list $a_1 \le \cdots \le a_{10}$. Then $a_5 + a_6 = 16$ and the sum is 100, so we minimise $a_1 + \cdots + a_9$. *Case* $a_5 = a_6 = 8$. Now 8 appears at least twice, so 5 appears at

least 3 times, all among $a_1, \ldots, a_4$. If 5 appears exactly 3 times, 8 may appear at most twice, so $a_7, a_8, a_9 \geq 9$ with no value three times: the least is $9, 9, 10$. The fourth small number is at least 1. So $a_1..a_9 \geq 1, 5, 5, 5, 8, 8, 9, 9, 10$, sum 60. If 5 appears 4 times, the least is $5, 5, 5, 5, 8, 8, 8, 9, 9$, sum 62. *Case* $a_5 = 7, a_6 = 9$. With 5 three times: $1, 5, 5, 5, 7, 9, 9, 10, 10$, sum 61 (a third 9 or 10 would tie the mode). With 5 only twice, every other value appears at most once and the sum is larger. *Case* $a_5 = 6, a_6 = 10$. At least two 5s sit in $a_1, \ldots, a_4$. With three or four 5s, $a_1 + \cdots + a_5 \geq 1+5+5+5+6 = 22$ and $a_6, \ldots, a_9 \geq 10, 10, 11, 11$ (no value three times), sum at least 64. With exactly two 5s every other value appears once: $1+2+5+5+6+10+11+12+13 = 65$. *Case* $a_5 = 5, a_6 = 11$. Now $a_6, \ldots, a_9 \geq 11, 11, 12, 12$ (sum 46) unless 5 appears four times, and $a_1 + \cdots + a_5 \geq 1+1+5+5+5 = 17$, so the sum is at least 63 (four 5s: at least $1+5\cdot4+11\cdot3+12 = 66$). *Case* $a_5 \leq 4$. Then $a_6 \geq 12$ and no entry equals 5, impossible. The minimum is 60, so the biggest number is at most $100 - 60 = 40$, reached by $1, 5, 5, 5, 8, 8, 9, 9, 10, 40$.

Check. In $1, 5, 5, 5, 8, 8, 9, 9, 10, 40$: the sum is 100 (mean 10), the middle two are 8 and 8 (median 8), and 5 appears three times while 8 and 9 appear only twice. Choosing $7, 9$ in the middle gives only 39; using four 5s gives 38.

22. Answer: (E) 90**. The idea.** Fill the board row by row. Once the first two rows are placed, the column counts that are still needed decide everything; split on how much row 2 overlaps row 1.

Work it out. Row 1: $\binom{4}{2} = 6$ choices. By symmetry, say row 1 uses columns $1, 2$. Row 2 has 6 choices, in three kinds. *Same columns* $\{1, 2\}$ (1 way): columns $1, 2$ are full, so rows 3 and 4 must both be $\{3, 4\}$: 1 completion. *No shared column,* $\{3, 4\}$ (1 way): every column still needs exactly one more counter. Row 3 picks any 2 columns (6 ways) and row 4 takes the other two: 6 completions. *One shared column* (4 ways, e.g. $\{1, 3\}$): column 1 is full, column 2 and 3 need one more, column 4 needs two. So both rows 3 and 4 use column 4, and they split columns 2 and 3: 2 completions. Total for row 2 onward: $1 \cdot 1 + 1 \cdot 6 + 4 \cdot 2 = 15$. Overall $6 \cdot 15 = 90$.

Check. The three kinds of second row cover all $1 + 1 + 4 = 6$ choices, so nothing is missed. Counting by brute force agrees: of the $6^4 = 1296$ boards with two counters in each row, exactly 90 also have two in each column.

23. Answer: (B) 1001**. The idea.** "n^2 ends in n" means 1000 divides $n^2 - n = n(n-1)$. The two factors n and $n-1$ are consecutive, so they share no prime factor; hence all the 2s of $1000 = 8 \cdot 125$ must sit in one factor and all the 5s in one factor.

Work it out. Neither n nor $n-1$ can be a multiple of 1000 (for $100 \leq n \leq 999$), so one of them is a multiple of 125 and the other a multiple of 8. The multiples of 125 are $125, 250, 375, 500, 625, 750, 875$. *If* $n = 125k$, we need $n - 1$ divisible by 8: $124, 249, 374, 499, 624, 749, 874$; only $624 = 8 \cdot 78$ works, so $n = 625$. *If* $n-1 = 125k$, then $n \in \{126, 251, 376, 501, 626, 751, 876\}$ must be divisible by 8: only $376 = 8 \cdot 47$ works. So $n = 376$ or $n = 625$, and the sum is $376 + 625 = 1001$.

Check. $376^2 = 141376$ and $625^2 = 390625$. Note $376+625 = 1001 \equiv 1 \pmod{1000}$: if n works then so does $1001 - n$, since $(1-n)^2 - (1-n) = n^2 - n$. A tempting miss is 876: $876^2 = 767376$ ends in 376, not 876.

24. Answer: (B) $\frac{4}{25}$**. The idea.** First find how far apart two touching circles meet the line. If circles of radii p and q both touch the line and each other, join their centres (length $p + q$) and drop to a right triangle with vertical leg $|p - q|$: the horizontal leg, which is the distance between their points of contact with the line, is $\sqrt{(p+q)^2 - (p-q)^2} = 2\sqrt{pq}$. A circle squeezed into the gap splits that distance into two such pieces.

Work it out. The circles of radius 4 and 1 touch the line $2\sqrt{4 \cdot 1} = 4$ apart. Let circle X have radius x; its contact point lies between theirs, so $2\sqrt{4x} + 2\sqrt{x} = 4$, i.e. $6\sqrt{x} = 4$, $\sqrt{x} = \frac{2}{3}$ and $x = \frac{4}{9}$. Now circle X and the radius-1 circle touch the line $2\sqrt{\frac{4}{9} \cdot 1} = \frac{4}{3}$ apart. Let Y have radius y: $2\sqrt{\frac{4}{9}y} + 2\sqrt{y} = \frac{4}{3}$, i.e. $\frac{4}{3}\sqrt{y} + 2\sqrt{y} = \frac{4}{3}$, so $\frac{10}{3}\sqrt{y} = \frac{4}{3}$, $\sqrt{y} = \frac{2}{5}$ and $y = \frac{4}{25}$.

Check. Dividing each equation by $2\sqrt{\text{product}}$ gives the pattern $\frac{1}{\sqrt{\text{new}}} = \frac{1}{\sqrt{P}} + \frac{1}{\sqrt{q}}$: $\frac{1}{\sqrt{x}} = \frac{1}{2} + 1 = \frac{3}{2}$ and $\frac{1}{\sqrt{y}} = \frac{3}{2} + 1 = \frac{5}{2}$, so $y = \frac{4}{25}$. Coordinates confirm it: with the big circle centred at $(0, 4)$ and the small one at $(4, 1)$, circle Y is centred at $(\frac{16}{5}, \frac{4}{25})$, which is $1 + \frac{4}{25}$ from $(4, 1)$.

25. Answer: (D) 728**. The idea.** Watch the largest disc. It must travel $A \to B$ and later $B \to C$, and each time it moves, every smaller disc must be out of the way on the one peg it is not using. That forces the smaller tower to make three complete end-to-end trips, which gives a recursion.

Work it out. Let $f(n)$ be the fewest moves to carry n discs from one end peg to the other. For the big disc to move $A \to B$, all $n - 1$ smaller discs must be on C: that is a full end-to-end transfer, $f(n - 1)$ moves. Move the big disc (1). For it to move $B \to C$, the smaller discs must all be on A: another transfer $C \to A$, $f(n-1)$ moves. Move the big disc (1). Finally carry the small tower $A \to C$: $f(n - 1)$. So $f(n) = 3f(n - 1) + 2$ with $f(1) = 2$ (a single disc goes $A \to B \to C$). Adding 1: $f(n) + 1 = 3\,(f(n - 1) + 1)$, and $f(1) + 1 = 3$, so $f(n) = 3^n - 1$. For six discs, $f(6) = 729 - 1 = 728$. No solution is shorter: the big disc must move at least twice, and before its first move, between its first and last moves, and after its last move, the whole smaller tower must make an end-to-end trip (C, then A, then C), each costing at least $f(n - 1)$ moves.

Check. Small cases: $f(1) = 2$, $f(2) = 3 \cdot 2 + 2 = 8$ (the two-disc solution by hand takes 8 moves), and $3^2 - 1 = 8$. Also there are $3^6 = 729$ ways to place six discs legally, and the optimal solution visits every one of them, which is exactly 728 moves. Without the neighbour rule the answer would be $2^6 - 1 = 63$.

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