MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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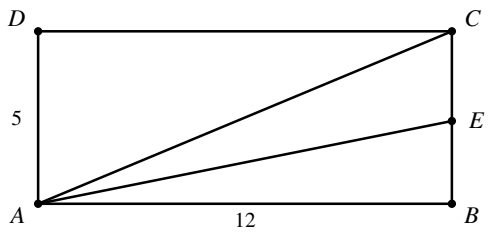
1. Real numbers a, b and c, all greater than 1, satisfy $\log_a b = \frac{3}{2}$ and $\log_b c = 4$. What is the value of $\log_{ab}(ac)$?

(A) $\frac{12}{5}$ **(B)** $\frac{14}{5}$ **(C)** $\frac{7}{2}$ **(D)** $\frac{11}{2}$ **(E)** 6

2. For how many integers n with $1 \le n \le 200$ do n^2 and n^3 have the same units digit?

(A) 40 **(B)** 60 **(C)** 80 **(D)** 100 **(E)** 120

3. Rectangle $ABCD$ has $AB = 12$ and $BC = 5$. The bisector of $\angle CAB$ (the angle between side $\overline{AB}$ and diagonal $\overline{AC}$) meets side $\overline{BC}$ at E. What is the length AE?



(A) $2\sqrt{37}$ **(B)** $\frac{12\sqrt{26}}{5}$ **(C)** $\frac{\sqrt{601}}{2}$ **(D)** $4\sqrt{10}$ **(E)** 13

4. An angle x with $0 < x < \pi$ satisfies $\sin x - \cos x = \frac{1}{5}$. What is $\tan \frac{x}{2}$?

(A) $\frac{1}{2}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{3}$ **(D)** 2 **(E)** 3

5. A flag consists of six vertical stripes in a row. Each stripe is coloured red, white or blue, and any two neighbouring stripes must have different colours. One of all the possible flags of this kind is chosen at random, each equally likely. What is the probability that the two end stripes (the first and the sixth) have the same colour?

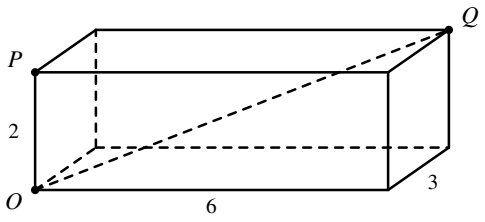
(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{11}{32}$ **(E)** $\frac{3}{8}$

6. Positive real numbers x, y and z satisfy the three equations
$x^2 + xy + xz + yz = 30$, $y^2 + xy + yz + xz = 24$, $z^2 + xz + yz + xy = 20$.
What is the value of xyz?

(A) $\frac{105}{8}$ **(B)** 15 **(C)** $\frac{105}{4}$ **(D)** $\frac{105}{2}$ **(E)** 120

7. A rectangular box has edges of lengths 2, 3 and 6. Let O be one vertex of the box, let Q be the vertex farthest from O, and let P be the vertex joined to O by an edge of length 2. What is the distance from P to the line OQ?

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(A) $\frac{3\sqrt{5}}{7}$ **(B)** $\frac{2\sqrt{5}}{3}$ **(C)** $\frac{12}{7}$ **(D)** $\frac{6\sqrt{5}}{7}$ **(E)** 2

8. A fair coin is flipped 10 times, and the results are written in a row. A block is a maximal run of consecutive heads, one that cannot be extended because it is bounded on each side by a tail or by an end of the row. What is the expected number of blocks that consist of exactly two heads?

(A) $\frac{7}{16}$ **(B)** $\frac{9}{16}$ **(C)** $\frac{5}{8}$ **(D)** $\frac{11}{16}$ **(E)** $\frac{3}{4}$

9. Each of the eight vertices of a regular octagon is coloured red or blue. The octagon is fixed in place, so two colourings that differ by a rotation or a reflection count as different. In how many colourings are there no three consecutive vertices (three vertices next to each other around the octagon) that all have the same colour?

(A) 23 **(B)** 40 **(C)** 42 **(D)** 44 **(E)** 46

10. For how many integers n with $1 \le n \le 1000$ are the sum of the digits of n and the sum of the digits of $n+1$ both odd?

(A) 45 **(B)** 46 **(C)** 50 **(D)** 90 **(E)** 100

11. The equation $(z-1)^4 = 16z^4$ has four different solutions in the complex numbers, and they are the vertices of a convex quadrilateral in the complex plane. What is the area of this quadrilateral?

(A) $\frac{4}{15}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{8}{15}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{16}{15}$

12. In triangle ABC, let M be the midpoint of side $\overline{BC}$. The median $\overline{AM}$ is perpendicular to side $\overline{AB}$. Over all such triangles, what is the largest possible value of $\tan \angle ACB$?

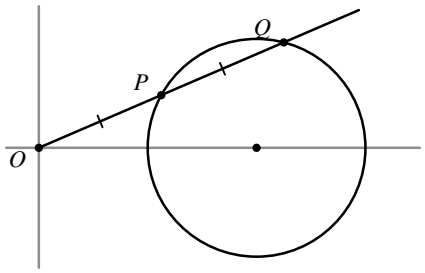
(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{2}}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\sqrt{3}}{3}$ **(E)** $\frac{\sqrt{2}}{2}$

13. Consider the 36 points (x, y) in the plane with $x, y \in \{1, 2, 3, 4, 5, 6\}$. The centroid of three points (x_1, y_1), (x_2, y_2), (x_3, y_3) is the point $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$. How many sets of three different points from these 36 have a centroid with both coordinates integers? (Three points in a line count too.)

(A) 420 **(B)** 548 **(C)** 612 **(D)** 768 **(E)** 804

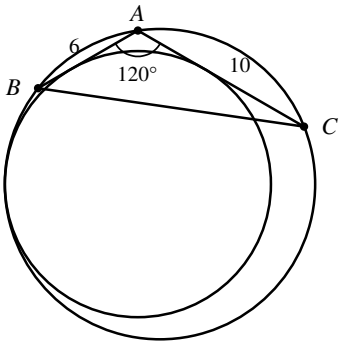
14. A line through the origin O with positive slope m meets the circle $(x-4)^2 + y^2 = 4$ at two points P and Q, where P lies between O and Q and $OP = PQ$. What is m?

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(A) $\frac{\sqrt{15}}{9}$ **(B)** $\frac{\sqrt{5}}{5}$ **(C)** $\frac{\sqrt{3}}{3}$ **(D)** $\frac{\sqrt{6}}{4}$ **(E)** $\frac{\sqrt{231}}{21}$

15. In triangle ABC, $AB = 6$, $AC = 10$ and $\angle BAC = 120°$. A circle is tangent to the sides $\overline{AB}$ and $\overline{AC}$ and is internally tangent to the circle that passes through A, B and C. What is the radius of this circle?



(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** $4\sqrt{3}$ **(D)** $\frac{14\sqrt{3}}{3}$ **(E)** $8\sqrt{3}$

16. Let
$$S = \sum_{n=1}^{20} \arctan \frac{2n}{n^4 - n^2 + 1},$$
where each arctan takes its value in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. What is $\tan S$?

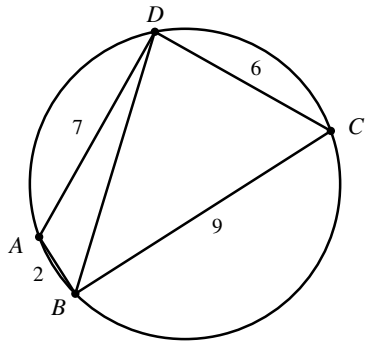
(A) 210 **(B)** 380 **(C)** 399 **(D)** 400 **(E)** 420

17. Call a positive integer *n lucky* if the number $2^n + 1$ is a multiple of 297. For example, $n = 1$ is not lucky, since $2^1 + 1 = 3$. How many lucky integers n satisfy $1 \le n \le 2026$?

(A) 23 **(B)** 45 **(C)** 68 **(D)** 113 **(E)** 203

18. Quadrilateral $ABCD$ is inscribed in a circle, with $AB = 2$, $BC = 9$, $CD = 6$ and $DA = 7$. What is the length of diagonal $\overline{BD}$?

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(A) $\frac{12\sqrt{85}}{17}$ **(B)** $\frac{4\sqrt{85}}{5}$ **(C)** $\frac{15\sqrt{85}}{17}$ **(D)** 9 **(E)** $\sqrt{85}$

19. A solid ball has center O and radius 5, and A is a point on its surface. A sphere of radius 6 is centered at A. What is the area of the part of this sphere (the one of radius 6) that lies inside the ball?

(A) $\frac{72\pi}{5}$ **(B)** $\frac{576\pi}{25}$ **(C)** 24π **(D)** $\frac{144\pi}{5}$ **(E)** 36π

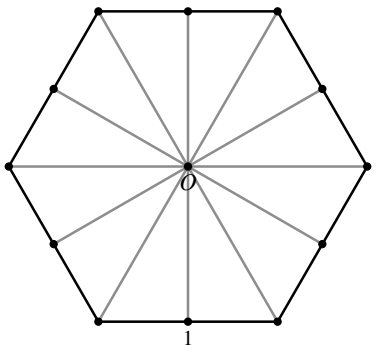
20. A path from $(0,0)$ to $(6,6)$ is made of 12 unit steps, each one unit to the right or one unit up, and one of the $\binom{12}{6} = 924$ such paths is chosen at random, all equally likely. On average, how many of the six points $(1,1), (2,2), \ldots, (6,6)$ does the chosen path pass through?

(A) $\frac{1979}{1024}$ **(B)** $\frac{562}{231}$ **(C)** 3 **(D)** $\frac{793}{231}$ **(E)** $\frac{1024}{231}$

21. A circle is centered at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ and passes through the point $(98, 98)$. How many points with integer coordinates lie on this circle?

(A) 9 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

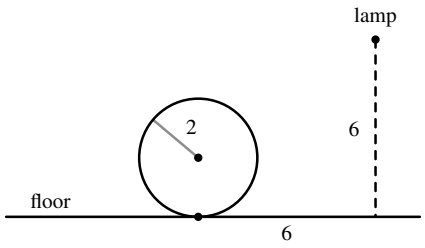
22. A regular hexagon has center O and side length 1. Consider the 12 vectors from O to the six vertices of the hexagon and to the six midpoints of its sides. Some of these vectors (at least one) are chosen and added together. What is the largest possible length of the sum?



(A) $2\sqrt{3}$ **(B)** $\frac{7}{2}$ **(C)** $\sqrt{13}$ **(D)** $\sqrt{3}+2$ **(E)** $\sqrt{2}+\sqrt{6}$

23. A ball of radius 2 rests on a flat, horizontal floor. A tiny lamp hangs at height 6 above the floor, at a horizontal distance of 6 from the point where the ball touches the floor. The ball's shadow is the set of

points P on the floor for which the segment from the lamp to P meets the ball; it is the region inside an ellipse. What is the area of the shadow?



(A) 9π **(B)** 12π **(C)** $12\sqrt{3}\pi$ **(D)** 24π **(E)** 48π

24. How many 15-element subsets S of $\{1, 2, 3, \ldots, 30\}$ have the property that no element of S divides a different element of S?

(A) 72 **(B)** 96 **(C)** 108 **(D)** 128 **(E)** 144

25. A permutation $(p_1, p_2, \ldots, p_7)$ of $1, 2, \ldots, 7$ is chosen uniformly at random. An entry p_i is a *left record* if it is larger than every entry to its left, and a *right record* if it is larger than every entry to its right. (So p_1 is always a left record, p_7 is always a right record, and the entry 7 is both.) What is the probability that the number of left records equals the number of right records?

(A) $\frac{1}{7}$ **(B)** $\frac{19}{120}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{77}{360}$

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Answer Key — AMC 12 Practice Contest 9

1. (B) $\frac{14}{5}$

2. (C) 80

3. (B) $\frac{12\sqrt{26}}{5}$

4. (A) $\frac{1}{2}$

5. (B) $\frac{5}{16}$

6. (A) $\frac{105}{8}$

7. (D) $\frac{6\sqrt{5}}{7}$

8. (D) $\frac{11}{16}$

9. (E) 46

10. (B) 46

11. (C) $\frac{8}{15}$

12. (B) $\frac{\sqrt{2}}{4}$

13. (E) 804

14. (A) $\frac{\sqrt{15}}{9}$

15. (C) $4\sqrt{3}$

16. (E) 420

17. (A) 23

18. (C) $\frac{15\sqrt{85}}{17}$

19. (D) $\frac{144\pi}{5}$

20. (D) $\frac{793}{231}$

21. (E) 36

22. (C) $\sqrt{13}$

23. (D) 24π

24. (A) 72

25. (E) $\frac{77}{360}$

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Solutions — AMC 12 Practice Contest 9

1. Answer: (B) $\frac{14}{5}$. The idea. Two logarithms with different bases are hard to combine directly, but every number here is a power of a. Rewrite b and c as powers of a and the question becomes a ratio of exponents.

Work it out. From $\log_a b = \frac{3}{2}$ we get $b = a^{3/2}$. From $\log_b c = 4$ we get $c = b^4 = a^6$. Then $ab = a^{1+3/2} = a^{5/2}$ and $ac = a^{1+6} = a^7$. So $\log_{ab}(ac) = \log_{a^{5/2}} a^7 = \dfrac{7}{5/2} = \dfrac{14}{5}$.

Check. Take $a = 4$: then $b = 8$, $c = 8^4 = 4096$, $ab = 32 = 2^5$ and $ac = 16384 = 2^{14}$, so $\log_{32} 16384 = \frac{14}{5}$. Adding the given logs ($\frac{11}{2}$) or multiplying them (6) are the traps; $\frac{12}{5}$ is $\log_{ab} c$, which forgets the factor a in ac.

2. Answer: (C) 80. The idea. The units digits agree exactly when $10 \mid n^3 - n^2 = n^2(n-1)$. Split $10 = 2 \cdot 5$ and handle each prime separately.

Work it out. Factor 2: one of n and $n-1$ is even, so $2 \mid n^2(n-1)$ always. Factor 5: 5 is prime, so $5 \mid n^2(n-1)$ exactly when $5 \mid n$ or $5 \mid n-1$, that is, $n \equiv 0$ or $1 \pmod 5$. So the condition holds for 2 residues out of every 5, and among $1, \ldots, 200$ (forty full blocks of 5) there are $40 \cdot 2 = 80$ such n.

Check. By units digit d: d^2 and d^3 end alike for $d = 0, 1, 5, 6$ (0, 0; 1, 1; 25, 125; 36, 216) and for no other digit (for example $4^2 = 16$, $4^3 = 64$). Four digits out of ten gives $\frac{4}{10} \cdot 200 = 80$. Stopping at $d = 0, 1$ gives the trap 40.

3. Answer: (B) $\frac{12\sqrt{26}}{5}$. The idea. Triangle ABC is right-angled at B, and E lies on its leg $\overline{BC}$. The angle-bisector theorem splits BC in the ratio of the sides $AB : AC$, and the hypotenuse AC is the diagonal.

Work it out. $AC = \sqrt{12^2 + 5^2} = 13$. In triangle ABC the bisector from A gives $\dfrac{BE}{EC} = \dfrac{AB}{AC} = \dfrac{12}{13}$, so $BE = \dfrac{12}{25} \cdot 5 = \dfrac{12}{5}$. Then right triangle ABE gives $AE = \sqrt{12^2 + \left(\frac{12}{5}\right)^2} = 12\sqrt{1 + \frac{1}{25}} = \dfrac{12\sqrt{26}}{5}$.

Check. With half-angles: $\tan \angle BAE = \tan \frac{\angle BAC}{2} = \dfrac{\sin \angle BAC}{1 + \cos \angle BAC} = \dfrac{5/13}{1 + 12/13} = \dfrac{1}{5}$, so $BE = 12 \cdot \frac{1}{5} = \frac{12}{5}$, the same. Taking E as the midpoint of BC instead gives the trap $\frac{\sqrt{601}}{2}$.

4. Answer: (A) $\frac{1}{2}$. The idea. Squaring the condition gives the product $\sin x \cos x$, and together with the difference that pins down $\sin x$ and $\cos x$ up to one swap. The range $0 < x < \pi$ decides between the two candidates; then $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$.

Work it out. Squaring: $1 - 2\sin x \cos x = \frac{1}{25}$, so $\sin x \cos x = \frac{12}{25}$. Hence $\sin x$ and $-\cos x$ have sum $\frac{1}{5}$ and product $-\frac{12}{25}$: they are the roots of $t^2 - \frac{1}{5}t - \frac{12}{25} = 0$, namely $\frac{4}{5}$ and $-\frac{3}{5}$. So either $(\sin x, \cos x) = \left(\frac{4}{5}, \frac{3}{5}\right)$ or $\left(-\frac{3}{5}, -\frac{4}{5}\right)$. Since $0 < x < \pi$, $\sin x > 0$, which leaves $\sin x = \frac{4}{5}$, $\cos x = \frac{3}{5}$. Then $\tan \frac{x}{2} = \dfrac{4/5}{1 + 3/5} = \dfrac{1}{2}$.

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Check. With $t = \tan\frac{x}{2} = \frac{1}{2}$: $\sin x = \frac{2t}{1+t^2} = \frac{4}{5}$ and $\cos x = \frac{1-t^2}{1+t^2} = \frac{3}{5}$, and $\frac{4}{5} - \frac{3}{5} = \frac{1}{5}$. The rejected pair gives $\tan\frac{x}{2} = -3$, whose sign-slip is the trap 3; using $\tan x = \frac{4}{3}$ in place of the half-angle is another trap.

5. Answer: (B) $\frac{5}{16}$. The idea. Build the flag from left to right: the first stripe has 3 choices and every later stripe has exactly 2 (anything except its left neighbour), so all $3 \cdot 2^5 = 96$ flags are equally likely when each stripe after the first is picked as a fair coin flip between its two allowed colours. Track p_n, the probability that stripe n matches stripe 1.

Work it out. If stripe n matches stripe 1, stripe $n{+}1$ cannot; if it does not, stripe $n{+}1$ matches stripe 1 with probability $\frac{1}{2}$ (the colour of stripe 1 is one of its two allowed colours). So $p_{n+1} = \frac{1}{2}(1-p_n)$. Starting from $p_1 = 1$: $p_2 = 0$, $p_3 = \frac{1}{2}$, $p_4 = \frac{1}{4}$, $p_5 = \frac{3}{8}$, $p_6 = \dfrac{5}{16}$.

Check. Count directly: with the first stripe red, the number of colourings ending in red is $\frac{2^5+2(-1)^5}{3} = 10$ out of $2^5 = 32$, again $\frac{10}{32} = \frac{5}{16}$. The naive answer $\frac{1}{3}$ treats the end stripes as independent; the answer is close to $\frac{1}{3}$ because $p_n \to \frac{1}{3}$, alternating above and below it.

6. Answer: (A) $\frac{105}{8}$. The idea. Each left side factors: $x^2 + xy + xz + yz = x(x+y) + z(x+y) = (x+y)(x+z)$, and likewise for the other two. So the system only involves the three pairwise sums $a = y+z$, $b = z+x$, $c = x+y$, and in those letters it says $cb = 30$, $ac = 24$, $ba = 20$: three products of two of a, b, c, which multiply to $(abc)^2$.

Work it out. The equations read $(x+y)(x+z) = 30$, $(y+z)(y+x) = 24$ and $(z+x)(z+y) = 20$, that is, $bc = 30$, $ca = 24$, $ab = 20$. Multiplying, $(abc)^2 = 30 \cdot 24 \cdot 20 = 14400$, and $a, b, c > 0$, so $abc = 120$. Then $a = \frac{120}{30} = 4$, $b = \frac{120}{24} = 5$, $c = \frac{120}{20} = 6$. Half the total gives $x+y+z = \frac{4+5+6}{2} = \frac{15}{2}$, so $x = \frac{15}{2} - a = \frac{7}{2}$, $y = \frac{15}{2} - b = \frac{5}{2}$, $z = \frac{15}{2} - c = \frac{3}{2}$. Hence $xyz = \frac{7 \cdot 5 \cdot 3}{8} = \dfrac{105}{8}$.

Check. With $x = \frac{7}{2}$, $y = \frac{5}{2}$, $z = \frac{3}{2}$: $x^2 + xy + xz + yz = \frac{49+35+21+15}{4} = 30$, $y^2 + xy + yz + xz = \frac{25+35+15+21}{4} = 24$ and $z^2 + xz + yz + xy = \frac{9+21+15+35}{4} = 20$. Answering with the product $abc = 120$ of the pair sums or with $a + b + c = 15$ are the traps; halving only one factor of $7 \cdot 5 \cdot 3$ instead of all three gives $\frac{105}{2}$, and halving only two gives $\frac{105}{4}$.

7. Answer: (D) $\frac{6\sqrt{5}}{7}$. The idea. The required distance is an altitude of triangle OPQ. That triangle is right-angled at P: edge $\overline{OP}$ is perpendicular to the whole face of the box containing P but not O, and $\overline{PQ}$ lies in that face.

Work it out. $OP = 2$. Segment $\overline{PQ}$ is a diagonal of a 3×6 face, so $PQ = \sqrt{9+36} = 3\sqrt{5}$. The space diagonal is $OQ = \sqrt{4+9+36} = 7$ (and indeed $2^2 + 45 = 49$). The altitude from the right angle is $\dfrac{OP \cdot PQ}{OQ} = \dfrac{2 \cdot 3\sqrt{5}}{7} = \dfrac{6\sqrt{5}}{7}$.

Check. Coordinates: $O = (0,0,0)$, $Q = (2,3,6)$, $P = (2,0,0)$. The projection of $\vec{OP}$ on the unit vector $\frac{1}{7}(2,3,6)$ is $\frac{4}{7}$, so the distance squared is $4 - \frac{16}{49} = \frac{180}{49}$, giving $\frac{6\sqrt{5}}{7} \approx 1.92$, a little less than $OP = 2$ as it must be.

8. Answer: (D) $\frac{11}{16}$. The idea. Linearity of expectation: every block of exactly two heads has a starting position, so add up, over all possible starting positions, the probability that a two-head

block starts there. The ends of the row need fewer tails to close the block off, which is where the count differs from the naive one.

Work it out. A block of length two can start at positions $1, 2, \ldots, 9$. Interior starts $i = 2, \ldots, 8$ (seven of them) need the pattern T H H T in positions $i-1, \ldots, i+2$: probability $\frac{1}{16}$ each. Start $i = 1$ needs H H T in positions $1, 2, 3$: probability $\frac{1}{8}$. Start $i = 9$ needs T H H in positions $8, 9, 10$: probability $\frac{1}{8}$. Expected number $= 7 \cdot \frac{1}{16} + 2 \cdot \frac{1}{8} = \frac{7}{16} + \frac{4}{16} = \dfrac{11}{16}$.

Check. Small case, 3 flips: two-head blocks occur only in HHT and THH, so the expectation is $\frac{2}{8} = \frac{1}{4}$, and the formula gives $0 \cdot \frac{1}{16} + 2 \cdot \frac{1}{8} = \frac{1}{4}$. Treating all nine starts as interior gives the trap $\frac{9}{16}$; dropping the ends gives $\frac{7}{16}$.

9. Answer: (E) 46**. The idea.** Going around the octagon, the colours form blocks of equal colour, and the condition says exactly that every block has length 1 or 2. Instead of the colours, count where the colour changes: a colouring is determined by the set of sides of the octagon whose two ends differ (the change sides) together with the colour of one vertex.

Work it out. Going around once the colour must come back to where it started, so the number of change sides is even, and the blocks between consecutive change sides have lengths 1 or 2 adding up to 8. That leaves three possibilities for the number k of change sides. $k = 8$: every side is a change side, 1 way. $k = 4$: four blocks of length 2, so the change sides alternate with the others, 2 ways. $k = 6$: two sides are not change sides, and they must not be next to each other (two neighbouring ones would create a block of length 3), so there are $\binom{8}{2} - 8 = 20$ ways. That is $1 + 2 + 20 = 23$ sets of change sides, and each allows 2 choices for the colour of a fixed vertex, giving $2 \cdot 23 = 46$ colourings.

Check. A computer check of all $2^8 = 256$ colourings finds 46. Forgetting the factor 2 gives the trap 23; missing the alternating case $k = 4$ gives 42, and missing $k = 8$ gives 44. Counting only the case $k = 6$ gives 40.

10. Answer: (B) 46**. The idea.** Track how the digit sum changes when you add 1. If n ends in exactly t nines, adding 1 turns those nines into zeros and raises the next digit by 1, so the digit sum changes by $1 - 9t$. Both sums odd means the change is even, so t must be odd.

Work it out. For $n \le 1000$, t odd means $t = 1$ or $t = 3$. Case $t = 3$: only $n = 999$, digit sum 27 (odd), and 1000 has digit sum 1 (odd): it counts. Case $t = 1$: n ends in 9 and its tens digit is not 9. Write $n = \overline{ab9}$ with $a \in \{0, \ldots, 9\}$ (leading zeros allowed) and $b \in \{0, \ldots, 8\}$; its digit sum $a + b + 9$ is odd exactly when $a + b$ is even. For each of the 9 values of b, exactly 5 of the 10 values of a have the right parity, so there are 45 such n. (Then $n + 1$ has digit sum $a + b + 1$, odd as well.) Total $45 + 1 = 46$.

Check. Examples: $9 \to 10$ (sums 9, 1) counts; $19 \to 20$ (sums 10, 2) does not; $29 \to 30$ (sums 11, 3) counts. Forgetting 999 gives 45; counting all numbers ending in exactly one 9 gives 90. A direct computer check over $1 \le n \le 1000$ gives 46.

11. Answer: (C) $\frac{8}{15}$**. The idea.** Do not expand the quartic. Since $z = 0$ is not a solution, divide by z^4: the equation says $\left(\frac{z-1}{z}\right)^4 = 16$, so $\frac{z-1}{z}$ is one of the four fourth roots of 16. Each of those gives one z by a simple division.

Work it out. Let $w = \frac{z-1}{z} = 1 - \frac{1}{z}$, so $z = \dfrac{1}{1-w}$, with $w \in \{2, -2, 2i, -2i\}$. Then $w = 2$ gives

$z = -1$; $w = -2$ gives $z = \frac{1}{3}$; $w = 2i$ gives $z = \dfrac{1}{1-2i} = \dfrac{1+2i}{5}$; $w = -2i$ gives $z = \dfrac{1-2i}{5}$. Two vertices, -1 and $\frac{1}{3}$, lie on the real axis, and the other two, $\frac{1}{5} \pm \frac{2}{5}i$, are mirror images across it. So the quadrilateral is a kite whose diagonals are perpendicular: one runs from -1 to $\frac{1}{3}$ (length $\frac{4}{3}$), the other from $\frac{1}{5} - \frac{2}{5}i$ to $\frac{1}{5} + \frac{2}{5}i$ (length $\frac{4}{5}$), and they cross at $\frac{1}{5}$, which lies between -1 and $\frac{1}{3}$. The area is $\frac{1}{2} \cdot \frac{4}{3} \cdot \frac{4}{5} = \dfrac{8}{15}$.

Check. Shoelace with the vertices in order $(-1, 0)$, $\left(\frac{1}{5}, -\frac{2}{5}\right)$, $\left(\frac{1}{3}, 0\right)$, $\left(\frac{1}{5}, \frac{2}{5}\right)$ gives $\frac{8}{15}$ again. Also every solution satisfies $|z-1| = 2|z|$, so all four lie on one circle (centre $-\frac{1}{3}$, radius $\frac{2}{3}$); for instance $\left|\frac{1}{5} + \frac{1}{3} + \frac{2}{5}i\right| = \frac{1}{15}\sqrt{8^2 + 6^2} = \frac{2}{3}$. Forgetting the $\frac{1}{2}$ in the kite formula gives the trap $\frac{16}{15}$.

12. Answer: (B) $\frac{\sqrt{2}}{4}$. The idea. The condition fixes the shape up to one parameter. Put the right angle at the origin: then C is the reflection of B through M, and $\tan C$ becomes a simple function of one ratio, which AM-GM maximises.

Work it out. Place $A = (0, 0)$, $B = (c, 0)$ and, since $AM \perp AB$, $M = (0, m)$ with $c, m > 0$. Then $C = 2M - B = (-c, 2m)$. From C, the vectors to A and B are $(c, -2m)$ and $(2c, -2m)$. Their cross product has size $|c(-2m) - (-2m)(2c)| = 2cm$ and their dot product is $2c^2 + 4m^2 > 0$, so $\angle C$ is acute with $\tan C = \dfrac{2cm}{2c^2 + 4m^2} = \dfrac{t}{1 + 2t^2}$, where $t = \dfrac{m}{c}$. Now $\dfrac{t}{1 + 2t^2} = \dfrac{1}{\frac{1}{t} + 2t}$ and by AM-GM $\frac{1}{t} + 2t \geq 2\sqrt{2}$, with equality at $t = \frac{1}{\sqrt{2}}$. So the largest value is $\dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$.

Check. With $c = \sqrt{2}$, $m = 1$: $AB = \sqrt{2}$, $AC = \sqrt{6}$, $BC = 2\sqrt{3}$. The law of cosines gives $\cos C = \dfrac{6 + 12 - 2}{2 \cdot \sqrt{6} \cdot 2\sqrt{3}} = \dfrac{2\sqrt{2}}{3}$, so $\sin C = \dfrac{1}{3}$ and $\tan C = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$. In angle terms the condition says $\tan A = -2 \tan B$ (angle A is obtuse), and $\tan C = -\tan(A + B) = \dfrac{\tan B}{1 + 2\tan^2 B}$, the same function. Forgetting the factor 2 in $1 + 2t^2$ gives the trap $\frac{1}{2}$.

13. Answer: (E) 804**. The idea.** Only the coordinates modulo 3 matter. Three residues mod 3 add up to a multiple of 3 exactly when they are all equal or all different: $r + r + r = 3r$ and $0 + 1 + 2 = 3$, while two equal and one different gives $2a + b \equiv b - a \not\equiv 0$. So sort the points into 9 classes by $(x \bmod 3,\ y \bmod 3)$. A set works exactly when, in each coordinate separately, the three classes agree completely or differ completely.

Work it out. Each residue appears twice among $1, \ldots, 6$, so each of the 9 classes holds exactly $2 \cdot 2 = 4$ points. Picture the classes as a 3×3 array.

All three points in one class. Both coordinates agree, so this always works: $9 \cdot \binom{4}{3} = 36$ sets.

Two points in one class, the third elsewhere. The third class differs from the other two in some coordinate, where the pattern is two equal and one different. None of these work.

Three different classes. The coordinate patterns can't both be "all equal" (the classes would coincide). If the first coordinates agree, the second ones are all different: that's one of the 3 rows of the array. Similarly, there are 3 columns. If both coordinates are all different, the second coordinates are a rearrangement of $0, 1, 2$ matched to the first ones, which gives $3! = 6$ class triples. These $3 + 3 + 6 = 12$ triples are the rows, the columns and the six diagonals of the array (two ordinary diagonals and four that wrap around the edges, three in each direction). Each one gives $4^3 = 64$ sets of points, so $12 \cdot 64 = 768$ sets.

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The total is $36 + 768 = 804$.

Check. There are $\binom{36}{3} = 7140$ sets in all, and $804/7140 \approx 0.113$, close to the $\frac{1}{9}$ you'd guess if each coordinate condition held with probability $\frac{1}{3}$ independently. The guess can't be exact, since $7140/9$ is not an integer. Leaving out the same-class sets gives 768. Counting only the rows, the columns and the two ordinary diagonals of the array gives $36 + 8 \cdot 64 = 548$, and keeping the diagonals in only one direction gives $36 + 9 \cdot 64 = 612$. Counting only the rows and the columns gives $36 + 6 \cdot 64 = 420$.

14. Answer: (A) $\frac{\sqrt{15}}{9}$**. The idea.** The power of the point O with respect to the circle is the same for every line through O: $OP \cdot OQ = OC^2 - r^2$, where $C = (4, 0)$ is the centre and $r = 2$. Since $OQ = 2\,OP$, this gives the chord length, and the chord length tells us how far the line is from the centre.

Work it out. The power is $4^2 - 2^2 = 12$, so $OP \cdot 2\,OP = 12$, $OP^2 = 6$ and the chord is $PQ = OP = \sqrt{6}$. The distance d from C to the line satisfies $d^2 = r^2 - \left(\frac{PQ}{2}\right)^2 = 4 - \frac{6}{4} = \frac{5}{2}$. The distance from $(4, 0)$ to the line $mx - y = 0$ is $\frac{4m}{\sqrt{1+m^2}}$, so $\frac{16m^2}{1+m^2} = \frac{5}{2}$, that is $32m^2 = 5 + 5m^2$, $m^2 = \frac{5}{27}$ and $m = \frac{\sqrt{5}}{3\sqrt{3}} = \frac{\sqrt{15}}{9}$.

Check. Substituting $y = mx$ gives $(1+m^2)x^2 - 8x + 12 = 0$. With $m^2 = \frac{5}{27}$ this is $\frac{32}{27}x^2 - 8x + 12 = 0$, or $8x^2 - 54x + 81 = 0$, with roots $x = \frac{9}{4}$ and $x = \frac{9}{2}$: one is exactly twice the other, as required. Adding instead of subtracting in $d^2 = r^2 - \left(\frac{PQ}{2}\right)^2$ gives $d^2 = \frac{11}{2}$, $m^2 = \frac{11}{21}$ and the wrong slope $\frac{\sqrt{231}}{21}$.

15. Answer: (C) $4\sqrt{3}$**. The idea.** A circle tangent to both sides of angle A has its center P on the bisector of that angle, and if $AP = d$ its radius is $\rho = d \sin 60^\circ = \frac{\sqrt{3}}{2}d$, since the bisector makes 60° with each side. Internal tangency to the circumcircle (center O, radius R) says $OP = R - \rho$. Put the bisector on the x-axis and this is one equation in d.

Work it out. By the law of cosines $BC^2 = 36 + 100 + 60 = 196$, so $BC = 14$ and $R = \frac{BC}{2\sin 120^\circ} = \frac{14}{\sqrt{3}}$. Place A at the origin with the bisector along the positive x-axis, so $B = (3, -3\sqrt{3})$ and $C = (5, 5\sqrt{3})$. The circumcenter $O = (p, q)$ is as far from A as from B and from C: this gives $p - \sqrt{3}\,q = 6$ and $p + \sqrt{3}\,q = 10$, so $O = \left(8, \frac{2}{\sqrt{3}}\right)$, and indeed $64 + \frac{4}{3} = \frac{196}{3} = R^2$. With $P = (d, 0)$, the tangency condition $OP^2 = (R - \rho)^2$ reads

$(d - 8)^2 + \frac{4}{3} = \left(\frac{14}{\sqrt{3}} - \frac{\sqrt{3}}{2}d\right)^2$, that is, $d^2 - 16d + \frac{196}{3} = \frac{196}{3} - 14d + \frac{3}{4}d^2$.

So $\frac{1}{4}d^2 = 2d$, giving $d = 8$ and $\rho = \frac{\sqrt{3}}{2} \cdot 8 = 4\sqrt{3}$.

Check. The circle touches AB and AC at distance $d\cos 60^\circ = 4$ from A, inside both sides ($4 < 6$ and $4 < 10$), as required. A known shortcut agrees: this circle's radius is $\frac{r}{\cos^2(A/2)}$, where r is the inradius; here $r = \frac{\text{area}}{s} = \frac{15\sqrt{3}}{15} = \sqrt{3}$ and $\cos^2 60^\circ = \frac{1}{4}$, giving $4\sqrt{3}$. The inradius $\sqrt{3}$ and the circumradius $\frac{14\sqrt{3}}{3}$ are the tempting wrong answers, dividing by $\cos 60^\circ$ only once gives $2\sqrt{3}$, and using $\rho = d\tan 60^\circ$ in place of $d \sin 60^\circ$ gives $8\sqrt{3}$.

16. Answer: (E) 420**. The idea.** The subtraction formula $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ turns a difference of two arctangents into a single one. Look for numbers x_n with $x_n - x_{n-1} = 2n$ and $1 + x_n x_{n-1} =$

n^4-n^2+1. The first condition suggests $x_n = n(n+1)$, and then $1+x_n x_{n-1} = 1+n(n+1)\cdot(n-1)n = 1+n^2(n^2-1) = n^4-n^2+1$. It fits.

Work it out. Since $0 \le x_{n-1} < x_n$, the angle $\arctan x_n - \arctan x_{n-1}$ lies in $\left(0, \frac{\pi}{2}\right)$, and its tangent is $\frac{2n}{n^4-n^2+1}$. So each term is $\arctan(n(n+1)) - \arctan((n-1)n)$, and the sum telescopes:

$S = \arctan(20 \cdot 21) - \arctan 0 = \arctan 420$.

Hence $\tan S = 420$.

Check. The first term is $\arctan 2 = \arctan 2 - \arctan 0$. The second is $\arctan \frac{4}{13}$, and indeed $\tan(\arctan 6 - \arctan 2) = \frac{6-2}{1+12} = \frac{4}{13}$. Ending at 20^2 instead of $20 \cdot 21$ gives 400, stopping one term early gives $19 \cdot 20 = 380$, and halving (as if the terms were $\arctan \frac{n}{n^4-n^2+1}$) gives 210.

17. Answer: (A) 23**. The idea.** $297 = 27 \cdot 11$, so we need $2^n \equiv -1$ modulo 27 and modulo 11 separately. For each modulus, find the order of 2; then $2^n \equiv -1$ exactly when n is an odd multiple of half the order.

Work it out. Modulo 11: $2^5 = 32 \equiv -1$, so 2 has order 10 and $2^n \equiv -1 \pmod{11}$ exactly when $n \equiv 5 \pmod{10}$. Modulo 27: $2^9 = 512 = 19 \cdot 27 - 1 \equiv -1$, while $2^3 = 8$ and $2^6 = 64 \equiv 10$ are not ± 1, so 2 has order 18 and $2^n \equiv -1 \pmod{27}$ exactly when $n \equiv 9 \pmod{18}$, that is, n is an odd multiple of 9. Together: n is odd, divisible by 5 and divisible by 9, so n is an odd multiple of 45, i.e. $n \equiv 45 \pmod{90}$. The values $45, 135, \ldots, 45 + 90k \le 2026$ need $k \le 22$, giving 23 values.

Check. Another route to the factor 27: write $2^n + 1 = (3-1)^n + 1$ with n odd and expand; the power of 3 dividing it is 1 more than the power of 3 dividing n, so 27 needs $9 \mid n$. Using only $3 \mid n$ gives 68; ignoring 11 gives the 113 odd multiples of 9; ignoring 27 gives 203; and forgetting that n must be odd gives the 45 multiples of 45.

18. Answer: (C) $\frac{15\sqrt{85}}{17}$**. The idea.** Notice that $AB^2 + BC^2 = 4 + 81 = 85$ and $CD^2 + DA^2 = 36 + 49 = 85$. Triangles ABC and ADC share the side AC, and the angles at B and D add to $180°$. If $AC^2 = 85$, both angles B and D are right angles, and that is consistent; the law of cosines shows it is forced. Then AC is a diameter and Ptolemy's theorem gives the other diagonal.

Work it out. By the law of cosines, $AC^2 = 85 - 36\cos B$ and $AC^2 = 85 - 84\cos D = 85 + 84\cos B$. So $\cos B = 0$, the angles B and D are right angles, and $AC = \sqrt{85}$. Ptolemy's theorem for a cyclic quadrilateral says $AC \cdot BD = AB \cdot CD + BC \cdot DA = 2 \cdot 6 + 9 \cdot 7 = 75$. So $BD = \frac{75}{\sqrt{85}} = \frac{15\sqrt{85}}{17}$.

Check. Use triangles ABD and CBD instead: $BD^2 = 4 + 49 - 28\cos A = 81 + 36 + 108\cos A$, since $\cos C = -\cos A$. This gives $\cos A = -\frac{8}{17}$ and $BD^2 = 53 + \frac{224}{17} = \frac{1125}{17}$, so $BD = \frac{15\sqrt{5}}{\sqrt{17}} = \frac{15\sqrt{85}}{17}$.

Pairing the sides wrongly in Ptolemy's theorem ($2 \cdot 9 + 6 \cdot 7 = 60$) gives $\frac{12\sqrt{85}}{17}$, and $\sqrt{85}$ is the other diagonal.

19. Answer: (D) $\frac{144\pi}{5}$**. The idea.** The two spheres meet in a circle, and the part of the big sphere inside the ball is the cap cut off by the plane of that circle. Archimedes' hat-box theorem says a spherical cap of height h on a sphere of radius R has area $2\pi Rh$, no matter where it sits. So we only need the distance from A to the plane of the circle.

Work it out. Let X be a point on both spheres, so triangle OAX has $OA = OX = 5$ and $AX = 6$. Let $\theta = \angle OAX$. The law of cosines in this triangle gives $25 = 36 + 25 - 60\cos\theta$, so $\cos\theta = \frac{3}{5}$. The projection of X onto line AO is at distance $6\cos\theta = \frac{18}{5}$ from A, and this is where the plane

of the circle crosses AO. The points of the radius-6 sphere inside the ball are the ones beyond this plane on the side of O. They form a cap of height $6 - \frac{18}{5} = \frac{12}{5}$, with area $2\pi \cdot 6 \cdot \frac{12}{5} = \frac{144\pi}{5}$.

Check. The cap is $\frac{12/5}{12} = \frac{1}{5}$ of the whole sphere's area 144π, which matches. The answer 36π is the other cap: the part of the ball's surface within distance 6 of A, with height $\frac{18}{5}$ on the radius-5 sphere. The answer $\frac{576\pi}{25}$ is the flat disk bounded by the circle, whose radius is $6\sin\theta = \frac{24}{5}$.

20. Answer: (D) $\frac{793}{231}$**. The idea.** Linearity of expectation: the average number of diagonal points visited is the sum, over $k = 1, \ldots, 6$, of the probability that the path passes through (k, k). Those probabilities are easy to count, because a path through (k, k) is a path to (k, k) followed by a path from (k, k) to $(6, 6)$.

Work it out. The number of paths through (k, k) is $\binom{2k}{k}\binom{12-2k}{6-k}$. For $k = 1, \ldots, 6$ this is

$$2 \cdot 252 = 504, \quad 6 \cdot 70 = 420, \quad 20 \cdot 20 = 400, \quad 70 \cdot 6 = 420, \quad 252 \cdot 2 = 504, \quad 924 \cdot 1 = 924,$$

with total 3172. So the expected number is $\frac{3172}{924} = \frac{793}{231} \approx 3.43$.

Check. The list is symmetric (k and $6-k$ give the same count), as reversing a path shows, and the last term is 924 because every path ends at $(6, 6)$. There is also an identity: $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$, and $4^6 - \binom{12}{6} = 4096 - 924 = 3172$, matching. Counting the start $(0, 0)$ as well gives $\frac{1024}{231}$; leaving out the certain endpoint $(6, 6)$ gives $\frac{562}{231}$; and using the free random-walk chances $\binom{2k}{k}/4^k$, which ignore that the path must end at $(6, 6)$, gives $\frac{1979}{1024}$.

21. Answer: (E) 36**. The idea.** Double all coordinates to clear the halves. The point (x, y) is on the circle exactly when $(2x-1)^2 + (2y-1)^2 = 2 \cdot 195^2 = 76050$. So we count the ways to write 76050 as a sum of two squares of odd integers. The parity condition comes for free: $76050 \equiv 2 \pmod 4$, so in any representation $u^2 + v^2 = 76050$ both u and v are odd, and each one gives a lattice point $\left(\frac{u+1}{2}, \frac{v+1}{2}\right)$.

Work it out. $76050 = 2 \cdot 3^2 \cdot 5^2 \cdot 13^2$. A prime $\equiv 3 \pmod 4$ that divides $u^2 + v^2$ must divide both u and v, so $u = 3u'$ and $v = 3v'$ with $u'^2 + v'^2 = 8450 = 2 \cdot 5^2 \cdot 13^2$. The number of ordered integer pairs (signs included) with $u'^2 + v'^2 = N$ is 4 times the product of $(e+1)$ over the primes $\equiv 1 \pmod 4$ dividing N to the power e, provided every prime $\equiv 3 \pmod 4$ appears to an even power. (In Gaussian integers, each such prime $p = \pi\bar{\pi}$ can share out its e factors in $e+1$ ways, and there are 4 unit multiples.) Here that is $4 \cdot (2+1)(2+1) = 36$.

Check. Listing $u'^2 + v'^2 = 8450$ with $0 < u' \le v'$ gives $13^2 + 91^2$, $23^2 + 89^2$, $35^2 + 85^2$, $47^2 + 79^2$ and $65^2 + 65^2$. That is 9 ordered positive pairs, and with 4 sign choices (no zero coordinates, since 8450 is not a square) we get 36. Counting the factor 3^2 as if it were a prime $\equiv 1 \pmod 4$ gives 108; stopping at the first quadrant gives 9.

22. Answer: (C) $\sqrt{13}$**. The idea.** Look at a best choice, with sum $\vec{s}$. For any of the twelve vectors $\vec{v}$, $|\vec{s}-\vec{v}|^2 = |\vec{s}|^2 - 2\,\vec{v}\cdot\vec{s} + |\vec{v}|^2$ and $|\vec{s}+\vec{v}|^2 = |\vec{s}|^2 + 2\,\vec{v}\cdot\vec{s} + |\vec{v}|^2$. So if a chosen $\vec{v}$ had $\vec{v}\cdot\vec{s} \le 0$, dropping it would make the sum longer, and if an unchosen $\vec{v}$ had $\vec{v}\cdot\vec{s} \ge 0$, adding it would. Hence the best choice is exactly the set of vectors making an acute angle with $\vec{s}$, and no vector is perpendicular to $\vec{s}$: the chosen vectors are the ones on one side of a line through O that passes through none of them.

Work it out. The twelve vectors point in directions 30° apart, alternating between vertices (length 1) and side midpoints (length $\frac{\sqrt{3}}{2}$, the apothem). A line through O missing all twelve directions

leaves exactly six consecutive directions on each side. Up to rotations and reflections of the hexagon there is one such set: say vertices at $0°, 60°, 120°$ and midpoints at $30°, 90°, 150°$. (Six consecutive directions starting at a midpoint are the mirror image of six starting at a vertex.) In complex numbers the three vertices add to $1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}\,i$, and the three midpoints add to $\frac{\sqrt{3}}{2}\left(e^{i30°} + i + e^{i150°}\right) = \frac{\sqrt{3}}{2} \cdot 2i = \sqrt{3}\,i$. The sum is $1 + 2\sqrt{3}\,i$, of length $\sqrt{1 + 12} = \sqrt{13}$.

Check. Neighbouring choices are shorter, as the argument predicts: five consecutive vectors give $\frac{7}{2}$ (if they start at a vertex) or $2\sqrt{3}$, and seven give the same two values. If all twelve vectors had length 1, the answer would be $\sqrt{6} + \sqrt{2}$; adding the lengths 2 and $\sqrt{3}$ of the two partial sums, as if they pointed the same way, gives $2 + \sqrt{3}$.

23. Answer: (D) 24π**. The idea.** The rays from the lamp that just graze the ball form a circular cone with apex at the lamp, and the shadow's edge is where that cone meets the floor. Take coordinates and write the cone's condition directly: a floor point P is on the edge exactly when the angle between $\overrightarrow{LP}$ and $\overrightarrow{LO}$ (lamp L, ball center O) equals the cone's half-angle α, where $\sin\alpha = \frac{2}{LO}$.

Work it out. Put the contact point at the origin, so $O = (0, 0, 2)$ and $L = (6, 0, 6)$. Then $\overrightarrow{LO} = (-6, 0, -4)$, $LO^2 = 52$, $\sin^2\alpha = \frac{4}{52} = \frac{1}{13}$ and $\cos^2\alpha = \frac{12}{13}$. For $P = (x, y, 0)$, $\overrightarrow{LP} = (x - 6, y, -6)$ and $\overrightarrow{LP} \cdot \overrightarrow{LO} = 60 - 6x$. The edge condition $\left(\overrightarrow{LP} \cdot \overrightarrow{LO}\right)^2 = \cos^2\alpha\, LP^2\, LO^2$ becomes

$$(60 - 6x)^2 = 48\left[(x - 6)^2 + y^2 + 36\right].$$

Dividing by 12: $3(10 - x)^2 = 4(x - 6)^2 + 4y^2 + 144$, which simplifies to $x^2 + 12x + 4y^2 = 12$, or $(x + 6)^2 + 4y^2 = 48$. This ellipse has semi-axes $\sqrt{48} = 4\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$, so the shadow's area is $\pi \cdot 4\sqrt{3} \cdot 2\sqrt{3} = 24\pi$.

Check. A second route uses the ball as a Dandelin sphere: it touches the floor at a focus of the ellipse. In the vertical plane through L and O, the two grazing rays hit the floor at $x = -6 \pm 4\sqrt{3}$, so the major axis is $8\sqrt{3}$ with center $(-6, 0)$. The focus is the origin, so $c = 6$ and $b^2 = a^2 - c^2 = 48 - 36 = 12$, as above. The answer 9π is the shadow of the ball's horizontal great circle alone (radius $2 \cdot \frac{6}{4} = 3$), and 48π treats the ellipse as a circle of radius $4\sqrt{3}$.

24. Answer: (A) 72**. The idea.** Write every number as $2^e m$ with m odd. The numbers with the same odd part m form a chain $m, 2m, 4m, \ldots$ in which each number divides the next. There are exactly 15 odd parts $1, 3, \ldots, 29$, so a 15-element set with no divisibility must take exactly one number from each chain. Call it $2^{e_m} m$. For two different odd parts, $2^a m \mid 2^b m'$ happens exactly when $m \mid m'$ and $a \le b$. So the condition is: whenever m properly divides m', the exponent must strictly drop, $e_m > e_{m'}$.

Work it out. The odd parts $17, 19, 21, 23, 25, 27, 29$ have one-element chains, so $e = 0$ for each of them. Now push the constraints downward.

$7 \mid 21$ forces $e_7 \ge 1$: 7 is replaced by 14 or 28 (2 ways). $9 \mid 27$ forces $e_9 \ge 1$, and 9's chain is only $9, 18$, so 18 is chosen. $5 \mid 25$ forces $e_5 \ge 1$: 10 or 20. $3 \mid 9$ forces $e_3 > e_9 = 1$: 12 or 24. The odd part 15 (chain $15, 30$) only needs $e_{15} < e_5$ and $e_{15} < e_3$, and the second holds automatically. The odd parts 11 and 13 (chains $11, 22$ and $13, 26$) divide nothing else in range, so each has 2 free choices. Finally 1 divides everything, so e_1 must exceed every other exponent. All of them are at most 2 except e_3, which is 2 or 3, so the condition is just $e_1 > e_3$, with $e_1 \le 4$.

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Count the pieces. The pair (e_3, e_1) is $(2,3)$, $(2,4)$ or $(3,4)$: 3 ways (that is, 12 with 8 or 16, or 24 with 16). The pair (e_5, e_{15}) is $(1,0)$, $(2,0)$ or $(2,1)$: 3 ways. Then 7 gives 2 ways, and 11 and 13 give $2 \cdot 2 = 4$. The total is $3 \cdot 3 \cdot 2 \cdot 4 = 72$.

Check. The same method on $\{1, \ldots, 20\}$ with 10-element sets gives 26, which matches a computer search, as does 72 here. One valid set is $\{16, 24, 18, 20, 30, 28, 22, 26, 17, 19, 21, 23, 25, 27, 29\}$. Skipping a constraint overcounts: letting 1's number be 8 or 16 whatever 3's number is (ignoring $e_1 > e_3$) gives 96, and ignoring $e_{15} < e_5$ as well gives 128; letting 7 stay as it is (forgetting $7 \mid 21$) gives 108, and forgetting $7 \mid 21$ while also ignoring $e_1 > e_3$ gives 144.

25. Answer: (E) $\frac{77}{360}$**. The idea.** Split the permutation at the entry 7. The left records are 7 together with the left records of the block before it, and the right records are 7 together with the right records of the block after it. Given the position of 7 and which numbers go to each side, the two blocks are independent uniformly random arrangements. So we need the distribution of the number of records in a random arrangement of m distinct numbers, and then a convolution.

Work it out. In a random arrangement of m numbers, the j-th entry is a record with probability $\frac{1}{j}$, and these events are independent. So the number of records has generating function $\prod_{j=1}^{m} \frac{(j-1)+z}{j}$, i.e. the coefficient of z^r is $\frac{c(m,r)}{m!}$, where the counts $c(m,r)$ for $m = 0, \ldots, 6$ are

$m = 1 : (1)$; $m = 2 : (1,1)$; $m = 3 : (2,3,1)$; $m = 4 : (6,11,6,1)$; $m = 5 : (24,50,35,10,1)$; $m = 6 : (120,274,225,85,15,1)$

(listing $r = 1, 2, \ldots$; an empty block has 0 records). If 7 is in position k, the blocks have sizes $m = k-1$ and $6-m$, and we need equal record counts. Call this probability $q(m, 6-m)$:

$q(0,6) = q(6,0) = 0$, since an empty block has 0 records and a nonempty block has at least 1;

$q(1,5) = q(5,1) = \frac{24}{120} = \frac{1}{5}$;

$q(2,4) = q(4,2) = \frac{1}{2} \cdot \frac{6}{24} + \frac{1}{2} \cdot \frac{11}{24} = \frac{17}{48}$;

$q(3,3) = \frac{2^2+3^2+1^2}{36} = \frac{7}{18}$.

Each position of 7 has probability $\frac{1}{7}$, so the answer is $\frac{1}{7}\left(\frac{2}{5} + \frac{17}{24} + \frac{7}{18}\right) = \frac{1}{7} \cdot \frac{144+255+140}{360} = \frac{1}{7} \cdot \frac{539}{360} = \frac{77}{360}$.

Check. Check $c(m,r)$ with the recursion $c(m+1,r) = c(m,r-1) + m\,c(m,r)$: the newest entry is a record with probability $\frac{1}{m+1}$. For example, $c(4,2) = c(3,1) + 3c(3,2) = 2 + 9 = 11$. A direct count over all 5040 permutations finds $1078 = 5040 \cdot \frac{77}{360}$ with equal counts. Leaving out the case with 7 in the middle gives $\frac{1}{7}\left(\frac{2}{5} + \frac{17}{24}\right) = \frac{19}{120}$. The tempting guess $\frac{1}{7}$ ("7 must be in the middle") is wrong both ways: 7 in the middle can give unequal counts, and 7 off-center can give equal ones. The same computation for 6 entries gives $\frac{2}{9}$.

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