MathCastle

AMC 8

Practice Contest

25 questions • 40 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 40 minutes to finish the test.

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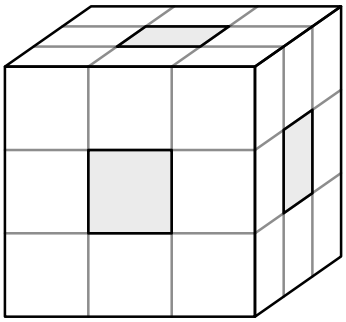
1. A carpenter takes 12 minutes to saw a board into 4 equal pieces, using straight cuts across the board, one at a time. Every cut takes her the same amount of time. How many minutes does she need to saw an identical board into 10 equal pieces?

(A) 36 **(B)** 40 **(C)** 44 **(D)** 108 **(E)** 120

2. Drink A is $\frac{1}{3}$ juice and $\frac{2}{3}$ water. Drink B is $\frac{3}{4}$ juice and $\frac{1}{4}$ water. Lena pours some of drink A and some of drink B into a pitcher, 60 mL in all, and the mixture in the pitcher is exactly half juice. How many mL of drink A did she pour?

(A) 20 **(B)** 24 **(C)** 30 **(D)** 36 **(E)** 40

3. A $3 \times 3 \times 3$ cube is glued together from 27 unit cubes. Then the unit cube at the very center and the six unit cubes at the centers of the six faces are removed, leaving three straight tunnels that run all the way through the solid and cross in the middle. What is the total surface area of the remaining solid, counting the inside walls of the tunnels as well as the outside?



(A) 48 **(B)** 54 **(C)** 66 **(D)** 72 **(E)** 84

4. The pages of a book are numbered $1, 2, 3, \ldots, n$, one number per page. When all the page numbers are printed, the digit 7 appears exactly 25 times in total. What is the largest possible value of n?

(A) 146 **(B)** 147 **(C)** 150 **(D)** 156 **(E)** 157

5. Tom, Ann, Ben and their dog Rex want to weigh themselves, but the scale only works with at least three on it. So each of the four possible groups of three steps on the scale once. The four readings, in pounds and in no particular order, are 140, 165, 180 and 220. Rex is the lightest of the four. How many pounds does Rex weigh?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 30 **(E)** 45

6. Call a two-digit number "loyal" if neither of its digits is 0 and the number is divisible by each of its own digits. For example, 24 is loyal because 24 is divisible by both 2 and 4, while 26 is not, since 26 is not divisible by 6. How many two-digit numbers are loyal?

(A) 9 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

7. Three fair coins are tossed and one fair six-sided die (faces 1 through 6) is rolled, all independently. What is the probability that the number of coins showing heads is equal to the number rolled on the die?

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(A) $\frac{1}{8}$ **(B)** $\frac{7}{48}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{7}{36}$ **(E)** $\frac{1}{4}$

8. In triangle ABC, $AB = AC$. Point D lies on side $\overline{AB}$, strictly between A and B, and $CD = CB$. If $\angle ACD = 30°$, what is the measure of $\angle BAC$ in degrees?

(A) 10 **(B)** 20 **(C)** 30 **(D)** 35 **(E)** 40

9. When Sam walks to school at a steady 4 km/h, he arrives 5 minutes late. When he walks at a steady 6 km/h, he arrives 5 minutes early. He always leaves home at the same time. At what steady speed, in km/h, should he walk to arrive exactly on time?

(A) 4.8 **(B)** 5 **(C)** 5.2 **(D)** 5.4 **(E)** 5.5

10. A club has 100 students. Exactly 50 of them play soccer, 40 play basketball and 30 play tennis. Exactly 10 students play all three sports and exactly 20 students play none of them. How many students play exactly two of the three sports?

(A) 20 **(B)** 25 **(C)** 30 **(D)** 40 **(E)** 50

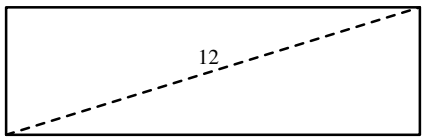
11. Five friends, Ava, Ben, Cal, Dev and Eli, sit in a row of five seats. Cal insists on sitting in one of the two end seats, and Ava and Ben refuse to sit in adjacent seats. How many different seatings satisfy both conditions?

(A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 72

12. A sequence of positive integers starts with any two positive integers, and from then on every term is the sum of the two terms just before it. For example, starting with 2 and 5 gives $2, 5, 7, 12, 19, 31, \ldots$. For how many such sequences is the eighth term equal to 2026?

(A) 8 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 51

13. A rectangular garden is enclosed by exactly 30 metres of fencing. A straight gravel path runs from one corner of the garden to the opposite corner, and the path is 12 metres long. The side lengths of the garden turn out not to be whole numbers. What is the area of the garden, in square metres?



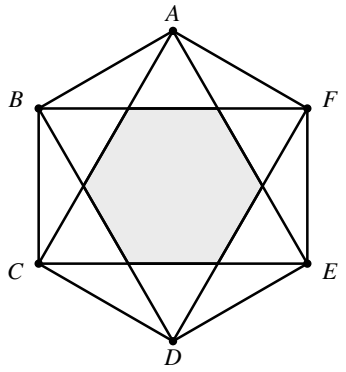
(A) $\frac{81}{2}$ **(B)** $\frac{225}{4}$ **(C)** 72 **(D)** 81 **(E)** $\frac{213}{2}$

14. At exactly 3:00 the hour hand of a clock points at the 3 and the minute hand points at the 12. Both hands

move steadily. How many minutes after 3:00 is the first moment when the two hands point in exactly opposite directions (forming a straight line through the center)?

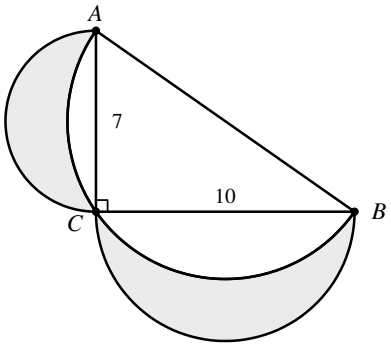
(A) 45 **(B)** 49 **(C)** $49\frac{1}{11}$ **(D)** 50 **(E)** $54\frac{6}{11}$

15. Regular hexagon $ABCDEF$ has area 60. The triangles ACE and BDF are drawn inside it; together they form a six-pointed star, and the region inside both triangles is a smaller regular hexagon. What is the area of that smaller hexagon?



(A) 15 **(B)** 20 **(C)** 24 **(D)** 30 **(E)** 40

16. Right triangle ABC has its right angle at C, with legs $AC = 7$ and $BC = 10$. On each leg a semicircle is drawn outward, away from the triangle, with that leg as its diameter. A third semicircle is drawn with the hypotenuse $\overline{AB}$ as its diameter, on the same side of $\overline{AB}$ as the triangle, so it passes through C. What is the total area of the two crescent-shaped regions that lie inside the two leg semicircles but outside the hypotenuse semicircle?



(A) $\frac{35}{2}$ **(B)** $\frac{149\pi}{16}$ **(C)** 35 **(D)** $\frac{149\pi}{8}$ **(E)** 70

17. Ten people sit around a round table. Each is either a knight, who always tells the truth, or a knave, who always lies. Every one of the ten says: "Both of the people sitting next to me are knaves." What is the smallest possible number of knights at the table?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

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18. Each positive integer n from 1 to 2026 is raised to its own power, and the results are added: $1^1 + 2^2 + 3^3 + \cdots + 2026^{2026}$. What is the units digit of this sum?

(A) 1 **(B)** 3 **(C)** 4 **(D)** 7 **(E)** 9

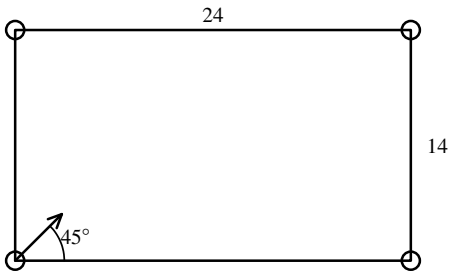
19. A fair six-sided die is rolled three times. Let D be the number of different values that appear among the three rolls (so D is 1, 2 or 3; for example, rolls $4, 2, 4$ give $D = 2$). What is the expected value of D?

(A) $\frac{91}{216}$ **(B)** 2 **(C)** $\frac{7}{3}$ **(D)** $\frac{5}{2}$ **(E)** $\frac{91}{36}$

20. How many ordered pairs (m, n) of positive integers satisfy $\frac{1}{m} + \frac{1}{n} = \frac{1}{12}$? (The pairs $(13, 156)$ and $(156, 13)$ both count.)

(A) 6 **(B)** 7 **(C)** 8 **(D)** 14 **(E)** 15

21. A rectangular billiard table measures 24 by 14, with a pocket at each of its four corners and no other pockets. A ball (treat it as a point) is shot out of one corner pocket at a 45° angle to the sides. Each time it hits a side cushion it bounces off at the same angle, and it keeps rolling until it reaches a corner, where it drops into the pocket. How many times does the ball bounce off a cushion before it drops?



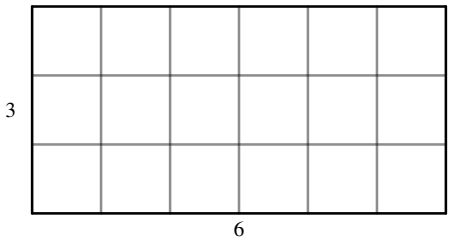
(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

22. A bag holds 4 red, 3 blue and 2 green marbles. Marbles are drawn one at a time, at random and without replacement, until the bag is empty. What is the probability that the red marbles run out first, that is, the last red marble is drawn before the last blue marble and also before the last green marble?

(A) $\frac{13}{63}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5}{21}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{5}{9}$

23. A rectangular floor 3 units wide and 6 units long is divided into 18 unit squares. It is to be covered completely by nine 1-by-2 tiles, each covering two neighbouring unit squares, with no overlaps and nothing hanging off the floor; tiles may be placed either way round. Two coverings are different if some tile position differs (the floor is not turned or flipped). How many different coverings are there?

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(A) 27 **(B)** 33 **(C)** 36 **(D)** 40 **(E)** 41

24. A deck of 14 cards, numbered 1 to 14 from the top, is given a perfect shuffle: it is cut into the top 7 cards and the bottom 7 cards, and the two halves are interleaved one card at a time so that the old top card stays on top and the old bottom card stays on the bottom. (One shuffle turns the order $1, 2, \ldots, 14$ into $1, 8, 2, 9, 3, 10, \ldots, 7, 14$.) The same shuffle is done 100 times in a row. Which card is then fifth from the top?

(A) 2 **(B)** 5 **(C)** 7 **(D)** 11 **(E)** 13

25. Two players take turns removing counters from a single pile. On each turn a player must remove exactly 1, 3 or 4 counters (never more than are left). The player who takes the last counter LOSES. For how many starting sizes n with $1 \le n \le 100$ can the second player (the one who does not move first) guarantee a win?

(A) 14 **(B)** 15 **(C)** 28 **(D)** 29 **(E)** 34

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Answer Key — AMC 8 Practice Contest 2

1. (A) 36

2. (D) 36

3. (D) 72

4. (D) 156

5. (B) 15

6. (E) 14

7. (B) $\frac{7}{48}$

8. (E) 40

9. (A) 4.8

10. (A) 20

11. (B) 24

12. (C) 20

13. (A) $\frac{81}{2}$

14. (C) $49\frac{1}{11}$

15. (B) 20

16. (C) 35

17. (C) 4

18. (B) 3

19. (E) $\frac{91}{36}$

20. (E) 15

21. (C) 17

22. (A) $\frac{13}{63}$

23. (E) 41

24. (D) 11

25. (D) 29

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Solutions — AMC 8 Practice Contest 2

1. **Answer: (A)** 36**. The idea.** Count cuts, not pieces. Each cut across a board turns one piece into two, so a board cut into k pieces needs exactly $k - 1$ cuts.

   Work it out. Four pieces take 3 cuts, so one cut takes $12 \div 3 = 4$ minutes. Ten pieces take 9 cuts, which is $9 \times 4 = 36$ minutes.

   Check. The tempting answer 40 uses the right 4 minutes per cut but then makes 10 cuts, pretending the number of cuts equals the number of pieces; 120 does the same with 12 minutes. Picture 2 pieces: that is one cut, not two.

2. **Answer: (D)** 36**. The idea.** Think of a balance around the target $\frac{1}{2}$. Drink A is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ below it and drink B is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ above it. To balance, the amounts must be in the inverse ratio of those distances: $\text{A} : \text{B} = \frac{1}{4} : \frac{1}{6} = 3 : 2$.

   Work it out. Splitting 60 mL in the ratio $3 : 2$ gives 36 mL of A and 24 mL of B.

   Check. Juice: $\frac{1}{3} \cdot 36 + \frac{3}{4} \cdot 24 = 12 + 18 = 30$ mL, which is half of 60. Algebraically, $\frac{a}{3} + \frac{3(60-a)}{4} = 30$ gives $4a + 540 - 9a = 360$, so $a = 36$. The distractor 24 is the amount of B.

3. **Answer: (D)** 72**. The idea.** Instead of adding up outside area and tunnel area separately (easy to double count), count the exposed faces of each of the 20 unit cubes that remain.

   Work it out. The remaining cubes are the 8 corner cubes and the 12 edge cubes. A corner cube touches three edge cubes, which hide three of its faces, and its other 3 faces are on the outside. An edge cube touches two corner cubes (hiding 2 faces), has 2 faces on the outside, and its other 2 faces used to touch face-center cubes that are now gone, so they are tunnel walls. It shows 4 faces. Total: $8 \cdot 3 + 12 \cdot 4 = 24 + 48 = 72$.

   Check. Outside: each big face lost its center square, leaving $6 \cdot 8 = 48$. Tunnels: each of the 3 tunnels is 3 units long, but its middle unit is the empty center where the tunnels meet and has no walls; each end unit has 4 walls, so a tunnel has $2 \cdot 4 = 8$, and $3 \cdot 8 = 24$. Total $48 + 24 = 72$.

4. **Answer: (D)** 156**. The idea.** Count sevens place by place, then walk forward until the 26th seven would appear. The largest n is the page just before that.

   Work it out. From 1 to 99: the units digit is 7 ten times ($7, 17, \ldots, 97$) and the tens digit is 7 ten times (70 to 79), so 20 sevens (the page 77 is counted once in each list, correctly, since it has two sevens). Pages 100 to 106 add none. Then $107, 117, 127, 137, 147$ bring the total to $21, 22, 23, 24, 25$. The next seven is on page 157, which would make 26. So the count is exactly 25 for every n from 147 to 156, and the largest is 156.

   Check. 147 is the smallest possible n, not the largest (a common slip), and 157 already has 26 sevens.

5. **Answer: (B)** 15**. The idea.** Each individual stands on the scale in three of the four weighings (every group except the one that leaves them out). So adding all four readings counts every weight exactly three times.

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Work it out. $140+165+180+220 = 705$, so all four together weigh $705 \div 3 = 235$ pounds. Each weight is 235 minus the reading of the group that leaves that one out. The lightest individual is left out of the heaviest group, so Rex weighs $235 - 220 = 15$ pounds.

Check. The four weights are $235 - 140 = 95$, $235 - 165 = 70$, $235 - 180 = 55$ and 15. Triples: $95+70+55 = 220$, $95+70+15 = 180$, $95+55+15 = 165$, $70+55+15 = 140$. All four readings match.

6. Answer: (E) 14**. The idea.** Write the number as $10t + u$. Divisibility by t means t divides u (since t already divides $10t$), so u is a multiple of t. Then only a few candidates remain, and we test each for divisibility by u.

Work it out. $t = 1$: every u works for t; test $u \mid 10 + u$, i.e. $u \mid 10$: $u = 1, 2, 5$, giving $11, 12, 15$. $t = 2$: $u \in \{2, 4, 6, 8\}$ and $u \mid 20$: $22, 24$. $t = 3$: $u \in \{3, 6, 9\}$ and $u \mid 30$: $33, 36$ ($36 = 6 \cdot 6$; 39 fails). $t = 4$: $u \in \{4, 8\}$, $u \mid 40$: $44, 48$. $t = 5$: only 55. $t = 6$: 66. $t = 7$: 77. $t = 8$: 88. $t = 9$: 99. In all: $11, 12, 15, 22, 24, 33, 36, 44, 48, 55, 66, 77, 88, 99$, which is 14 numbers.

Check. Note $u \mid 10t + u$ is the same as $u \mid 10t$. For 36: $10 \cdot 3 = 30$ is divisible by 6. For $t \geq 5$ the only multiple of t that is a digit is t itself, leaving just the repdigits.

7. Answer: (B) $\frac{7}{48}$**. The idea.** Condition on the number of heads. The die can only match 1, 2 or 3 heads (it never shows 0), and whatever that number is, the die matches it with probability $\frac{1}{6}$.

Work it out. With three coins, $P(0) = \frac{1}{8}$, $P(1) = \frac{3}{8}$, $P(2) = \frac{3}{8}$, $P(3) = \frac{1}{8}$. The chance of at least one head is $1 - \frac{1}{8} = \frac{7}{8}$, and then the die matches with probability $\frac{1}{6}$. So the answer is $\frac{7}{8} \cdot \frac{1}{6} = \frac{7}{48}$.

Check. Count outcomes: $8 \cdot 6 = 48$ equally likely outcomes. Matches: 3 coin results with one head (die 1), 3 with two heads (die 2), 1 with three heads (die 3): $3 + 3 + 1 = 7$. So $\frac{7}{48}$.

8. Answer: (E) 40**. The idea.** There are two isosceles triangles, ABC and CDB, and they share the angle at B. Name that angle once and express everything else through it.

Work it out. Let $\angle ABC = \angle ACB = \beta$. In triangle CDB, $CD = CB$, so $\angle CDB = \angle CBD = \beta$ and $\angle DCB = 180^\circ - 2\beta$. Since D is on AB, ray CD lies inside $\angle ACB$, so $\angle ACD = \angle ACB - \angle DCB = \beta - (180^\circ - 2\beta) = 3\beta - 180^\circ$. Setting this equal to 30° gives $\beta = 70^\circ$, and $\angle BAC = 180^\circ - 2 \cdot 70^\circ = 40^\circ$.

Check. Triangle ADC: $\angle A = 40^\circ$, $\angle ACD = 30^\circ$, so $\angle ADC = 110^\circ$, and indeed $\angle ADC = 180^\circ - \angle CDB = 180^\circ - 70^\circ$. Everything is consistent.

9. Answer: (A) 4.8**. The idea.** First find the distance from the difference of the two travel times. Then the on-time travel time is exactly halfway between the two times, and the speed is distance over that time. The trap: the right speed is not the average of 4 and 6.

Work it out. The two trips differ by 10 minutes = $\frac{1}{6}$ hour, so $\frac{d}{4} - \frac{d}{6} = \frac{1}{6}$, i.e. $\frac{d}{12} = \frac{1}{6}$ and $d = 2$ km. At 4 km/h the trip takes $\frac{1}{2}$ hour = 30 minutes, which is 5 minutes late, so the on-time trip takes 25 minutes = $\frac{5}{12}$ hour. The speed is $2 \div \frac{5}{12} = \frac{24}{5} = 4.8$ km/h.

Check. At 6 km/h the trip takes 20 minutes, 5 early, consistent. Averaging the speeds gives 5 km/h, a trip of 24 minutes, which is 1 minute early. Averaging the times, not the speeds, is what is right here.

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10. Answer: (A) 20**. The idea.** We cannot find each two-sport group separately, but we do not need to: count "sport memberships" two ways. A student in one sport contributes 1, in exactly two sports 2, in all three 3.

Work it out. 80 students play at least one sport. Let x play exactly one sport and y exactly two. Then $x+y+10 = 80$, and counting memberships, $x+2y+3\cdot 10 = 50+40+30 = 120$. Subtracting the first equation from the second: $y + 20 = 40$, so $y = 20$.

Check. Inclusion-exclusion: $80 = 120 -$ (sum of pairwise overlaps) $+ 10$ gives pairwise overlaps totalling 50. Each all-three student is inside all three overlaps, so the exactly-two count is $50 - 3 \cdot 10 = 20$. The answer 50 forgets to remove the triple-counted students.

11. Answer: (B) 24**. The idea.** Place the most restricted person first. Once Cal takes an end seat, the other four people fill four seats in a row, and we subtract the arrangements where Ava and Ben are together.

Work it out. Cal has 2 choices. The remaining four seats are consecutive, so the other four can sit in $4! = 24$ ways. In how many are Ava and Ben adjacent? Glue them into a block: the block and Dev and Eli make 3 units, arranged in $3! = 6$ ways, and the block can be Ava-Ben or Ben-Ava, so 12. That leaves $24 - 12 = 12$ good seatings for each end, and $2 \cdot 12 = 24$ in all.

Check. Count directly with Cal in seat 1: Ava and Ben must take two of seats $2, 3, 4, 5$ that are not adjacent: $\{2,4\}, \{2,5\}, \{3,5\}$, that is 3 pairs, times 2 orders, times 2 ways for Dev and Eli: 12. Doubling for the other end gives 24.

12. Answer: (C) 20**. The idea.** Call the first two terms a and b and write every later term in terms of them; the coefficients grow like the sequence itself. The eighth term turns out to be $8a + 13b$, so the question becomes: how many pairs of positive integers solve $8a + 13b = 2026$?

Work it out. The terms are $a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ 5a+8b,\ 8a+13b$. For $8a = 2026 - 13b$ to be a multiple of 8, look at remainders on division by 8: 2026 leaves 2 and $13b$ leaves the same remainder as $5b$, so we need $5b$ to leave remainder 2. Trying $b = 0, 1, \ldots, 7$ shows this happens only for $b = 2$ ($5 \cdot 2 = 10$), so b must be $2, 10, 18, \ldots$, that is $b = 2 + 8k$. Also $a \geq 1$ needs $13b \leq 2018$, so $b \leq 155$. The allowed values are $b = 2 + 8k$ for $k = 0, 1, \ldots, 19$ (the last is $b = 154$), which is 20 sequences.

Check. The two extremes: $b = 2$ gives $a = (2026-26)/8 = 250$, and $250, 2, 252, 254, 506, 760, 1266, 2026$ really ends in 2026; $b = 154$ gives $a = (2026 - 2002)/8 = 3$. Each step of 8 in b lowers a by 13. The answer 19 loses one end of the list, 51 is the count for the seventh term $5a + 8b = 2026$, and 8 counts only the sequences with $a \leq b$.

13. Answer: (A) $\frac{81}{2}$**. The idea.** We never need the side lengths (they are messy irrational numbers). The perimeter tells us $a+b$ and the diagonal tells us a^2+b^2; the area ab is hiding in the difference, because $(a+b)^2 = a^2 + 2ab + b^2$.

Work it out. Let the sides be a and b. The fence gives $2(a+b) = 30$, so $a+b = 15$. The path is the diagonal, so by Pythagoras $a^2+b^2 = 12^2 = 144$. Then $2ab = (a+b)^2-(a^2+b^2) = 225-144 = 81$, so the area is $ab = \frac{81}{2} = 40.5$ square metres. (Forgetting to halve gives the trap answer 81.)

Check. Such a rectangle really exists: a and b are the roots of $x^2 - 15x + 40.5 = 0$, namely $\frac{15 \pm \sqrt{63}}{2} \approx 11.47$ and 3.53. Then $a + b = 15$, $ab = 40.5$, and $a^2 + b^2 \approx 131.6 + 12.5 \approx 144$, as required. A square with perimeter 30 would have area 56.25, the largest possible, and our rectangle is long and thin, so a smaller area makes sense.

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14. Answer: (C) $49\frac{1}{11}$**. The idea.** Think of a race. The minute hand moves 6° per minute and the hour hand $\frac{1}{2}^\circ$ per minute, so the minute hand gains $5\frac{1}{2}^\circ$ every minute. We only need to know how many degrees it has to gain.

Work it out. At 3:00 the minute hand is 90° behind the hour hand (measured in the direction the hands turn). To point exactly opposite, it must end up 180° ahead of the hour hand, so it has to gain $90 + 180 = 270°$. That takes $270 \div \frac{11}{2} = \frac{540}{11} = 49\frac{1}{11}$ minutes.

The tempting answer 45 ignores that the hour hand keeps moving: at 3:45 the minute hand is at the 9, but the hour hand has crept past the 3 to $3\frac{3}{4}$, so the hands are not yet opposite.

Check. At $t = \frac{540}{11}$ minutes the minute hand is at $6t = \frac{3240}{11} \approx 294.5°$ and the hour hand at $90 + \frac{t}{2} = 90 + \frac{270}{11} \approx 114.5°$ (both measured clockwise from 12). The difference is exactly $\frac{3240-1260}{11} = \frac{1980}{11} = 180°$. Before this moment the difference was smaller than 180°, so this is the first time.

15. Answer: (B) 20**. The idea.** First find the area of triangle ACE as a fraction of the hexagon, then see how the second triangle chops ACE into small equilateral triangles of one size.

Work it out. Cut the big hexagon into 6 equilateral triangles meeting at the center O. Triangle ACE is made of the three triangles OAC, OCE, OEA, and each of these has the same area as one of the 6 central triangles (for example OAC is half of rhombus $OABC$, which is two central triangles). So $[ACE] = 3 \cdot \frac{60}{6} = 30$, half the hexagon.

The sides of BDF cut each side of ACE into three equal parts (by symmetry the six crossing points are equally spaced). Drawing those lines divides ACE into 9 congruent small equilateral triangles with side one third of AC. The three corner triangles at A, C, E are the star's points that stick out of BDF; the other 6 small triangles form the inner hexagon. So its area is $\frac{6}{9} \cdot 30 = 20$.

Check. The star is triangle ACE plus three more small triangles (the points of BDF), so its area is $30 + 3 \cdot \frac{30}{9} = 40$, and star $= [ACE] + [BDF] - \text{inner} = 30 + 30 - 20 = 40$. Consistent. A coordinate computation of the six crossing points gives the same ratio $\frac{1}{3}$ of the big hexagon.

16. Answer: (C) 35**. The idea.** By Pythagoras the two small semicircles together have exactly the same area as the big one. Add the triangle to the small semicircles and subtract the big semicircle: the overlapping pieces cancel and only the crescents and the triangle's area remain balanced against each other.

Work it out. A semicircle on a segment of length d has area $\frac{\pi}{8}d^2$. The hypotenuse satisfies $AB^2 = 7^2 + 10^2 = 149$, so the leg semicircles have total area $\frac{\pi}{8}(49 + 100) = \frac{149\pi}{8}$, which equals the area of the hypotenuse semicircle $\frac{\pi}{8} \cdot 149$. Now look at the whole figure made of the triangle plus the two leg semicircles. It is the hypotenuse semicircle plus the two crescents (the big semicircle is the triangle together with the two slivers between the legs and the arc, and those slivers are inside the leg semicircles). So

$$\text{triangle} + \frac{149\pi}{8} = \frac{149\pi}{8} + \text{crescents},$$

and the crescents have total area equal to the triangle: $\frac{1}{2} \cdot 7 \cdot 10 = 35$.

Check. Computing one crescent directly: it is the leg semicircle minus the circular segment of the big circle (radius $\frac{\sqrt{149}}{2}$) cut off by that leg. Doing this with the segment formula for both legs, all the π terms and angle terms cancel and the total comes out to exactly 35, with no π in it, which is the surprising part of the problem.

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17. Answer: (C) 4**. The idea.** Turn the two kinds of speaker into two seating rules. A knight tells the truth, so both of a knight's neighbours are knaves: no two knights sit together. A knave lies, so it is false that both of its neighbours are knaves: every knave sits next to at least one knight.

Work it out. Each knight "looks after" itself and its two neighbours, at most 3 seats. Every one of the 10 seats must be looked after (a knight looks after itself, and each knave needs a knight beside it). So $3k \geq 10$ and $k \geq 4$; three knights cover at most 9 seats, which is not enough.

Four knights do work. Number the seats 1 to 10 around the table and put knights in seats $1, 4, 7, 9$. No two knights are adjacent (9 and 1 are separated by seat 10). Each knave has a knight beside it: seat 2 (next to 1), 3 (next to 4), 5 (next to 4), 6 (next to 7), 8 (next to 7 and 9), 10 (next to 9 and 1). So the smallest number is 4.

Check. To see directly why 3 knights fail: with 3 knights, the gaps of knaves between consecutive knights add to 7, so some gap has at least 3 knaves, and the middle knave of that gap has two knave neighbours, making its statement true, which a knave cannot do. A computer search over all 2^{10} seatings confirms that the only possible numbers of knights are 4 and 5.

18. Answer: (B) 3**. The idea.** The units digit of n^n depends only on the units digit of the base (which repeats every 10) and on the exponent modulo 4 (units digits of powers repeat in cycles of length dividing 4). Both patterns line up every 20 values of n, so the units digits of n^n repeat in blocks of 20.

Work it out. Replacing n by $n + 20$ keeps the last digit of the base and adds 20, a multiple of 4, to the exponent, so the units digit is unchanged. For $n = 1, 2, \ldots, 20$ the units digits of n^n are

$1, 4, 7, 6, 5, 6, 3, 6, 9, 0, \quad 1, 6, 3, 6, 5, 6, 7, 4, 9, 0,$

which add to $47 + 47 = 94$. (For instance 7^7: powers of 7 end $7, 9, 3, 1$, and $7 \equiv 3 \pmod 4$, so it ends in 3; 12^{12} ends like 2^{12}, and $12 \equiv 0 \pmod 4$ gives 6.)

Now $2026 = 101 \cdot 20 + 6$. The 101 full blocks contribute a units digit of $101 \cdot 94 = 9494$, i.e. 4. The leftover terms $2021^{2021}, \ldots, 2026^{2026}$ behave like $n = 1, \ldots, 6$: $1 + 4 + 7 + 6 + 5 + 6 = 29$, units digit 9. Total units digit: $4 + 9 = 13$, so 3.

Check. A direct computer sum of n^n mod 10 for all 2026 terms gives units digit 3. Common slips: dropping the leftover six terms gives 4; counting only the leftovers gives 9.

19. Answer: (E) $\frac{91}{36}$**. The idea.** Instead of working out the chance of each value of D, count face by face. D is the number of faces (out of $1, \ldots, 6$) that show up at least once, and the expected value of a count is the sum of the probabilities of its pieces (linearity of expectation).

Work it out. A particular face, say the 5, fails to appear in all three rolls with probability $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$, so it appears at least once with probability $1 - \frac{125}{216} = \frac{91}{216}$. Each of the 6 faces contributes this much, so

$E[D] = 6 \cdot \frac{91}{216} = \frac{91}{36} \approx 2.53.$

Check. Count directly over the 216 outcomes: all three equal in 6 outcomes ($D = 1$); all three different in $6 \cdot 5 \cdot 4 = 120$ outcomes ($D = 3$); the remaining 90 have $D = 2$. Then $E[D] = \frac{6 \cdot 1 + 90 \cdot 2 + 120 \cdot 3}{216} = \frac{546}{216} = \frac{91}{36}$. The trap answer 2 treats the three cases $D = 1, 2, 3$ as equally likely.

20. Answer: (E) 15**. The idea.** Clear the fractions and complete the product: an equation of the form $mn - 12m - 12n = 0$ factors after adding 144 to both sides (Simon's Favourite Factoring

Trick). Then solutions match up with divisors.

Work it out. Multiply by $12mn$: $12n + 12m = mn$, so $mn - 12m - 12n + 144 = 144$, i.e.

$(m - 12)(n - 12) = 144$.

Since $\frac{1}{m} < \frac{1}{12}$, we need $m > 12$, and likewise $n > 12$; so both factors are positive. (Both negative is impossible anyway: they would each be at least -11, and their product at most $121 < 144$.) Every positive divisor d of 144 gives exactly one ordered pair, $m = 12 + d$, $n = 12 + \frac{144}{d}$. Since $144 = 2^4 \cdot 3^2$ has $(4 + 1)(2 + 1) = 15$ divisors, there are 15 ordered pairs.

Check. $d = 1$ gives $(13, 156)$: $\frac{1}{13} + \frac{1}{156} = \frac{12+1}{156} = \frac{1}{12}$. $d = 12$ gives $(24, 24)$, the one symmetric pair. So there are $\frac{15-1}{2} = 7$ unordered pairs with $m \neq n$ plus $(24, 24)$, i.e. 8 unordered pairs, the trap answer for anyone ignoring order. A search over m up to 5000 finds exactly 15 pairs.

21. Answer: (C) 17**. The idea.** Instead of bending the ball's path at each bounce, reflect the table. Tile the plane with mirror copies of the table; the ball then travels in one straight line at 45°, and every bounce becomes a crossing of a grid line between copies.

Work it out. Put the starting corner at $(0, 0)$ with the table $0 \le x \le 24$, $0 \le y \le 14$. In the unfolded picture the ball moves along $y = x$. Corners of copies sit at points where x is a multiple of 24 and y is a multiple of 14, so the ball first reaches a corner at $x = y = \mathrm{lcm}(24, 14) = 168$.

On the way it crosses the vertical lines $x = 24, 48, \ldots, 144$, which is $\frac{168}{24} - 1 = 6$ lines, and the horizontal lines $y = 14, 28, \ldots, 154$, which is $\frac{168}{14} - 1 = 11$ lines. It never crosses a vertical and a horizontal line at the same point before 168 (that would be an earlier corner). Each crossing is one bounce, so there are $6 + 11 = 17$ bounces.

Check. Shrinking the table to 12 by 7 (dividing by the common factor 2) does not change the path's shape, and gives $(7 - 1) + (12 - 1) = 17$ again. On a tiny 2 by 1 table the same count gives $(1 - 1) + (2 - 1) = 1$ bounce, and indeed the ball goes from $(0, 0)$ to $(1, 1)$, bounces off the top, and drops into the corner $(2, 0)$. A step-by-step simulation of the 24 by 14 table also gives 17.

22. Answer: (A) $\frac{13}{63}$**. The idea.** "Red runs out first" is a statement about the END of the sequence: the colour that runs out last is the colour of the very last marble. Read the sequence backwards: the last marble must not be red, and then, among the other two colours, red must again not be the one whose last marble comes later.

Work it out. Red runs out first exactly when the last marble is blue and green runs out after red, or the last marble is green and blue runs out after red.

Case 1: the last marble is blue, with probability $\frac{3}{9}$. Given that, the 4 red and 2 green marbles still appear in a uniformly random relative order, and red runs out before green exactly when the last of those 6 is green: probability $\frac{2}{6}$. This case contributes $\frac{3}{9} \cdot \frac{2}{6} = \frac{1}{9} = \frac{7}{63}$.

Case 2: the last marble is green, probability $\frac{2}{9}$. Then among the 4 red and 3 blue marbles the last must be blue: $\frac{3}{7}$. This contributes $\frac{2}{9} \cdot \frac{3}{7} = \frac{2}{21} = \frac{6}{63}$.

Total: $\frac{7}{63} + \frac{6}{63} = \frac{13}{63}$.

Check. There are $\frac{9!}{4!\,3!\,2!} = 1260$ equally likely colour sequences; counting those where red finishes first gives $260 = \frac{13}{63} \cdot 1260$. Also, the only way red can finish first is if the last marble is not red, probability $\frac{5}{9}$, and our answer is well below that, as it should be.

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23. Answer: (E) 41**. The idea.** Build the tiling from the left end and keep track of two kinds of board: $f(n)$, the number of tilings of a full 3-by-n board, and $g(n)$, the number of tilings of a 3-by-n board whose top-left square has been removed. Each kind reduces to smaller boards of the two kinds.

Work it out. Full board: look at the three squares of the leftmost column. Either all three are covered by horizontal tiles, leaving a full 3-by-$(n-2)$ board, or a vertical tile covers two of them (the top two or the bottom two) and the third is covered by a horizontal tile, which leaves a 3-by-$(n-1)$ board missing a corner square: $g(n-1)$ ways, times 2 by the up-down mirror symmetry. So $f(n) = f(n-2) + 2g(n-1)$.

Board missing its top-left square: the leftmost column has two squares. Either one vertical tile covers both, leaving a full 3-by-$(n-1)$ board, or both are covered by horizontal tiles; then the top square of the second column can only be covered by a horizontal tile (the square below it is taken), which leaves a 3-by-$(n-2)$ board missing its top-left square. So $g(n) = f(n-1) + g(n-2)$.

Start with $f(0) = 1$ (the empty board) and $g(1) = 1$. Then

$f(2) = f(0) + 2g(1) = 3, \quad g(3) = f(2) + g(1) = 4, \quad f(4) = f(2) + 2g(3) = 11,$

$g(5) = f(4) + g(3) = 15, \quad f(6) = f(4) + 2g(5) = 11 + 30 = 41.$

Check. The 3-by-2 board really has 3 tilings (three horizontal tiles, or a vertical pair with a horizontal tile on top or bottom). A tempting shortcut splits the 3-by-6 board into three 3-by-2 blocks for $3^3 = 27$ tilings, but that misses every tiling in which a tile crosses a dividing line. Eliminating g gives $f(n) = 4f(n-2) - f(n-4)$, and $4 \cdot 11 - 3 = 41$ agrees. A computer search over all coverings also finds 41.

24. Answer: (D) 11**. The idea.** Nobody wants to shuffle 100 times, so first find how often the deck comes back to its starting order. Number the positions from the top as $0, 1, \ldots, 13$ instead of 1 to 14: then a shuffle simply doubles each position, if we only care about the remainder when dividing by 13. That shows the deck repeats every 12 shuffles, so 100 shuffles do the same as 4. Then run the doubling backwards to see which card lands in the fifth spot.

Work it out. A card in the top half at position i $(0 \le i \le 6)$ moves to position $2i$. A card in the bottom half at position $7 + j$ $(0 \le j \le 6)$ moves to position $2j + 1$, and $2(7 + j) = 13 + (2j + 1)$. So a card at position $p \le 12$ moves to the remainder of $2p$ on division by 13, and the bottom card (position 13) never moves.

Follow the card at position 1: it visits $2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1$, returning after exactly 12 shuffles. That single cycle passes through all 12 positions 1 to 12, so every card is home after 12 shuffles (and card 1 at position 0 never moves). Since $100 = 8 \cdot 12 + 4$, the 100 shuffles leave the deck exactly as 4 shuffles do.

After 4 shuffles, the card at position p has moved to $16p$, that is $3p$, modulo 13. The fifth card from the top is at position 4, so we need $3p \equiv 4 \pmod{13}$, giving $p = 10$ (as $3 \cdot 10 = 30 = 26 + 4$). Position 10 holds card 11 at the start, so card **11** is fifth from the top.

Check. Follow card 11 forward: positions $10 \to 20 - 13 = 7 \to 14 - 13 = 1 \to 2 \to 4$, fifth from the top after 4 shuffles, and the 96 shuffles before those change nothing. A direct simulation of 100 shuffles ends with the order $1, 10, 6, 2, 11, 7, 3, 12, 8, 4, 13, 9, 5, 14$. Traps: 13 is where card 5 ends up (running the question the wrong way round); 2 comes from assuming the deck repeats every 14 (or 7) shuffles, 7 from assuming every 13, and 5 from assuming 100 shuffles restore the deck.

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25. Answer: (D) 29**. The idea.** Label each pile size as L (the player about to move will lose against best play) or W (the player to move can win). A size is W exactly when some move leads to an L size, and L when every move leads to a W size. Work upward from small piles, spot the repeating pattern, and make sure it really repeats. The second player wins exactly when the starting size is L.

Work it out. Taking the last counter loses, so an empty pile is good for the player to move: the opponent has just lost. Call 0 a W size.

$n = 1$: the only move takes the last counter: L. $n = 2$: take 1, leaving 1 (L): W. $n = 3$: the moves reach 2 (W) and 0 (W): L. $n = 4$: reach 3 (L): W. $n = 5$: reach 1 (L): W. $n = 6$: reach 3 (L): W. $n = 7$: reach 3 (L): W. $n = 8$: reach 7, 5, 4, all W: L. $n = 9$: reach 8: W. $n = 10$: reach 9, 7, 6, all W: L. $n = 11$ to 14: each can reach 10 or 8: W. $n = 15$: reach 14, 12, 11: L.

So the L sizes start $1, 3, 8, 10, 15, 17, \ldots$: exactly the numbers that leave remainder 1 or 3 when divided by 7. Why does this continue forever? From $n = 5$ on, all three moves are allowed, so the label of n depends only on the labels of $n - 1$, $n - 3$ and $n - 4$. The labels of 8, 9, 10, 11 (L, W, L, W) are the same as those of $1, 2, 3, 4$, so from then on every label is a copy of the one 7 earlier.

Count from 1 to 100: remainder 1 gives $1, 8, \ldots, 99$, which is 15 numbers; remainder 3 gives $3, 10, \ldots, 94$, which is 14 numbers. Total 15 + 14 = 29.

Check. The strategy is concrete: from a size with remainder 1 or 3, the second player answers any move so the pile returns to remainder 1 or 3 (if the pile is 1 more than a multiple of 7 and the opponent takes 1, 3 or 4, reply with 4, 4 or 1; if it is 3 more than a multiple of 7 and the opponent takes 1, 3 or 4, reply with 1, 4 or 3; for example $15 \to 14 \to 10$). A computer evaluation of all sizes up to 100 confirms 29 losing starting sizes. (In the normal game where the last counter WINS, the losing sizes are those with remainder 0 or 2 mod 7; the rule change shifts the pattern.)

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