MathCastle

AMC 8

Practice Contest

25 questions • 40 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 40 minutes to finish the test.

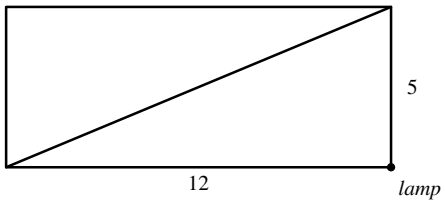
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Ten boxes stand in a row, and each holds a number. For every three consecutive boxes, the three numbers add up to exactly 15. The first box holds 4 and the eighth box holds 7. What is the sum of all ten numbers?

(A) 45 **(B)** 49 **(C)** 50 **(D)** 52 **(E)** 55

2. A rectangular park measures 12 m by 5 m. A straight path runs across it from one corner to the opposite corner. A lamp stands at one of the other two corners. What is the shortest distance, in metres, from the lamp to the path?



(A) $\frac{25}{13}$ **(B)** $\frac{30}{13}$ **(C)** $\frac{60}{17}$ **(D)** 4 **(E)** $\frac{60}{13}$

3. A club has 5 boys and 4 girls. A team of 3 members is chosen, and the team must include at least one boy and at least one girl. In addition, one of the girls, Ann, and one of the boys, Ben, refuse to be on the same team. How many different teams are possible?

(A) 40 **(B)** 63 **(C)** 70 **(D)** 74 **(E)** 140

4. In a 200 m race, when runner A crosses the finish line, runner B is exactly 40 m behind. In a separate 150 m race, when B crosses the finish line, runner C is exactly 30 m behind. Each runner always runs at the same constant speed. If A and C race 250 m against each other, how many metres behind will C be when A finishes?

(A) 90 **(B)** 100 **(C)** 120 **(D)** 150 **(E)** 160

5. All apples weigh the same, all bananas weigh the same, and all melons weigh the same. On a pan balance, 3 apples and 2 bananas exactly balance 1 melon. Also, 1 apple and 1 melon together exactly balance 5 bananas. How many bananas exactly balance 4 melons?

(A) 11 **(B)** 12 **(C)** 15 **(D)** 16 **(E)** 17

6. Three gears sit in a chain: gear A has 12 teeth and meshes with gear B, which has 18 teeth and also meshes with gear C, which has 30 teeth. Each gear has an arrow painted on it, and at the start all three arrows point straight up. Gear A is turned steadily. How many full turns does gear A make before all three arrows point straight up at the same time for the first time?

(A) 6 **(B)** 10 **(C)** 15 **(D)** 30 **(E)** 180

7. Square $ABCD$ has its vertices labelled in order around the square. An equilateral triangle ABE is built on side $\overline{AB}$, outside the square. What is the measure, in degrees, of $\angle DEC$?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

8. Three players play a single round of rock–paper–scissors at the same moment. Each independently shows rock, paper or scissors, each with probability $\frac{1}{3}$. (Rock beats scissors, scissors beats paper, paper beats rock; equal symbols tie.) What is the probability that exactly one player's symbol beats the symbols of both other players?

(A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{7}{9}$

9. A plane leaves city P at 9:40 a.m. (P time) and lands in city Q at 1:10 p.m. the same day (Q time). The return flight leaves Q at 3:00 p.m. (Q time) and lands in P at 11:30 p.m. the same day (P time). Both flights take exactly the same amount of time, and the clocks in the two cities differ by a fixed amount. How many hours does each flight take?

(A) $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** 5 **(D)** 6 **(E)** $\frac{17}{2}$

10. How many of the integers $1, 2, 3, \ldots, 100$ can be written in the form $a^2 - b^2$, where a and b are both positive integers?

(A) 25 **(B)** 50 **(C)** 73 **(D)** 74 **(E)** 75

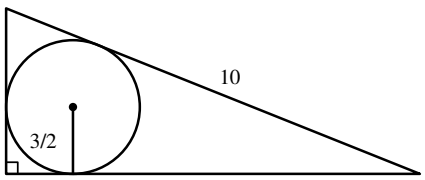
11. Two candles have the same length. The first burns down completely in 4 hours and the second in 3 hours, each at a steady rate. The first candle is lit at noon, and the second is lit 30 minutes later. How many minutes after noon is one candle exactly twice as long as the other?

(A) 144 **(B)** 162 **(C)** 174 **(D)** 192 **(E)** 210

12. Three fair six-sided dice, with faces numbered 1 to 6, are rolled. What is the probability that the product of the three numbers rolled is a multiple of 4?

(A) $\frac{5}{8}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{13}{16}$ **(E)** $\frac{7}{8}$

13. A right triangle has a hypotenuse of length 10. The circle inscribed in the triangle (touching all three sides) has radius $\frac{3}{2}$. What is the area of the triangle?



(A) $\frac{69}{8}$ **(B)** 15 **(C)** $\frac{69}{4}$ **(D)** $\frac{39}{2}$ **(E)** $\frac{69}{2}$

14. In a town, 22% of the adults own a cat. Among the adults who own a dog, 25% own a cat, while among

the adults who do not own a dog, only 20% own a cat. What fraction of the town's cat owners also own a dog?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{5}{11}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5}{6}$

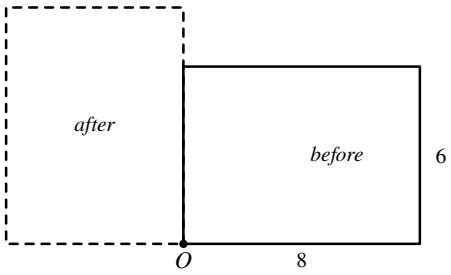
15. Ten identical candies are shared among four children, Ana, Ben, Cal and Dee, and all ten candies are handed out. Every child gets at least 1 candy and no child gets more than 4. In how many ways can this be done? (Two ways are different if some child gets a different number of candies.)

(A) 40 **(B)** 44 **(C)** 68 **(D)** 80 **(E)** 84

16. A sequence begins $a_1 = 3$ and $a_2 = 8$. Every later term is the previous term minus the term before that: $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$. So $a_3 = 5$ and $a_4 = -3$. What is $a_1 + a_2 + a_3 + \cdots + a_{2026}$?

(A) 0 **(B)** 5 **(C)** 11 **(D)** 13 **(E)** 16

17. An 8 by 6 rectangle lies with one corner at the point O. The rectangle turns steadily through 90° counterclockwise about O. What is the area of the region it sweeps out, that is, the set of all points covered by the rectangle at some moment during the turn (start and finish included)?



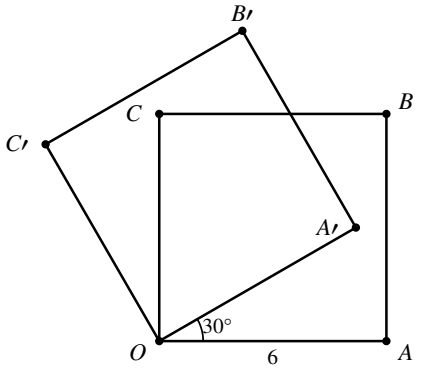
(A) $9\pi + 48$ **(B)** 25π **(C)** $48 + 16\pi$ **(D)** $24 + 25\pi$ **(E)** $48 + 25\pi$

18. In how many ways can 2025 be written as a sum of two or more consecutive positive integers? For example, 2025 = 1012 + 1013 is one way. (Two ways are the same if they use the same numbers.)

(A) 7 **(B)** 8 **(C)** 13 **(D)** 14 **(E)** 15

19. Square $OABC$ has side length 6, with its vertices labelled counterclockwise. The square is rotated 30° counterclockwise about its corner O, giving square $OA'B'C'$. What is the area of the region that lies inside both squares?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.



(A) $6\sqrt{3}$ **(B)** $36 - 12\sqrt{3}$ **(C)** $9\sqrt{3}$ **(D)** 18 **(E)** $12\sqrt{3}$

20. Three different vertices of a regular octagon are chosen at random, and all sets of three vertices are equally likely. What is the probability that the triangle they form contains the octagon's center strictly inside it (not on one of its sides)?

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{7}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{7}$

21. For each positive integer k, let $f(k)$ be the largest whole number whose square is at most k. For example, $f(1) = f(2) = f(3) = 1$ and $f(4) = 2$. What is the smallest positive integer N for which $f(1) + f(2) + \cdots + f(N)$ is greater than 2026?

(A) 195 **(B)** 216 **(C)** 217 **(D)** 218 **(E)** 224

22. A lattice point is a point both of whose coordinates are integers. How many lattice points lie strictly inside the triangle with vertices $(0, 0)$, $(15, 6)$ and $(4, 10)$? (Points on the sides do not count.)

(A) 58 **(B)** 59 **(C)** 60 **(D)** 61 **(E)** 63

23. For some even numbers n, the whole numbers $1, 2, \ldots, n$ can be split into $\frac{n}{2}$ pairs, each number used exactly once, so that the two numbers in every pair add up to a perfect square. For example, $n = 8$ works: $1 + 8$, $2 + 7$, $3 + 6$ and $4 + 5$ all equal 9. For how many even numbers n with $2 \le n \le 24$ is such a split possible?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

24. Let $S(n)$ be the sum of the digits of the positive integer n. For example, $S(2026) = 10$. What is the smallest positive integer n for which $S(n)$ is divisible by 11 and $S(n + 1)$ is divisible by 13?

(A) 589 **(B)** 4899999 **(C)** 29999999 **(D)** 39999999 **(E)** 48999999

25. A frog starts at 0 on a number line and only jumps to the right. Each jump is 1 unit or 2 units, each with probability $\frac{1}{2}$, independently of all other jumps, and the frog keeps jumping forever. What is the probability that the frog lands on 10 at some point but never lands on 5?

(A) $\frac{121}{1024}$ **(B)** $\frac{121}{512}$ **(C)** $\frac{441}{1024}$ **(D)** $\frac{121}{256}$ **(E)** $\frac{683}{1024}$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — AMC 8 Practice Contest 3

1. (B) 49

2. (E) $\frac{60}{13}$

3. (B) 63

4. (A) 90

5. (E) 17

6. (C) 15

7. (B) 30

8. (A) $\frac{1}{3}$

9. (D) 6

10. (C) 73

11. (D) 192

12. (A) $\frac{5}{8}$

13. (C) $\frac{69}{4}$

14. (C) $\frac{5}{11}$

15. (B) 44

16. (D) 13

17. (E) $48 + 25\pi$

18. (D) 14

19. (E) $12\sqrt{3}$

20. (A) $\frac{1}{7}$

21. (C) 217

22. (D) 61

23. (A) 5

24. (E) 48999999

25. (B) $\frac{121}{512}$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — AMC 8 Practice Contest 3

1. Answer: (B) 49**. The idea.** Two overlapping triples share two boxes, so comparing them shows that box k and box $k+3$ must hold the same number. The whole row is then a repeating pattern of length 3.

Work it out. Boxes $1, 2, 3$ and boxes $2, 3, 4$ both sum to 15; subtracting, box 4 equals box 1. In the same way box $k+3$ equals box k every time. So boxes $1, 4, 7, 10$ are equal (all 4), boxes $2, 5, 8$ are equal (all 7, because box 8 holds 7), and boxes $3, 6, 9$ are equal to $15 - 4 - 7 = 4$. The row is $4, 7, 4, 4, 7, 4, 4, 7, 4, 4$. Boxes 1 to 9 are three disjoint triples, worth $3 \cdot 15 = 45$, and box 10 adds 4, for a total of 49.

Check. Adding the row directly: three blocks $4 + 7 + 4 = 15$ plus a final 4 gives 49. A tempting slip is $10 \cdot 15/3 = 50$, which pretends ten boxes split into triples evenly.

2. Answer: (E) $\frac{60}{13}$**. The idea.** The shortest distance to the path is the height of a triangle drawn to the path as its base. Compute that triangle's area in two ways.

Work it out. The path splits the park into two right triangles, each with legs 12 and 5 and area $\frac{1}{2} \cdot 12 \cdot 5 = 30$. The lamp is the right-angle corner of one of them. By Pythagoras the path has length $\sqrt{12^2 + 5^2} = 13$. Using the path as the base, $\frac{1}{2} \cdot 13 \cdot h = 30$, so $h = \frac{60}{13}$ m, a little over 4.6 m.

Check. In coordinates the path is the line $5x - 12y = 0$ and the lamp is at $(12, 0)$; the distance is $\frac{|60|}{\sqrt{25+144}} = \frac{60}{13}$. It is less than the shorter side 5, as it must be.

3. Answer: (B) 63**. The idea.** Count all teams, remove those that break a rule. Removing the single-gender teams is easier than building the mixed ones, and a team containing both Ann and Ben is automatically mixed, so the two rules don't overlap.

Work it out. There are $\binom{9}{3} = 84$ teams in all. All-boy teams: $\binom{5}{3} = 10$; all-girl teams: $\binom{4}{3} = 4$. That leaves $84 - 10 - 4 = 70$ mixed teams. A team containing both Ann and Ben is mixed and its third member is any of the other 7 people, so 7 of the 70 are forbidden. The answer is $70 - 7 = 63$.

Check. Count directly: two boys and a girl gives $\binom{5}{2} \cdot 4 = 40$ teams, of which 4 contain Ben and Ann (Ben, Ann, and one of the other 4 boys), leaving 36. One boy and two girls gives $5 \cdot \binom{4}{2} = 30$, of which 3 contain Ben and Ann, leaving 27. Total $36 + 27 = 63$. (Choosing "a boy, a girl, then anyone" gives $5 \cdot 4 \cdot 7 = 140$, which counts most teams twice.)

4. Answer: (A) 90**. The idea.** Margins from races of different lengths can't be added, and even percentages don't add: the speed ratios multiply. In any stretch of time B covers a fixed fraction of A's distance, and C a fixed fraction of B's.

Work it out. In the first race B runs 160 m while A runs 200 m, so B covers $\frac{160}{200} = \frac{4}{5}$ of A's distance in the same time. In the second race C runs 120 m while B runs 150 m, so C covers $\frac{120}{150} = \frac{4}{5}$ of B's distance. Hence C covers $\frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25}$ of A's distance. While A runs 250 m, C runs $\frac{16}{25} \cdot 250 = 160$ m, so C is $250 - 160 = 90$ m behind.

Check. Give A a speed of 25 m/s, so B runs 20 m/s and C runs 16 m/s. Race one: A needs 8 s, when B has run 160 m, 40 m behind. Race two: B needs 7.5 s, when C has run 120 m, 30 m

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

behind. Race three: A needs 10 s, when C has run 160 m, 90 m behind. Adding the percentages (20% + 20%) gives 100, and 160 is how far C runs, not how far behind C is; but C loses 20% of B's distance, which is only 16% of A's, so the total loss is 36% of 250, which is 90.

5. Answer: (E) 17**. The idea.** Replace the melon in the second balance by what it equals in the first. That leaves only apples and bananas, so everything can be written in banana units.

Work it out. Write a, b, m for the weights: $3a + 2b = m$ and $a + m = 5b$. Substituting, $a + 3a + 2b = 5b$, so $4a = 3b$: four apples weigh as much as three bananas, and $a = \frac{3}{4}b$. Then $m = 3 \cdot \frac{3}{4}b + 2b = \frac{9}{4}b + \frac{8}{4}b = \frac{17}{4}b$. Four melons weigh $17b$: they balance 17 bananas.

Check. Take a banana to weigh 4, so an apple weighs 3 and a melon 17. Then $3 \cdot 3 + 2 \cdot 4 = 17$ and $3 + 17 = 20 = 5 \cdot 4$, both balances hold, and $4 \cdot 17 = 68 = 17 \cdot 4$.

6. Answer: (C) 15**. The idea.** Meshing gears pass teeth at the same rate: whenever one tooth of A passes its contact point, one tooth of B passes it, and B passes one tooth to C as well. So count everything in "teeth passed", which is the same number N for all three gears.

Work it out. A gear with n teeth is back to its starting position exactly when N is a multiple of n. All three are back together when N is a common multiple of 12, 18 and 30; the first time is $N = \operatorname{lcm}(12, 18, 30)$. With $12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$, $30 = 2 \cdot 3 \cdot 5$, the lcm is $2^2 \cdot 3^2 \cdot 5 = 180$. Gear A has then made $180/12 = 15$ turns.

Check. At that moment B has made $180/18 = 10$ turns and C has made $180/30 = 6$, all whole numbers. At any earlier multiple of 12, say $N = 60$ or $N = 90$, some gear is mid-turn ($60/18$ and $90/12$ are not whole). The middle gear only reverses direction; it does not change the count.

7. Answer: (B) 30**. The idea.** The side of the square and the side of the triangle are equal, so $AD = AB = AE$. That makes triangle DAE isosceles, and its apex angle is easy to find.

Work it out. At A, the angle between $\overline{AD}$ and $\overline{AE}$ is the square's 90° plus the triangle's 60°, so $\angle DAE = 150°$. Triangle DAE is isosceles with $AD = AE$, so its base angles are $\frac{180° - 150°}{2} = 15°$; in particular $\angle AED = 15°$. By the same argument at B, $\angle BEC = 15°$. Both rays ED and EC lie inside $\angle AEB = 60°$, so $\angle DEC = 60° - 15° - 15° = 30°$.

Check. Coordinates: $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$, $E = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Then $\overrightarrow{ED} = (-\frac{1}{2}, 1 + \frac{\sqrt{3}}{2})$ and $\overrightarrow{EC} = (\frac{1}{2}, 1 + \frac{\sqrt{3}}{2})$, and the cosine of the angle between them works out to $\frac{\sqrt{3}}{2}$, which is 30°.

8. Answer: (A) $\frac{1}{3}$**. The idea.** Each symbol beats exactly one other symbol. So if one player beats both others, the other two must have shown the same symbol, namely the one the winner beats.

Work it out. Choose the winning player (3 ways) and the winner's symbol (3 ways). The other two players are then forced to show the single symbol it beats. That gives $3 \cdot 3 = 9$ outcomes. They are all different, and in each one exactly one player beats both others (the two losers tie each other). There are $3^3 = 27$ equally likely outcomes, so the probability is $\frac{9}{27} = \frac{1}{3}$.

Check. Sort the 27 outcomes: 3 have all symbols equal, 6 have all three different (then nobody beats both), and 18 use exactly two symbols. In a two-symbol outcome either one player beats a pair (9 cases) or a pair beats one player (the other 9, where two players each beat only one opponent). $3 + 6 + 9 + 9 = 27$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

9. Answer: (D) 6**. The idea.** Each flight's apparent duration (arrival clock minus departure clock) is off by the time difference, but in opposite directions for the two flights. Averaging the two apparent durations cancels the time difference.

Work it out. Let the flight take t hours and let Q's clocks be d hours behind P's. Outbound, the clocks show 9:40 $\to$ 1:10, an apparent $3\frac{1}{2}$ h, which is $t - d$ (flying toward the clocks that read earlier). Return, the clocks show 3:00 $\to$ 11:30, an apparent $8\frac{1}{2}$ h, which is $t+d$. Adding, $2t = 12$, so $t = 6$ hours, and $d = 2\frac{1}{2}$ hours.

Check. Leave P at 9:40 and fly 6 h: it is 3:40 p.m. in P, which is 1:10 p.m. in Q. Leave Q at 3:00 p.m., which is 5:30 p.m. in P; 6 h later it is 11:30 p.m. in P. Both match. The options $3\frac{1}{2}$ and $8\frac{1}{2}$ come from forgetting the time zones.

10. Answer: (C) 73**. The idea.** Factor: $a^2 - b^2 = (a - b)(a + b)$. The two factors $a - b$ and $a + b$ differ by $2b$, an even number, so they have the same parity.

Work it out. If both factors are odd, the number is odd; if both are even, it is a multiple of 4. So a number that is 2 more than a multiple of 4 never works: that rules out the 25 numbers $2, 6, \ldots, 98$. Conversely, an odd $n = 2k + 1$ equals $(k + 1)^2 - k^2$, which uses $b = k \geq 1$ as long as $n \geq 3$; and a multiple of four $n = 4k$ equals $(k + 1)^2 - (k - 1)^2$, which needs $b = k - 1 \geq 1$, so $n \geq 8$. The two exceptions are 1 and 4. Could they be written another way? $1 = (a - b)(a + b)$ forces $a + b = 1$, impossible with $a, b \geq 1$; 4 would need two same-parity factors with product 4 and $a + b > a - b$, and $2 \cdot 2$ gives $b = 0$. So both fail. The count is $100 - 25 - 2 = 73$.

Check. With $b = 0$ allowed the answer would be 75; the word "positive" is what removes $1 = 1^2 - 0^2$ and $4 = 2^2 - 0^2$. Spot checks: $3 = 2^2 - 1^2$, $8 = 3^2 - 1^2$, $12 = 4^2 - 2^2$, and 6 and 10 have no representation.

11. Answer: (D) 192**. The idea.** Write each length as a fraction of the original length, as a function of time. The faster candle starts ahead but shrinks faster, so it is the one that ends up half as long as the other.

Work it out. Let t be the hours after noon. The first candle has $1 - \frac{t}{4}$ left; the second, lit at $t = \frac{1}{2}$, has $1 - \frac{t-1/2}{3}$ left. Before the second is lit, the first is still longer than $\frac{7}{8}$ while the second is 1, so neither is twice the other. The second candle burns out at $t = 3\frac{1}{2}$ while the first lasts until 4, so the second can't be twice the first near the end, and setting "second = 2× first" gives $t = 5$, too late. So solve $1 - \frac{t}{4} = 2\left(1 - \frac{2t-1}{6}\right) = \frac{7-2t}{3}$. Multiply by 12: $12 - 3t = 28 - 8t$, so $5t = 16$ and $t = \frac{16}{5}$ h = 192 minutes, i.e. 3:12 p.m.

Check. At $t = \frac{16}{5}$ the first candle has $1 - \frac{4}{5} = \frac{1}{5}$ left and the second has $1 - \frac{27/10}{3} = \frac{1}{10}$ left, exactly half. Without the 30-minute delay the answer would be 144 minutes; the delay costs more than 30 extra minutes because the gap closes slowly.

12. Answer: (A) $\frac{5}{8}$**. The idea.** Count the failures instead. The product misses a factor of 4 only if it has at most one factor of 2 in total. Each face is odd (1, 3, 5), "singly even" (2, 6, one factor of 2), or 4 (two factors of 2).

Work it out. The product fails exactly when (i) all three dice are odd, $3^3 = 27$ outcomes, or (ii) exactly one die is 2 or 6 and the other two are odd: 3 choices of which die, 2 faces for it, $3 \cdot 3$ for the others, 54 outcomes. A 4 anywhere, or two even dice, always succeeds. Failures: $27+54 = 81$ of 216, so the probability of success is $1 - \frac{81}{216} = 1 - \frac{3}{8} = \frac{5}{8}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Check. Count successes directly: at least one 4 gives $216 - 125 = 91$ outcomes; with no 4, at least two dice from $\{2, 6\}$ and the rest from $\{1, 3, 5\}$ gives $3 \cdot 2^2 \cdot 3 + 2^3 = 36 + 8 = 44$. Total $135 = \frac{5}{8} \cdot 216$. Only excluding "all odd" would give $\frac{7}{8}$, the most common slip.

13. Answer: (C) $\frac{69}{4}$**. The idea.** The legs here are not whole numbers, so do not hunt for them. Tangent lengths give the sum of the legs directly, and the area of any triangle equals its inradius times its semiperimeter.

Work it out. Call the legs a, b and the hypotenuse $c = 10$. The two tangent segments from any vertex to the incircle are equal. At the right angle, the two tangent segments and the two radii to the touching points form a square, so both tangent segments from that corner have length $r = \frac{3}{2}$. The rest of leg a and the rest of leg b are the two tangent segments from the acute vertices, and together these make up the hypotenuse. So $a + b = c + 2r = 10 + 3 = 13$.

Cut the triangle into three triangles with apex at the incenter. Each has height r above one side, so the area is $\frac{1}{2}r(a + b + c) = r \cdot s$, where $s = \frac{13+10}{2} = \frac{23}{2}$. The area is $\frac{3}{2} \cdot \frac{23}{2} = \frac{69}{4}$.

Check. The area is also $\frac{1}{2}ab$, so $ab = \frac{69}{2}$. Then $a^2 + b^2 = (a + b)^2 - 2ab = 169 - 69 = 100 = c^2$, as it should be. The legs are $\frac{13 \pm \sqrt{31}}{2}$, both positive. Common slips: $ab = \frac{69}{2}$ is twice the area, and $r \cdot c = 15$ uses the wrong length.

14. Answer: (C) $\frac{5}{11}$**. The idea.** Two steps. First, the overall cat rate 22% is a weighted average of 25% (dog owners) and 20% (everyone else), and where it sits between them tells us how many adults own a dog. Then turn the question around: we know what fraction of dog owners have cats, but we want the fraction of cat owners who have dogs, so count people in a concrete town of 100 adults.

Work it out. Say d of the 100 adults own a dog. The cat owners number $\frac{1}{4}d + \frac{1}{5}(100 - d) = 20 + \frac{d}{20}$, and this must be 22, so $d = 40$. (Equivalently, 22% is $\frac{2}{5}$ of the way from 20% to 25%, so $\frac{2}{5}$ of the adults own a dog.) Then 25% of the 40 dog owners, 10 people, own a cat, and 20% of the other 60, 12 people, own a cat. That is 22 cat owners, as required, and 10 of them own a dog. The fraction is $\frac{10}{22} = \frac{5}{11}$.

Check. The answer does not depend on the town having 100 adults: with N adults every count scales by $\frac{N}{100}$ and the ratio stays $\frac{10}{22}$. The trap $\frac{2}{5}$ is the fraction of adults who own a dog, not of cat owners. $\frac{1}{4}$ answers the question the wrong way round, and $\frac{10}{12} = \frac{5}{6}$ divides by the wrong group.

15. Answer: (B) 44**. The idea.** First give every child the 1 candy they are guaranteed. Then count the ways to share the rest with stars and bars, and subtract the ways that break the upper limit.

Work it out. After each child gets 1 candy, 6 candies are left and each child may get at most 3 more. Without the limit, sharing 6 identical candies among 4 children can be done in $\binom{6+3}{3} = \binom{9}{3} = 84$ ways. A sharing is bad if some child gets 4 or more of the extra candies. Two children cannot both do that, because $4 + 4 > 6$, so each bad sharing is counted once. If Ana is the one, hand her 4 extra first and share the other 2 freely: $\binom{2+3}{3} = 10$ ways. By symmetry there are $4 \cdot 10 = 40$ bad sharings, which leaves $84 - 40 = 44$.

Check. Count by the multiset of amounts, each between 1 and 4, summing to 10: $\{4, 4, 1, 1\}$ gives 6 orders, $\{4, 3, 2, 1\}$ gives 24, $\{4, 2, 2, 2\}$ gives 4, $\{3, 3, 3, 1\}$ gives 4 and $\{3, 3, 2, 2\}$ gives 6. The total is $6 + 24 + 4 + 4 + 6 = 44$. Reading "more than 4" as "4 extra is still fine" gives $84 - 16 = 68$ instead.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

16. Answer: (D) 13**. The idea.** Write out a few more terms. The sequence comes back to $3, 8$ after six steps, so it repeats with period 6, and each full block of six terms adds to 0.

Work it out. The terms are $3, 8, 5, -3, -8, -5, 3, 8, \ldots$. Once two consecutive terms repeat, every later term repeats too, because each term depends only on the two before it. One block sums to $3 + 8 + 5 - 3 - 8 - 5 = 0$. Since $2026 = 6 \cdot 337 + 4$, the first 2022 terms make 337 blocks with total 0. The last four terms are $a_{2023}, \ldots, a_{2026} = 3, 8, 5, -3$, so the sum is $3 + 8 + 5 - 3 = 13$.

Check. There is also a shortcut. Adding $a_n = a_{n-1} - a_{n-2}$ for $n = 3$ to N gives $a_1 + \cdots + a_N = a_2 + a_{N-1}$. For $N = 2026$ this is $8 + a_{2025} = 8 + 5 = 13$. Stopping one or two terms early or late gives the distractors 16, 11 and 5.

17. Answer: (E) $48 + 25\pi$**. The idea.** Find the point of the rectangle farthest from O, which is the opposite corner P at distance $\sqrt{8^2 + 6^2} = 10$. The diagonal OP sweeps a quarter disk of radius 10. Every other part of the swept region is covered by the starting or the finishing rectangle.

Work it out. Put O at the origin, with the rectangle at the start filling $0 \le x \le 8, 0 \le y \le 6$, so $P = (8, 6)$. The diagonal OP splits the rectangle into two right triangles of area 24. Before the turn, the lower triangle lies clockwise of OP. After the turn, the upper triangle lies counterclockwise of the final diagonal OP'. In between, the diagonal OP sweeps out the quarter disk between OP and OP', of radius 10. Every point of the quarter disk is at distance at most 10 from O, and at some moment the diagonal points straight at it, so the rectangle covers it. No point of the rectangle ever gets farther than 10 from O, and every point clockwise of OP or counterclockwise of OP' that gets covered is covered by the start or end position. The swept area is therefore $24+25\pi+24 = 48+25\pi$.

Check. Look at the region circle by circle. At distance ρ from O the rectangle covers an arc of some angle. The turn stretches that arc by exactly $90°$, so the swept area is the rectangle's own area plus $\int_0^{10} \frac{\pi}{2}\rho \, d\rho = 25\pi$, which is again $48 + 25\pi$. Sweeping with the long side (radius 8) instead of the diagonal gives $48 + 16\pi$, and counting only one of the two leftover triangles gives $24 + 25\pi$.

18. Answer: (D) 14**. The idea.** If k consecutive integers start at a, their sum is $\frac{k(2a+k-1)}{2}$. The two factors k and $2a + k - 1$ have opposite parity. Since 2025 is odd, both of them must be odd divisors, and each way to write $2 \cdot 2025$ as a product of the two factors gives exactly one sum.

Work it out. We need $k(2a + k - 1) = 4050$ with $k \ge 2$ and $a \ge 1$, so $2a + k - 1 > k$. Write $4050 = 2 \cdot 2025$, where $2025 = 3^4 \cdot 5^2$. The factors k and $m = 2a + k - 1$ have opposite parity, so exactly one of them is even and the other is an odd divisor d of 2025. Given d, the two factors are d and $\frac{4050}{d}$. Choose k to be the smaller one and m the larger (they are never equal). Then $a = \frac{m-k+1}{2}$ is a positive integer. The only failure is $k = 1$, which happens when $d = 1$ (the pair $1, 4050$). 2025 has $(4 + 1)(2 + 1) = 15$ divisors, so there are $15 - 1 = 14$ ways.

Check. A few of them: $d = 3$ gives $k = 3$ and $674 + 675 + 676$. $d = 5$ gives $k = 5$ and $403 + \cdots + 407$. $d = 2025$ gives $k = 2$ and $1012 + 1013$. $d = 45$ gives $k = 45$ and $23 + \cdots + 67$. Forgetting to throw out $d = 1$ gives 15.

19. Answer: (E) $12\sqrt{3}$**. The idea.** The overlap is a kite with one corner at O. Its diagonal from O splits it into two congruent right triangles. Each has a leg of 6 and a $30°$ angle at O, because the two squares' right angles at O overlap in a $60°$ angle, and the diagonal cuts that angle in half.

Work it out. Place O at the origin with $A = (6, 0)$ and $C = (0, 6)$. After the rotation, side OA' points at $30°$ and side OC' points at $120°$. So near O the common region lies between the ray at

30° (along OA') and the ray at 90° (along OC), an angle of 60°. It is closed off by side CB (the line $y = 6$) and by side $A'B'$. These two sides meet at a point Q. Triangles OCQ and $OA'Q$ are right triangles, with right angles at C and A', and they share the hypotenuse OQ and have legs $OC = OA' = 6$. So they are congruent, and OQ bisects the 60° angle. In triangle OCQ, the angle at O is 30°, so $CQ = 6\tan 30° = 2\sqrt{3}$. The overlap has area $2 \cdot \frac{1}{2} \cdot 6 \cdot 2\sqrt{3} = 12\sqrt{3}$.

Check. In coordinates, $A' = (3\sqrt{3}, 3)$ lies inside the first square because $3\sqrt{3} < 6$. The line $A'B'$ is $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 6$, and it meets $y = 6$ at $x = 2\sqrt{3}$, so $Q = (2\sqrt{3}, 6)$. The shoelace formula on O, A', Q, C gives $\frac{1}{2}\left|(3\sqrt{3})(6) - 3(2\sqrt{3}) + 2\sqrt{3} \cdot 6 - 0\right| = \frac{1}{2} \cdot 24\sqrt{3} = 12\sqrt{3} \approx 20.8$, a bit over half the square, as the picture suggests.

20. Answer: (A) $\frac{1}{7}$. The idea. All eight vertices lie on a circle around the center. A triangle inscribed in a circle contains the center strictly inside exactly when it is acute. It has the center on a side when it is right-angled (a side is a diameter), and it misses the center when it is obtuse (all three vertices fit in an open half of the circle). Count the right and obtuse triangles, then subtract.

Work it out. There are $\binom{8}{3} = 56$ triangles. *Right triangles:* the octagon has 4 diameters joining opposite vertices, and each can be paired with any of the other 6 vertices, giving $4 \cdot 6 = 24$. *Obtuse triangles:* going counterclockwise, give each such triangle its first vertex V. Then the other two vertices are among the next 3 vertices after V (the fourth one after V is opposite V). That gives $8 \cdot \binom{3}{2} = 24$. So $56 - 24 - 24 = 8$ triangles are acute, and the probability is $\frac{8}{56} = \frac{1}{7}$.

Check. List the acute triangles directly. Number the vertices 0 to 7. The gaps (in steps) between consecutive chosen vertices must add to 8, and each gap must be less than 4. The only way is $3 + 3 + 2$ in some cyclic order. Each of the 8 rotations of $\{0, 3, 6\}$ is a different triangle, which again makes 8. Counting the right triangles as "inside" gives $\frac{32}{56} = \frac{4}{7}$ instead.

21. Answer: (C) 217. The idea. f is constant on long runs. It equals j for every k from j^2 to $(j+1)^2 - 1$, which is a run of $2j + 1$ numbers, so each run contributes $j(2j + 1)$. Add whole runs until the next one would pass 2026, then finish one term at a time.

Work it out. The run totals $j(2j+1)$ for $j = 1, 2, \ldots$ are 3, 10, 21, 36, 55, 78, 105, 136, 171, 210, 253, 300, 351, 406 Their running totals are 3, 13, 34, 70, 125, 203, 308, 444, 615, 825, 1078, 1378, 1729, 2135. So the sum up to $k = 13^2 + 26 = 195$ is 1729, and the full next run (up to 224) would reach $2135 > 2026$. From $k = 196$ onward each term adds 14. We need $1729 + 14t > 2026$, that is $14t > 297$. Since $14 \cdot 21 = 294$ is too small and $14 \cdot 22 = 308$ works, $t = 22$ and $N = 195 + 22 = 217$.

Check. The sum up to 216 is $1729 + 294 = 2023$, which is not greater than 2026. The sum up to 217 is 2037, which is. Stopping at $t = 21$ gives the off-by-one 216, and stopping at a run boundary gives 195 or 224.

22. Answer: (D) 61. The idea. Pick's theorem: for a polygon whose corners are lattice points, Area $= I + \frac{B}{2} - 1$, where I counts the lattice points inside and B counts those on the boundary. The area comes from the shoelace formula, and a side from (x_1, y_1) to (x_2, y_2) holds $\gcd(|x_2 - x_1|, |y_2 - y_1|)$ lattice points if you count one endpoint and leave out the other.

Work it out. Area: $\frac{1}{2}|15 \cdot 10 - 6 \cdot 4| = \frac{1}{2} \cdot 126 = 63$. Boundary: the side to $(15, 6)$ gives $\gcd(15, 6) = 3$, the side from $(15, 6)$ to $(4, 10)$ gives $\gcd(11, 4) = 1$, and the side from $(4, 10)$ back to the origin gives $\gcd(4, 10) = 2$. So $B = 3 + 1 + 2 = 6$, namely $(0, 0), (5, 2), (10, 4), (15, 6), (4, 10), (2, 5)$. Then $63 = I + 3 - 1$, so $I = 61$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Check. Count column by column. The lower side is $y = \frac{2x}{5}$. The upper side is $y = \frac{5x}{2}$ for $x \le 4$ and $y = 10 - \frac{4(x-4)}{11}$ for $x \ge 4$. The number of integers strictly between them for $x = 1, 2, \ldots, 14$ is $2, 4, 6, 8, 7, 7, 6, 5, 5, 3, 3, 3, 1, 1$, and these add to 61. Using the area 63 as the count, or getting the sign of the -1 wrong (59), are the usual slips.

23. Answer: (A) 5**. The idea.** Some numbers have very few possible partners, and that settles each case quickly. Two numbers that both need the same single partner make a split impossible. When a split does exist, following the forced partners usually builds it.

Work it out. The partners of 2 are 7 ($2+7=9$), 14 (16) and 23 (25). $n = 2, 4, 6$*:* 2 has no partner, so these fail. $n = 8$*:* works, as shown. $n = 10, 12$*:* 2 must pair with 7, but 9's only partner up to 12 is also 7 ($9 + 16 = 25$ would need 16). Both fail. $n = 14$*:* $(1, 8)$, $(2, 14)$, $(3, 13)$, $(4, 12)$, $(5, 11)$, $(6, 10)$, $(7, 9)$. $n = 16$*:* $(1, 8)$, $(2, 7)$, $(3, 6)$, $(4, 5)$, $(9, 16)$, $(10, 15)$, $(11, 14)$, $(12, 13)$. $n = 18$*:* $(1, 15)$, $(2, 14)$, $(3, 13)$, $(4, 12)$, $(5, 11)$, $(6, 10)$, $(7, 18)$, $(8, 17)$, $(9, 16)$. $n = 20$*:* 18's only partner is 7 ($18 + 18$ is not allowed). Then 2 must take 14, and 9 (partners 7 and 16) must take 16. That forces 20 (partners 5 and 16) to take 5. Now 11 needs 5 or 14 (or 25, which is too big), and both are taken, so $n = 20$ fails. $n = 22$*:* the same chain applies. 18 takes 7, 9 takes 16, 20 takes 5, and 2 takes 14 because 23 is still too big. Again 11 is left with no partner, so it fails. $n = 24$*:* pair k with $25 - k$. The values that work are 8, 14, 16, 18, 24, which makes 5.

Check. A computer search over all pairings agrees: the splits exist exactly for $n = 8, 14, 16, 18, 24$ in this range. For $n = 16$ the split is even unique, because 16 must pair with 9 and 8 must pair with 1, and the rest follows. Missing that $n = 20$ and 22 fail (they look like they should work) gives 6 or 7, and assuming every n from 8 up works gives 9.

24. Answer: (E) 48999999**. The idea.** Adding 1 usually raises the digit sum by 1, but every trailing 9 turns into a 0 and carries. So split n as a prefix P followed by exactly k nines, where the last digit of P is not 9. Then $n + 1$ is $P + 1$ followed by k zeros, and $S(n + 1) = S(P) + 1$, while $S(n) = S(P) + 9k$. The two conditions become one condition on $S(P)$ and one linking $S(P)$ to k.

Work it out. The prefix cannot be empty: if $n = 99\ldots9$ then $S(n+1) = 1$. We need $S(P)+1 \equiv 0 \pmod{13}$, so $S(P)$ is one of $12, 25, 38, 51, 64, 77, \ldots$, and $S(P) + 9k \equiv 0 \pmod{11}$. Try the smallest, $S(P) = 12$. Since $12 \equiv 1 \pmod{11}$, we need $9k \equiv 10 \pmod{11}$. The values of $9k$ for $k = 0, 1, \ldots, 6$ are $0, 9, 18, 27, 36, 45, 54$, and $54 = 44 + 10$ is the first that works, so $k = 6$. A digit sum of 12 needs at least two digits. The smallest two-digit numbers with digit sum 12 are 39 and 48, but 39 ends in 9 (it would add a seventh trailing nine), so $P = 48$ and $n = 48999999$, with 8 digits.

Nothing else is that short. If $S(P) = 25, 38, 51$ or 64, the same computation gives $k = 7, 8, 9, 10$, and P needs at least 3 digits, so n has at least 10 digits. If $S(P) \ge 77$, then P alone needs at least 9 digits, because eight digits add to at most 72. So the answer is 48999999.

Check. $S(48999999) = 4 + 8 + 54 = 66 = 6 \cdot 11$, and $n + 1 = 49000000$ has $S = 13$. The trap 39999999 also has $S = 66$, but $n + 1 = 40000000$ has digit sum 4. The trap 589 comes from writing $S(n + 1) = S(P)$ and forgetting the +1: $S(589) = 22$, but 590 has digit sum 14. 4899999 uses five nines, one too few: its digit sum is 57. 29999999 has digit sum 65. A computer search over every n below 60 million confirms 48999999 is the first.

25. Answer: (B) $\frac{121}{512}$**. The idea.** Let p_n be the probability that the frog ever lands on n. The frog skips n exactly when it lands on $n - 1$ and then jumps 2. So $1 - p_n = \frac{1}{2}p_{n-1}$, and all the p_n follow

quickly. To skip 5, the frog must stand on 4 and jump to 6. After that the jumps start fresh, and landing on 10 is just landing 4 units further on.

Work it out. Start with $p_0 = 1$. Using $p_n = 1 - \frac{1}{2}p_{n-1}$: $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$, $p_3 = \frac{5}{8}$, $p_4 = \frac{11}{16}$, $p_5 = \frac{21}{32}$. Never landing on 5 means landing on 4 (probability $\frac{11}{16}$) and then jumping 2 (probability $\frac{1}{2}$), which puts the frog on 6 with probability $\frac{11}{32}$. From 6, the chance of later landing on 10 is the chance of ever landing on 4 from a fresh start, $p_4 = \frac{11}{16}$. The answer is $\frac{11}{32} \cdot \frac{11}{16} = \frac{121}{512}$.

Check. Count the complement another way. Continuing the recursion, $p_{10} = \frac{683}{1024}$. Landing on both 5 and 10 has probability $p_5 \cdot p_5 = \frac{441}{1024}$, since after reaching 5 the frog starts fresh 5 short of 10. So $\frac{683-441}{1024} = \frac{242}{1024} = \frac{121}{512}$, a match. (The closed form $p_n = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n$ gives the same values.) Forgetting the forced jump of 2 gives $\frac{121}{256}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.