MathCastle

AMC 8

Practice Contest

25 questions • 40 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 40 minutes to finish the test.

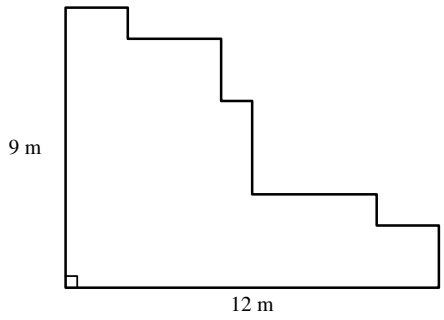
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A store raises the price of a jacket by 25%. A week later Lena buys it with a coupon that takes a certain percentage off the new, raised price. She ends up paying exactly 10% less than the jacket's original price (before the increase). What percentage off did the coupon give?

(A) 15% **(B)** 20% **(C)** 25% **(D)** 28% **(E)** 35%

2. A room's floor plan is bounded by three parts: a straight left wall 9 m tall, a straight floor edge 12 m long meeting the wall at a right angle at the bottom-left corner, and a staircase edge that runs from the top of the wall to the far end of the floor edge. Going from the top of the wall, the staircase edge alternates between a horizontal piece going right and a vertical piece going down, starting with a horizontal piece and ending with a vertical piece that lands exactly at the far end of the floor edge. There are 5 horizontal pieces and 5 vertical pieces, of various lengths that are not given. What is the perimeter of the floor plan, in metres?



(A) 21 **(B)** 42 **(C)** 51 **(D)** 54 **(E)** 60

3. A building has floors numbered 1 through 20. Its lift has only two buttons: "up 7", which moves the lift up exactly 7 floors, and "down 4", which moves it down exactly 4 floors. A button cannot be pressed if it would take the lift below floor 1 or above floor 20, so the lift always stays on floors 1 to 20. Starting on floor 1, what is the fewest number of button presses needed to reach floor 12?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

4. Maya starts Monday with some stickers. On each of Monday, Tuesday and Wednesday the same two things happen, in this order: first Maya gives away exactly half of the stickers she has plus 2 more, and then her friend gives her 3 new stickers. At the end of Wednesday Maya has 9 stickers. How many stickers did she have at the start of Monday?

(A) 30 **(B)** 34 **(C)** 46 **(D)** 52 **(E)** 58

5. A three-digit number is called monotone if its digits either strictly increase from left to right (like 358) or strictly decrease from left to right (like 950). How many monotone three-digit numbers have 5 as one of their digits?

(A) 28 **(B)** 56 **(C)** 64 **(D)** 72 **(E)** 84

6. Two fair six-sided dice, one red and one blue, are rolled. What is the probability that the product of the two numbers rolled is a perfect square? (Count 1 as a perfect square.)

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(A) $\frac{2}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{4}{9}$

7. Call a four-digit number balanced if the sum of its four digits equals the product of its four digits. For example, 2141 is balanced because $2 + 1 + 4 + 1 = 8$ and $2 \cdot 1 \cdot 4 \cdot 1 = 8$. How many balanced four-digit numbers are there?

(A) 12 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 36

8. $ABCDE$ is a regular pentagon, with its vertices labelled in order around it. The diagonals $\overline{AC}$ and $\overline{BD}$ cross at the point P. What is the measure, in degrees, of $\angle APE$, the angle at P between the segments $\overline{PA}$ and $\overline{PE}$?

(A) 36° **(B)** 45° **(C)** 54° **(D)** 72° **(E)** 108°

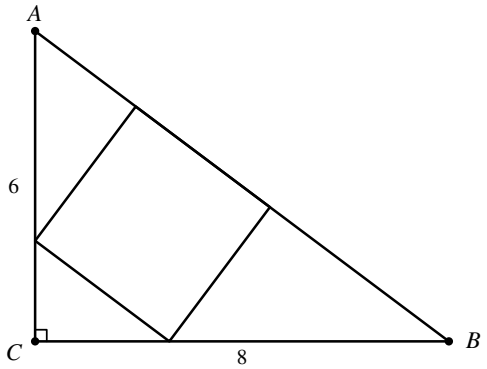
9. An airport moving walkway is 60 m long, and a person standing still on it is carried from one end to the other in 60 seconds. At the same moment, Jo steps onto the starting end of the walkway and Kim steps onto the far end. Jo walks in the direction the walkway moves and Kim walks against it. Both walk at the same steady pace, meaning the same speed measured relative to the walkway's moving surface. They meet 12 seconds later. How many seconds would it take Kim to walk the whole length of the walkway, against its motion, at this pace?

(A) 15 **(B)** 24 **(C)** 40 **(D)** 60 **(E)** 120

10. A 20-litre jug is completely full of pure orange juice. Sam pours out x litres and then fills the jug back up to 20 litres with water, stirring well. He then does exactly the same thing a second time: he pours out x litres of the mixture and tops the jug up with water again. After this the jug contains exactly 11.25 litres of pure juice (mixed into the water). What is x?

(A) 5 **(B)** $\frac{25}{4}$ **(C)** 8 **(D)** $\frac{35}{4}$ **(E)** 15

11. Right triangle ABC has its right angle at C, with legs $AC = 6$ and $BC = 8$. A square is drawn inside the triangle so that one whole side of the square lies along the hypotenuse $\overline{AB}$, one of its other two corners lies on leg $\overline{AC}$, and the last corner lies on leg $\overline{BC}$. What is the side length of the square?



(A) 3 **(B)** $\frac{120}{37}$ **(C)** $\frac{10}{3}$ **(D)** $\frac{24}{7}$ **(E)** $\frac{7}{2}$

12. Five soccer teams play a tournament in which every team plays every other team exactly once. A win earns 3 points and a loss earns 0 points; in a draw, each of the two teams earns 1 point. When the

tournament is over, the five teams' point totals turn out to be five consecutive whole numbers. How many of the games ended in a draw?

(A) 0 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

13. Pat is thinking of a positive integer n. Pat says: "Half of my number is a perfect square, and one third of my number is a perfect cube." (Both $\frac{n}{2}$ and $\frac{n}{3}$ are whole numbers.) What is the smallest number Pat could be thinking of?

(A) 72 **(B)** 162 **(C)** 216 **(D)** 324 **(E)** 648

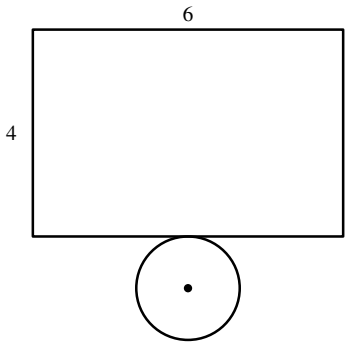
14. After several tests, Kai's average score is exactly 84. If he scores 100 on the next test, his average will rise to exactly 86. If instead he scores 60 on that next test, what will his average be?

(A) 76 **(B)** 78 **(C)** 80 **(D)** 81 **(E)** 82

15. A certain leap year (a year with 366 days, including February 29) begins on a Saturday, January 1. In how many of the twelve months of that year are there five Saturdays?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

16. A coin of radius 1 lies flat on a table next to a 4 by 6 rectangular card. The coin rolls once all the way around the outside of the card, always touching the card's edge, until it returns to its starting position. How far does the center of the coin travel?



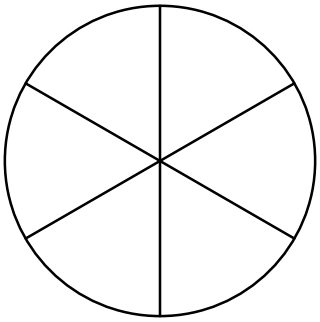
(A) $20+\pi$ **(B)** $20+2\pi$ **(C)** 28 **(D)** $20+4\pi$ **(E)** $28+2\pi$

17. A post office sells only 5-cent stamps and 8-cent stamps, in unlimited supply. An amount of postage is "possible" if it can be made exactly using some number of these stamps (you may use only one kind). Some positive whole-number amounts of cents are not possible. What is the sum of all of the amounts that are not possible?

(A) 108 **(B)** 127 **(C)** 136 **(D)** 151 **(E)** 154

18. A round flower bed is divided into 6 equal wedge-shaped sections around its center, so that each section shares a side with exactly two others. Each section is planted entirely with red, yellow or white flowers, and any two sections that share a side must have different colours. Not every colour has to be used. The sections are fixed in place, so two plantings that differ only by a rotation count as different. How many plantings are possible?

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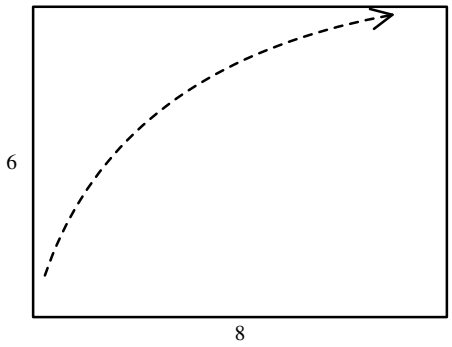


(A) 66 **(B)** 72 **(C)** 80 **(D)** 90 **(E)** 96

19. Two trains leave towns A and B at the same moment and travel toward each other on parallel tracks, each at its own constant speed. They pass each other, and each continues to the other town. The train from A reaches B exactly 4 hours after they pass, and the train from B reaches A exactly 9 hours after they pass. How many hours after starting did the trains pass each other?

(A) 5 **(B)** 6 **(C)** $\frac{13}{2}$ **(D)** 9 **(E)** 13

20. A rectangular sheet of paper measures 8 by 6. It is folded flat along a straight crease so that one corner lands exactly on the opposite corner (the corner diagonally across from it). The crease runs from one edge of the sheet to another. How long is the crease?



(A) 5 **(B)** 6 **(C)** $\frac{25}{4}$ **(D)** 7 **(E)** $\frac{15}{2}$

21. The ten numbers $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}$ are written on a board. A move consists of erasing any two numbers a and b on the board and writing the single number $a + b + ab$ in their place. After nine moves only one number remains. What is it? (It does not depend on the order of the moves.)

(A) $\frac{7381}{2520}$ **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

22. A hallway has 100 light bulbs numbered 1 to 100, all off. For each odd number $k = 1, 3, 5, \ldots, 99$, one student walks down the hallway and toggles (switches on if off, off if on) every bulb whose number is a multiple of k. After all 50 students have passed, how many bulbs are on?

(A) 7 **(B)** 10 **(C)** 14 **(D)** 16 **(E)** 17

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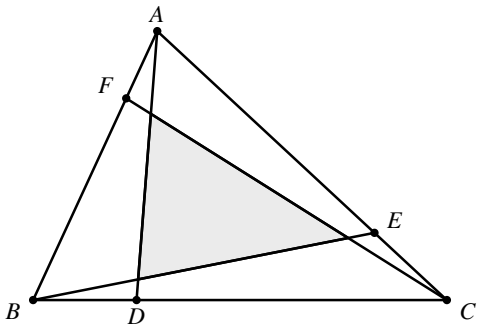
23. Each of the six faces of a cube is painted red, blue or green, so that exactly two faces are red, exactly two are blue and exactly two are green. Two painted cubes are considered the same if one can be turned (rotated in space, not reflected in a mirror) so that it looks exactly like the other. How many different painted cubes are there?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 15

24. Fifty children, numbered 1 to 50, stand in a circle in order, so that child 50 stands next to child 1. Going around the circle in increasing order and starting with child 2, every second child who is still in the circle leaves: first child 2 leaves, then child 4, then child 6, and so on, continuing around the circle again and again (skipping one child, removing the next) until only one child remains. Which child is the last one to leave, that is, the one who leaves when only two children are left?

(A) 1 **(B)** 5 **(C)** 17 **(D)** 21 **(E)** 37

25. Triangle ABC has area 130. Point D lies on side $\overline{BC}$ with $BD : DC = 1 : 3$, point E lies on side $\overline{CA}$ with $CE : EA = 1 : 3$, and point F lies on side $\overline{AB}$ with $AF : FB = 1 : 3$. The three segments $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ cross one another in pairs and enclose a small triangle in the middle of ABC. What is the area of that small triangle?



(A) 10 **(B)** 26 **(C)** 30 **(D)** 40 **(E)** 52

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Answer Key — AMC 8 Practice Contest 1

1. (D) 28%

2. (B) 42

3. (D) 11

4. (E) 58

5. (C) 64

6. (A) $\frac{2}{9}$

7. (A) 12

8. (C) 54°

9. (C) 40

10. (A) 5

11. (B) $\frac{120}{37}$

12. (D) 5

13. (E) 648

14. (D) 81

15. (C) 5

16. (B) $20 + 2\pi$

17. (E) 154

18. (A) 66

19. (B) 6

20. (E) $\frac{15}{2}$

21. (C) 10

22. (E) 17

23. (A) 6

24. (B) 5

25. (D) 40

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Solutions — AMC 8 Practice Contest 1

1. Answer: (D) 28%. **The idea.** Percentage changes multiply; they do not add. The tempting answer 25% + 10% = 35% is wrong because the coupon's percentage is taken from the larger, raised price.

Work it out. Say the original price is \$100. After the 25% increase it costs \$125. Lena pays 10% less than \$100, which is \$90. So the coupon removed \$125 − \$90 = \$35 from a price of \$125, and $\frac{35}{125} = \frac{7}{25} = 28\%$.

Check. In multipliers: $1.25 \times (1 - 0.28) = 1.25 \times 0.72 = 0.90$, which is exactly 10% below the original. With 35% off instead, she would pay $1.25 \times 0.65 = 0.8125$, i.e. about 19% below the original.

2. Answer: (B) 42. **The idea.** The lengths of the individual steps do not matter. The horizontal pieces of the staircase together cover the full width of the room, and the vertical pieces together cover its full height.

Work it out. The staircase starts at the top of the wall, at horizontal position 0, and ends at the far end of the floor, at horizontal position 12. Every horizontal piece goes right and no piece goes left, so the 5 horizontal pieces add up to exactly 12. In the same way, the staircase starts at height 9 and ends at height 0, always moving down, so the 5 vertical pieces add up to exactly 9. The perimeter is therefore wall + floor + treads + drops = 9 + 12 + 12 + 9 = 42.

Check. Try one concrete staircase: treads $2, 3, 1, 4, 2$ (sum 12) and drops $1, 2, 3, 1, 2$ (sum 9). The perimeter is $9 + 12 + (2 + 3 + 1 + 4 + 2) + (1 + 2 + 3 + 1 + 2) = 42$. It is also the perimeter of the 9 by 12 rectangle that encloses the room, $2(9 + 12) = 42$, just as sliding each step outward to the rectangle's corner would suggest.

3. Answer: (D) 11. **The idea.** If the lift goes up a times and down b times, it ends $7a - 4b$ floors higher, whatever the order. Solve $7a - 4b = 11$ in non-negative whole numbers, take the smallest $a + b$, and then check that some order of presses stays inside the building.

Work it out. We need $7a - 4b = 11$. Look at remainders when dividing by 4: $7a$ leaves the same remainder as $3a$, and 11 leaves remainder 3, so $3a$ must leave remainder 3, which happens exactly when a leaves remainder 1: $a = 1, 5, 9, \ldots$ With $a = 1$ we would need $4b = -4$, which is impossible. With $a = 5$ we get $4b = 24$, $b = 6$, so $a + b = 11$. Larger a only makes a and b both bigger. Now an order that stays on floors 1 to 20: $1 \to 8 \to 15 \to 11 \to 7 \to 3 \to 10 \to 6 \to 2 \to 9 \to 5 \to 12$. That is 5 ups and 6 downs, 11 presses.

Check. A tempting wrong idea is "up 7 twice and down 4 once" ($1 \to 8 \to 15 \to 11$), which reaches 11, not 12; the remainder argument shows no route with fewer than 5 ups can ever end exactly on 12. A computer search over all floors confirms 11 presses is the minimum.

4. Answer: (E) 58. **The idea.** Work backwards, undoing each day's steps in reverse order.

Work it out. If Maya has x stickers at the start of a day, she gives away $\frac{x}{2} + 2$, keeping $\frac{x}{2} - 2$, and then receives 3, ending with $\frac{x}{2} + 1$. To undo a day that ends with y stickers: subtract 1, then double, so $x = 2(y - 1)$. End of Wednesday: 9. Start of Wednesday: $2(9 - 1) = 16$. Start of Tuesday: $2(16 - 1) = 30$. Start of Monday: $2(30 - 1) = 58$.

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Check. Forwards: Monday, 58: she gives away $29+2=31$, leaving 27, then gets 3: 30. Tuesday: gives away $15+2=17$, leaving 13, gets 3: 16. Wednesday: gives away $8+2=10$, leaving 6, gets 3: 9. A common slip is to undo the steps in the wrong order (doubling before subtracting), which gives a different, wrong start.

5. Answer: (C) 64**. The idea.** A set of three different digits can be written in increasing order in exactly one way and in decreasing order in exactly one way, so counting monotone numbers means counting digit sets. The only catch is the digit 0, which cannot lead a number.

Work it out. Increasing numbers: the smallest digit comes first, so it cannot be 0, and in fact 0 cannot appear at all. Choose the two digits other than 5 from $\{1,2,\ldots,9\}\setminus\{5\}$: $\binom{8}{2}=28$ ways. Decreasing numbers: 0 can only be the last digit, which is allowed (e.g. 950). Choose the two other digits from $\{0,1,\ldots,9\}\setminus\{5\}$: $\binom{9}{2}=36$ ways. No number is both increasing and decreasing, so the total is $28+36=64$.

Check. Doubling the increasing count gives 56, which forgets the 8 decreasing numbers ending in 0: $5\,a\,0$ or $a\,5\,0$ with the other digit $a\neq 0,5$, i.e. $540, 530, 520, 510$ and $650, 750, 850, 950$. Indeed $56+8=64$.

6. Answer: (A) $\frac{2}{9}$**. The idea.** In a perfect square every prime appears an even number of times. Write each face as a perfect square times a "leftover" part: $1=1$, $2=2$, $3=3$, $4=4\cdot 1$, $5=5$, $6=6$. The product is a square exactly when the two leftover parts are equal.

Work it out. The leftover parts are $1,2,3,1,5,6$ for faces 1 to 6. Leftover 1 belongs to faces 1 and 4, giving $2\times 2=4$ ordered pairs: $(1,1),(1,4),(4,1),(4,4)$. Each other leftover belongs to one face only, giving the doubles $(2,2),(3,3),(5,5),(6,6)$. That is 8 of the 36 equally likely outcomes, so the probability is $\frac{8}{36}=\frac{2}{9}$.

Check. Listing products: the squares $1,4,9,16,25,36$ arise as $1\cdot 1$; $1\cdot 4, 4\cdot 1, 2\cdot 2$; $3\cdot 3$; $4\cdot 4$; $5\cdot 5$; $6\cdot 6$ (note $2\cdot 8$, $3\cdot 12$ and so on are impossible). That is $1+3+1+1+1+1=8$. Counting unordered pairs instead gives $\frac{7}{21}=\frac{1}{3}$, but the 21 unordered pairs are not equally likely: a double like $(2,2)$ can happen in only one way, while $\{1,4\}$ can happen in two.

7. Answer: (A) 12**. The idea.** Products grow much faster than sums, so most of the digits must be 1. Split on how many 1s there are.

Work it out. No digit is 0 (the product would be 0 but the sum is positive). Let the digits other than 1 be the "big" digits (each at least 2). Four 1s: sum 4, product 1. No. Three 1s and big digit x: sum $3+x$, product x. Never equal. Two 1s and big digits $x\le y$: we need $xy=x+y+2$, i.e. $(x-1)(y-1)=3$, so $x-1=1$, $y-1=3$: digits $\{1,1,2,4\}$. One 1 and big digits $x\le y\le z$: we need $xyz=1+x+y+z$. The product is at least $4z$ (since $xy\ge 4$) while the sum is at most $1+3z$, and $4z>1+3z$ since $z\ge 2$. No solutions. No 1s: the product is at least $8w$ for the largest digit w, while the sum is at most $4w$. No solutions. So the digits must be $1,1,2,4$, which can be arranged in $\frac{4!}{2!}=12$ ways, all valid four-digit numbers.

Check. 1124: sum 8, product 8. 1133: sum 8, product 9, so near misses really do fail. A computer check of all four-digit numbers finds exactly 12.

8. Answer: (C) 54°**. The idea.** The pentagon's mirror line through E swaps A with D and B with C, so it swaps diagonal $\overline{AC}$ with diagonal $\overline{DB}$. Their crossing point P must therefore lie on that mirror line, so $\overline{EP}$ cuts $\angle AED$ exactly in half.

Work it out. Each interior angle of a regular pentagon is $\frac{540^\circ}{5} = 108^\circ$. Triangle ABC is isosceles with $AB = BC$ and apex angle 108°, so $\angle BAC = \frac{180^\circ - 108^\circ}{2} = 36^\circ$. Hence $\angle EAP = \angle EAB - \angle BAC = 108^\circ - 36^\circ = 72^\circ$. By the symmetry above, $\angle AEP = \frac{1}{2}\angle AED = 54^\circ$. The angles of triangle AEP add to 180°, so $\angle APE = 180^\circ - 72^\circ - 54^\circ = 54^\circ$.

Check. Triangle AEP has two 54° angles, so it is isosceles with $AP = AE$: the piece of a diagonal from a vertex to the crossing point equals a side, a well-known property of the regular pentagon. A coordinate computation with the vertices on a circle also gives exactly 54°. (The angle $\angle APB$ at the same crossing is 72°, a tempting but different answer.)

9. Answer: (C) 40**. The idea.** The walkway pushes Jo forward and pushes Kim back by exactly the same amount, so it has no effect on how fast the gap between them closes. The closing speed is just twice the walking pace.

Work it out. The walkway moves 60 m in 60 s, so its speed is 1 m/s. Let the walking pace be p m/s. Jo's speed over the ground is $p + 1$ and Kim's is $p - 1$ (towards Jo), so the gap of 60 m closes at $(p + 1) + (p - 1) = 2p$ m/s. They meet after 12 s, so $2p = \frac{60}{12} = 5$ and $p = 2.5$. Walking against the walkway, Kim moves at $2.5 - 1 = 1.5$ m/s over the ground, so the whole 60 m takes $\frac{60}{1.5} = 40$ seconds.

Check. In the 12 seconds Jo covers $3.5 \times 12 = 42$ m and Kim covers $1.5 \times 12 = 18$ m, and $42 + 18 = 60$. They do not meet in the middle: assuming they do gives Jo 30 m in 12 s, a ground speed of 2.5, a pace of 1.5, and the wrong answer $\frac{60}{1.5-1} = 120$. Forgetting the walkway at the end gives $\frac{60}{2.5} = 24$, and forgetting to halve 5 gives $\frac{60}{5-1} = 15$.

10. Answer: (A) 5**. The idea.** Each pour removes the same fraction of whatever juice is in the jug, so the amount of juice is multiplied by the same factor each round. It is not a fixed number of litres of juice each time.

Work it out. Pouring out x of the 20 litres removes the fraction $\frac{x}{20}$ of the juice, leaving the fraction $1 - \frac{x}{20}$. After two rounds, the juice is $20\left(1 - \frac{x}{20}\right)^2 = 11.25$. So $\left(1 - \frac{x}{20}\right)^2 = \frac{11.25}{20} = \frac{9}{16}$, which gives $1 - \frac{x}{20} = \frac{3}{4}$ and $x = 5$.

Check. Round 1: pour out 5 L of pure juice, leaving 15 L of juice. Round 2: the jug is $\frac{15}{20} = \frac{3}{4}$ juice, so the 5 L poured out contain 3.75 L of juice, leaving $15 - 3.75 = 11.25$ L. The tempting answer $\frac{35}{4} = 8.75$ comes from setting $1 - \frac{x}{20} = \frac{9}{16}$, which forgets that the fraction $1 - \frac{x}{20}$ is applied twice.

11. Answer: (B) $\frac{120}{37}$**. The idea.** The side of the square opposite the hypotenuse cuts off a small triangle at C that is similar to ABC. Compare lengths measured along the altitude from C.

Work it out. The hypotenuse is $AB = \sqrt{6^2 + 8^2} = 10$, and the altitude from C to $\overline{AB}$ is $h = \frac{6 \cdot 8}{10} = \frac{24}{5}$ (area computed two ways). Let the square have side s. Its top side $\overline{PQ}$ (with P on $\overline{AC}$ and Q on $\overline{BC}$) is parallel to $\overline{AB}$, so triangle PQC is similar to triangle ABC. The square takes up height s above $\overline{AB}$, so triangle PQC has height $h - s$ from C, and by similarity $\frac{PQ}{AB} = \frac{h-s}{h}$. Since $PQ = s$: $\frac{s}{10} = \frac{24/5 - s}{24/5}$, so $\frac{24s}{5} = 10\left(\frac{24}{5} - s\right) = 48 - 10s$, giving $24s = 240 - 50s$, $74s = 240$ and $s = \frac{120}{37}$.

Check. In general $s = \frac{ch}{c+h} = \frac{10 \cdot 4.8}{14.8} = \frac{48}{14.8} = \frac{120}{37} \approx 3.24$. This is a little smaller than the square that sits in the right-angle corner, whose side is $\frac{6 \cdot 8}{6+8} = \frac{24}{7} \approx 3.43$, a tempting wrong answer.

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12. Answer: (D) 5**. The idea.** Count the points from the games' side instead of the teams' side. A decided game hands out 3 points in all and a drawn game hands out 2, so the grand total of points tells you how many draws there were. Then compare that total with the sum of five consecutive numbers.

Work it out. There are $\binom{5}{2} = 10$ games. With D draws the total is $3(10-D)+2D = 30-D$. Five consecutive numbers $m, m+1, \ldots, m+4$ add to $5m+10$, a multiple of 5, so D is 0, 5 or 10. If $D = 0$, every team's total is a multiple of 3, and two consecutive numbers can't both be multiples of 3. If $D = 10$, every team drew all 4 of its games and has 4 points, so the totals are all equal. That leaves $D = 5$, with totals $3, 4, 5, 6, 7$ (which add to $25 = 30 - 5$).

Check. It really can happen. Call the teams A, B, C, D, E: B beats A; A beats C and D; C beats D; D beats E; the five games between A and E, B and C, B and D, B and E, and C and E are draws. The totals are $A = 7$, $B = 6$, $C = 5$, $D = 4$, $E = 3$. The answer 0 comes from noticing that $30 = 4 + 5 + 6 + 7 + 8$ without checking that the totals could not all be multiples of 3.

13. Answer: (E) 648**. The idea.** Only the primes 2 and 3 are forced into n, and any other prime would only make n bigger, so write $n = 2^a 3^b$. Then each condition becomes a condition on the exponents: a square has all exponents even, a cube has all exponents multiples of 3.

Work it out. $\frac{n}{2} = 2^{a-1}3^b$ is a square, so $a-1$ is even (so a is odd) and b is even. $\frac{n}{3} = 2^a 3^{b-1}$ is a cube, so a is a multiple of 3 and $b-1$ is a multiple of 3. The smallest odd multiple of 3 is $a = 3$. The smallest even b with $b \equiv 1 \pmod 3$ is $b = 4$ ($b = 1$ is odd, $b = 4$ works). So $n = 2^3 \cdot 3^4 = 8 \cdot 81 = 648$.

Check. $648/2 = 324 = 18^2$ and $648/3 = 216 = 6^3$. The tempting $72 = 2^3 3^2$ gives $72/2 = 36$, a square, but $72/3 = 24$ is not a cube.

14. Answer: (D) 81**. The idea.** Think of every test as sitting at the old average 84. A new score only moves the average by its excess over 84, shared among all the tests. That tells us how many tests there are, and then the 60 is easy.

Work it out. Let Kai have taken n tests. A score of 100 is 16 above the average, and spread over $n+1$ tests it raises the average by $\frac{16}{n+1}$. This equals 2, so $n+1 = 8$. A score of 60 is 24 below the average, which lowers the average by $\frac{24}{8} = 3$, giving $84 - 3 = 81$.

Check. With $n = 7$ tests the total is $7 \cdot 84 = 588$. Then $\frac{588+100}{8} = \frac{688}{8} = 86$, as stated, and $\frac{588+60}{8} = \frac{648}{8} = 81$.

15. Answer: (C) 5**. The idea.** Count Saturdays for the whole year first. Every month has at least 28 days, so it has either 4 or 5 Saturdays; the extra Saturdays beyond 4 per month are exactly the five-Saturday months.

Work it out. $366 = 52 \cdot 7 + 2$, so the year is 52 full weeks plus two extra days. Those two extra days are the first two weekdays of the year, Saturday and Sunday, so the year has 53 Saturdays. Twelve months with 4 Saturdays each account for 48 of them, so $53 - 48 = 5$ months must have a fifth Saturday.

Check. Walk the calendar. Months start on: Jan Sat, Feb Tue, Mar Wed, Apr Sat, May Mon, Jun Thu, Jul Sat, Aug Tue, Sep Fri, Oct Sun, Nov Wed, Dec Fri. A 31-day month has five Saturdays when it starts on Thu, Fri or Sat; a 30-day month when it starts on Fri or Sat; a 29-day February only when it starts on Sat. That picks out January, April, July, September and December: 5 months.

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16. Answer: (B) $20 + 2\pi$. **The idea.** Split the trip into two kinds of motion: while the coin rolls along a side, its center moves parallel to that side; while it swings around a corner, the coin pivots on the corner and its center moves along a circular arc of radius 1 centered at that corner.

Work it out. Along the sides the center travels exactly the side lengths: $6 + 4 + 6 + 4 = 20$. At each corner the direction of travel turns by 90°, so the center traces a quarter circle of radius 1, of length $\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$. Four corners give $4 \cdot \frac{\pi}{2} = 2\pi$. The total is $20 + 2\pi$.

Check. The four quarter circles fit together into one whole circle of radius 1, because going once around any convex shape turns you through a full 360° in total. The wrong answer 28 comes from pretending the center goes around a 6 by 8 rectangle with sharp corners; the true path cuts those corners, and $2\pi < 8$.

17. Answer: (E) 154. **The idea.** Sort amounts by their remainder when divided by 5. Once some amount with a given remainder is possible, adding 5-cent stamps makes every larger amount with that remainder possible. So for each remainder we only need the smallest possible amount, and that comes from using as few 8-cent stamps as we can. Everything below that starting point in the same remainder class is impossible, and those amounts form a short list we can add up.

Work it out. Multiples of 8: $0, 8, 16, 24, 32$ have remainders $0, 3, 1, 4, 2$ when divided by 5, one of each. So remainder 0 starts at 0 (every multiple of 5 is possible), remainder 3 starts at 8, remainder 1 at 16, remainder 4 at 24 and remainder 2 at 32. An amount with a given remainder is possible exactly when it is at least the start of its class. The impossible amounts are therefore: remainder 3: 3, sum 3; remainder 1: 1, 6, 11, sum 18; remainder 4: 4, 9, 14, 19, sum 46; remainder 2: 2, 7, 12, 17, 22, 27, sum 87. The total is $3 + 18 + 46 + 87 = 154$.

Check. There are $1+3+4+6 = 14$ impossible amounts, matching the known count $\frac{(5-1)(8-1)}{2} = 14$, and the largest is $27 = 5 \cdot 8 - 5 - 8$; from 28 to 32 five consecutive amounts are possible ($28 = 8+20$, $29 = 24+5$, 30, $31 = 16+15$, 32), so nothing past 27 is missed. Forgetting the largest amount 27 gives 127; forgetting the lone amount 3 gives 151.

18. Answer: (A) 66. **The idea.** Go around the ring wedge by wedge. Each new wedge has 2 choices, except that the last wedge must also differ from the first, and whether that leaves it 1 or 2 choices depends on the wedge before it. So keep track of whether the current wedge matches wedge 1.

Work it out. Colour wedge 1 in any of 3 ways and then colour wedges $2, 3, \ldots$ in turn, each differing from its predecessor. Let s_k be the number of ways to colour wedges 2 through k so that wedge k has the same colour as wedge 1, and d_k the number where it differs. Wedge 2 must differ from wedge 1 and may be either of the other 2 colours, so $s_2 = 0$ and $d_2 = 2$. From a "same" wedge, the next wedge has 2 choices, both different; from a "different" wedge, the next has 1 choice equal to wedge 1 and 1 choice different. So $s_{k+1} = d_k$ and $d_{k+1} = 2s_k + d_k$. Then (s, d) runs $(0, 2), (2, 2), (2, 6), (6, 10), (10, 22)$ for $k = 2, 3, 4, 5, 6$. Wedge 6 must differ from wedge 1, so we want $d_6 = 22$, and with 3 choices for wedge 1 the total is $3 \cdot 22 = 66$.

Check. The known ring formula for n regions and 3 colours is $2^n + 2(-1)^n$, which for $n = 6$ is $64 + 2 = 66$. The tempting $3 \cdot 2^5 = 96$ forgets that wedge 6 also touches wedge 1.

19. Answer: (B) 6. **The idea.** The meeting point splits the track into two pieces, and each piece is covered by both trains: once before the meeting and once after. Comparing the two times for the same piece gives the ratio of the speeds in two different ways.

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Work it out. Let them meet t hours after starting, and let the speeds be u (from A) and v (from B). The piece between the meeting point and B was covered by the B-train in t hours and by the A-train in 4 hours, so $vt = 4u$, that is $\frac{u}{v} = \frac{t}{4}$. The piece between A and the meeting point was covered by the A-train in t hours and by the B-train in 9 hours, so $ut = 9v$, that is $\frac{u}{v} = \frac{9}{t}$. Hence $\frac{t}{4} = \frac{9}{t}$, $t^2 = 36$, and $t = 6$.

Check. Then $\frac{u}{v} = \frac{3}{2}$. Take $u = 3$ and $v = 2$: in 6 hours they cover $18 + 12 = 30$, so the towns are 30 apart. Afterwards the A-train has 12 left at speed 3: 4 hours; the B-train has 18 left at speed 2: 9 hours. The average 6.5 of 4 and 9 is the tempting wrong answer; the right one is their geometric mean.

20. Answer: (E) $\frac{15}{2}$**. The idea.** When a fold sends point P onto point Q, every point on the crease is equally far from P and Q, so the crease is the perpendicular bisector of $\overline{PQ}$. Here $\overline{PQ}$ is the diagonal, so the crease crosses the diagonal at the center at right angles, and similar triangles finish it off.

Work it out. Put the rectangle at $(0,0)$, $(8,0)$, $(8,6)$, $(0,6)$ and fold $(0,0)$ onto $(8,6)$. The diagonal has length $\sqrt{8^2+6^2} = 10$ and its midpoint is $M = (4,3)$. The crease through M is perpendicular to the diagonal (slope $\frac{3}{4}$), so it has slope $-\frac{4}{3}$: $y - 3 = -\frac{4}{3}(x-4)$. It meets the bottom edge $y = 0$ at $x = \frac{25}{4}$ and the top edge $y = 6$ at $x = \frac{7}{4}$. Those points differ by $\frac{18}{4} = \frac{9}{2}$ horizontally and 6 vertically, so the crease has length $\sqrt{\frac{81}{4} + 36} = \sqrt{\frac{225}{4}} = \frac{15}{2}$.

Check. By similar triangles: half the crease is to half the diagonal as the short side is to the long side, because the small right triangle at M is similar to the triangle with legs 8 and 6. So half the crease is $5 \cdot \frac{6}{8} = \frac{15}{4}$ and the crease is $\frac{15}{2}$. Also $(\frac{25}{4}, 0)$ is $\frac{25}{4}$ from $(0,0)$ and $\sqrt{(\frac{7}{4})^2 + 36} = \frac{25}{4}$ from $(8,6)$, as it should be.

21. Answer: (C) 10**. The idea.** Look for something that a move does not change. The trick is to add 1: $a + b + ab + 1 = (1+a)(1+b)$. So if we replace every number x on the board by $1 + x$, a move simply multiplies two of these together, and the product of all the $(1 + x)$ values never changes.

Work it out. At the start that product is $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\cdots\left(1 + \frac{1}{10}\right) = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{11}{10} = 11$, since everything cancels except $\frac{11}{1}$. At the end there is one number z, and the product is just $1 + z$. So $1 + z = 11$ and $z = 10$.

Check. Try a small case by hand: from 1 and $\frac{1}{2}$ we get $1 + \frac{1}{2} + \frac{1}{2} = 2$, and the rule predicts $\frac{2}{1} \cdot \frac{3}{2} - 1 = 2$. With $1, \frac{1}{2}, \frac{1}{3}$ the rule predicts $4 - 1 = 3$, and indeed 2 with $\frac{1}{3}$ gives $2 + \frac{1}{3} + \frac{2}{3} = 3$. The tempting 11 forgets to subtract the 1 back at the end.

22. Answer: (E) 17**. The idea.** Bulb n is toggled once for every odd divisor of n (every odd divisor of a number up to 100 is at most 99, so all of them get a student). It ends up on exactly when n has an odd number of odd divisors. The odd divisors of n are the divisors of its odd part m (what is left after removing all factors of 2), and a number has an odd number of divisors exactly when it is a perfect square, because divisors pair up as d and $\frac{m}{d}$ except when $d = \sqrt{m}$.

Work it out. So bulb n is on exactly when $n = 2^a \cdot s^2$ with s odd. Count these up to 100: $s = 1$: $1, 2, 4, 8, 16, 32, 64$ (seven); $s = 3$: $9, 18, 36, 72$ (four); $s = 5$: $25, 50, 100$ (three); $s = 7$: $49, 98$ (two); $s = 9$: 81 (one). Total $7 + 4 + 3 + 2 + 1 = 17$.

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Check. A number of the form $2^a s^2$ is a perfect square when a is even and twice a perfect square when a is odd. There are 10 squares up to 100 and 7 numbers of the form $2t^2$ (2, 8, 18, 32, 50, 72, 98), and $10 + 7 = 17$. The classic puzzle with all students would give only the 10 squares.

23. Answer: (A) 6**. The idea.** For each colour, its two faces are either opposite each other or adjacent (sharing an edge). Split into cases by how many colours sit on opposite pairs. It cannot be exactly two: if two colours each fill a pair of opposite faces, the last two faces are also opposite. So there are three cases: all three colours opposite, exactly one, or none.

Work it out. *All three opposite.* Red on top and bottom, and blue and green on the two remaining opposite pairs around the side. Any two such cubes match after a turn: 1 cube.

Exactly one opposite. Choose that colour in 3 ways and turn it to the top and bottom. The four side faces form a ring holding two of each other colour, with neither colour on opposite sides, so each colour fills two neighbouring side faces. Turning about the vertical axis makes any two such rings match: 3 cubes.

None opposite. Now each colour fills two faces that share an edge. Turn the cube so that red is on the top and front. The four faces left are bottom, back, left and right. Left and right are opposite, so they cannot have the same colour: blue takes one of them and green the other, and each also takes one of bottom and back. That gives 4 paintings: blue on left and bottom (green on right and back), blue on left and back, blue on right and bottom, or blue on right and back. The half turn about the line through the middles of the top-front edge and the bottom-back edge keeps red where it is but swaps top with front, left with right and bottom with back; it matches the first painting with the fourth and the second with the third. So there are 2 cubes here, and they are mirror images of each other that no rotation can match.

In total $1 + 3 + 2 = 6$.

Check. Burnside's lemma confirms it. There are $\frac{6!}{2!\,2!\,2!} = 90$ paintings of a fixed cube. The 24 rotations fix: identity 90; the 6 quarter turns about face axes 0; the 3 half turns about face axes 6 each (each colour fills a pair of opposite side faces, or one of the top/bottom); the 8 turns about corner axes 0; the 6 half turns about edge axes 6 each. So the count is $\frac{90+18+36}{24} = \frac{144}{24} = 6$. Stopping at $1 + 3 + 4 = 8$ forgets that the half turn matches the four "none opposite" paintings in pairs.

24. Answer: (B) 5**. The idea.** Circles whose size is a power of 2 are easy to follow. If 2^k children stand in a circle and the counting begins by skipping the child in position 1, then one lap removes exactly the children in the even positions $2, 4, \ldots, 2^k$; the last removal of the lap is the last child in the circle, so the next lap again begins by skipping position 1. So after each lap the survivors are the positions that leave remainder 1 when divided by 2, then by 4, then by 8, and so on. So first remove enough children to get down to a power of 2, then relabel.

Work it out. The largest power of 2 below 50 is 32, so first $50 - 32 = 18$ children leave: $2, 4, 6, \ldots, 36$. Now 32 children remain and the next one in line, child 37, is about to be skipped. Number the remaining children by position, starting from child 37: positions 1 to 14 are children $37, 38, \ldots, 50$, and positions $15, 16, 17, \ldots, 32$ are the odd children $1, 3, 5, \ldots, 35$. In this circle of 32, the laps leave the positions of remainder 1 mod 2, then mod 4, then mod 8, then mod 16: that is, positions 1 and 17. Position 1 is skipped and position 17 leaves last. Position 17 is child 5.

Check. The survivor is position 1, child 37, which matches the well-known answer $2(50-32)+1 =$

37; the question asks for the other one of the final pair. A small case: with 10 children, 2 and 4 leave first, then the circle $5, 6, 7, 8, 9, 10, 1, 3$ of size 8 ends with positions 1 and 5, so child 9 leaves last and child 5 survives. Doing it by hand: $2, 4, 6, 8, 10$ leave, then $3, 7, 1$, then 9, leaving 5. Answering 37 gives the survivor instead of the last to leave, and 17 confuses the position with the child's number.

25. Answer: (D) 40**. The idea.** By symmetry of the setup (each cevian cuts its side in the same ratio, going around the triangle the same way), it is enough to find how one cevian is cut by the other two. Then the middle triangle is what is left of ABC after removing three equal-area outer triangles. Area ratios along a common line do the work; since ratios of areas do not change when the triangle is stretched, we can use a convenient triangle.

Work it out. Use coordinates in which area ratios are easy: $B = (0, 0)$, $C = (4, 0)$, $A = (0, 4)$, of area 8 (we rescale at the end). Then $D = (1, 0)$, $E = C + \frac{1}{4}(A - C) = (3, 1)$ and $F = A + \frac{1}{4}(B - A) = (0, 3)$.

Line AD: $x + \frac{y}{4} = 1$, i.e. $4x + y = 4$. Line BE: $x = 3y$. Line CF: $\frac{x}{4} + \frac{y}{3} = 1$, i.e. $3x + 4y = 12$.

$AD \cap BE$: $12y + y = 4$, so $P = \left(\frac{12}{13}, \frac{4}{13}\right)$. $BE \cap CF$: $9y + 4y = 12$, so $Q = \left(\frac{36}{13}, \frac{12}{13}\right)$. $CF \cap AD$: from $y = 4 - 4x$, $3x + 16 - 16x = 12$, so $x = \frac{4}{13}$ and $R = \left(\frac{4}{13}, \frac{36}{13}\right)$.

Shoelace for PQR (all coordinates over 13, so the area is $\frac{1}{169}$ of the area with numerators $(12, 4)$, $(36, 12)$, $(4, 36)$): twice that area is $|12{\cdot}12 - 36{\cdot}4 + 36{\cdot}36 - 4{\cdot}12 + 4{\cdot}4 - 12{\cdot}36| = |144 - 144 + 1296 - 48 + 16 - 432| = 832$, so the area is $\frac{416}{169} = \frac{32}{13}$. Compared with the area 8 of this triangle, that is $\frac{32/13}{8} = \frac{4}{13}$. So in the real triangle the middle area is $\frac{4}{13} \cdot 130 = 40$.

Check. Along AD, the points are $A = (0, 4)$, R at $y = \frac{36}{13}$, P at $y = \frac{4}{13}$, D at $y = 0$, so $AR : RP : PD = 16 : 32 : 4 = 4 : 8 : 1$. Then triangle ABP has area $\frac{12}{13}$ of triangle ABD, which is $\frac{1}{4}$ of ABC, so $[ABP] = \frac{3}{13}[ABC]$; likewise for BCQ and CAR. The middle is $1 - 3 \cdot \frac{3}{13} = \frac{4}{13}$ of ABC, again 40. (Routh's theorem with ratio 3 gives $\frac{(3-1)^2}{3^2+3+1} = \frac{4}{13}$ as well.)

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