MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

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1. A real number x satisfies the equation
$3\sin x + 4\cos x = 5$.
This one equation turns out to pin down both $\sin x$ and $\cos x$ completely, even though x itself is only determined up to adding multiples of 2π. What is the value of $\tan x$?

(A) $\frac{3}{5}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** 1 **(E)** $\frac{4}{3}$

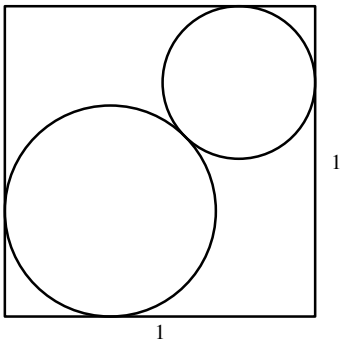
2. A real number $x > 1$ satisfies
$\log_2\left(\log_4 x\right) + \log_4\left(\log_2 x\right) = 2$.
What is x?

(A) 4 **(B)** 8 **(C)** 16 **(D)** 64 **(E)** 256

3. The graphs of $y = \sin x$ and $y = \cos 2x$ are drawn for $0 \le x < 2\pi$. What is the sum of the x-coordinates of all the points where the two graphs meet?

(A) π **(B)** $\frac{3\pi}{2}$ **(C)** $\frac{5\pi}{3}$ **(D)** 2π **(E)** $\frac{5\pi}{2}$

4. Two circles lie inside a square of side length 1. The first circle is tangent to the left side and to the bottom side of the square; the second is tangent to the right side and to the top side. The two circles are tangent to each other (externally). The radii of the circles are not given, but the sum of the two radii is determined by this information. What is that sum?



(A) $1 - \frac{\sqrt{2}}{2}$ **(B)** $-1 + \sqrt{2}$ **(C)** $\frac{1}{2}$ **(D)** $2 - \sqrt{2}$ **(E)** $\frac{\sqrt{2}}{2}$

5. For a real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ be its fractional part, so $0 \le \{x\} < 1$. How many real numbers x with $0 \le x < 100$ satisfy $\lfloor x \rfloor \cdot \{x\} = 20$?

(A) 78 **(B)** 79 **(C)** 80 **(D)** 99 **(E)** 100

6. Five balls, labelled 1 through 5, are placed into five boxes, labelled A through E. Each ball is put into a box chosen uniformly at random, independently of the other balls, and a box may hold any number of balls. What is the probability that exactly one of the five boxes ends up empty?

(A) $\frac{8}{125}$ **(B)** $\frac{48}{625}$ **(C)** $\frac{12}{125}$ **(D)** $\frac{24}{125}$ **(E)** $\frac{48}{125}$

7. Some positive integers can be written in the form $n = ab + 2a + 2b$, where a and b are positive integers (not necessarily different). For example, $n = 5$ (with $a = b = 1$) and $n = 11$ (with $a = 1$, $b = 3$) can be written this way. How many of the integers n with $1 \le n \le 50$ can *not* be written in this form?

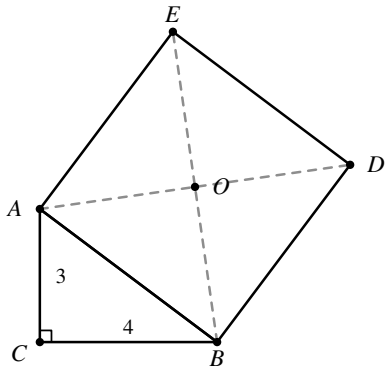
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(A) 8 **(B)** 14 **(C)** 22 **(D)** 23 **(E)** 27

8. What is the smallest positive integer n for which the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$ is divisible by 2026? (Note that $2026 = 2 \cdot 1013$ and 1013 is prime.)

(A) 1012 **(B)** 1013 **(C)** 1014 **(D)** 2025 **(E)** 2026

9. Triangle ABC has a right angle at C, with $CA = 3$ and $CB = 4$. A square $ABDE$ is built on the hypotenuse $\overline{AB}$, on the opposite side of line AB from C. Let O be the centre of the square (the point where its diagonals cross). What is the distance CO?



(A) $\frac{7\sqrt{2}}{2}$ **(B)** 5 **(C)** 6 **(D)** 7 **(E)** $7\sqrt{2}$

10. Six different books are handed out to three friends, Ana, Ben and Cy, so that every book goes to exactly one friend and each friend gets at least one book. In how many ways can this be done if Ana must get more books than Ben?

(A) 150 **(B)** 180 **(C)** 210 **(D)** 240 **(E)** 270

11. Consider the 121 numbers of the form $2^a \cdot 3^b$, where a and b are integers from 0 to 10 inclusive. How many of these numbers are exactly one less than a multiple of 5?

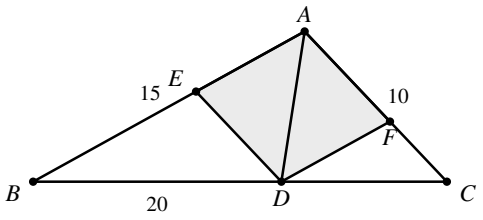
(A) 24 **(B)** 27 **(C)** 30 **(D)** 31 **(E)** 36

12. A circle has radius 5. Points B and C are fixed on the circle with $BC = 8$. A third point A may be placed anywhere on the circle other than B and C, forming triangle ABC. What is the largest possible perimeter of triangle ABC?

(A) $8 + 4\sqrt{5}$ **(B)** $8 + 4\sqrt{10}$ **(C)** 24 **(D)** $8 + 8\sqrt{5}$ **(E)** 28

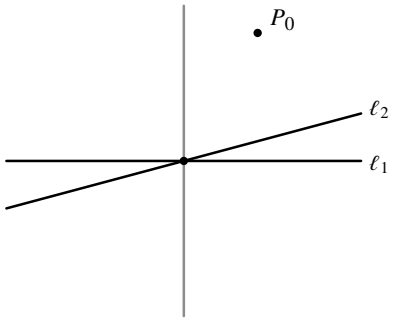
13. Triangle ABC has $AB = 15$, $AC = 10$ and $BC = 20$. The bisector of angle A meets side $\overline{BC}$ at D. The line through D parallel to $\overline{AC}$ meets side $\overline{AB}$ at E, and the line through D parallel to $\overline{AB}$ meets side $\overline{AC}$ at F. What is the area of quadrilateral $AEDF$?

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(A) $9\sqrt{15}$ **(B)** 36 **(C)** $12\sqrt{15}$ **(D)** $18\sqrt{15}$ **(E)** $\frac{75\sqrt{15}}{4}$

14. Let ℓ_1 be the x-axis and let ℓ_2 be the line $y = (2 - \sqrt{3})x$. Start with the point $P_0 = (1, \sqrt{3})$. For each $n \geq 1$, the point P_n is the reflection of P_{n-1} across ℓ_1 when n is odd, and across ℓ_2 when n is even. What is P_{2026}?



(A) $(2, 0)$ **(B)** $(-2, 0)$ **(C)** $(0, -2)$ **(D)** $(-\sqrt{3}, 1)$ **(E)** $(-\sqrt{3}, -1)$

15. Angles α and β satisfy $0 < \beta < \alpha < \frac{\pi}{2}$ and $\tan\alpha = 3\tan\beta$. What is the largest possible value of $\alpha - \beta$?

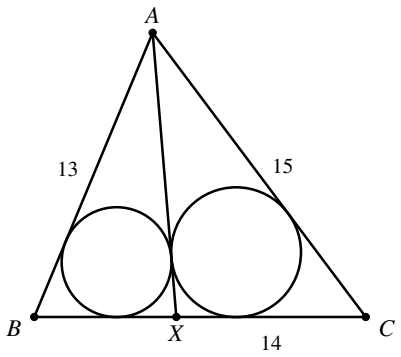
(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{5}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{\pi}{3}$ **(E)** $\frac{5\pi}{12}$

16. How many ordered pairs (A, B) of subsets of $\{1, 2, 3, \ldots, 10\}$ satisfy $A \subseteq B$ and have the property that B contains an even number of elements that are not in A? (Zero counts as even, and A or B may be empty.)

(A) 1024 **(B)** 19683 **(C)** 29523 **(D)** 29524 **(E)** 29525

17. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. A point X on side $\overline{BC}$ is chosen so that the incircle of triangle ABX and the incircle of triangle ACX touch segment $\overline{AX}$ at the same point. What is AX?

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(A) 12 **(B)** $\sqrt{145}$ **(C)** $2\sqrt{37}$ **(D)** $3\sqrt{17}$ **(E)** 13

18. For a certain real number k, the three roots of $z^3 - 6z^2 + kz - 10 = 0$ are the vertices of an equilateral triangle in the complex plane. What is the area of this triangle?

(A) $\frac{\sqrt{3}}{4}\sqrt[3]{4}$ **(B)** $\frac{3\sqrt{3}}{4}\sqrt[3]{2}$ **(C)** $\frac{3\sqrt{3}}{4}\sqrt[3]{4}$ **(D)** $\frac{3\sqrt{3}}{2}$ **(E)** $\frac{3\sqrt{3}}{2}\sqrt[3]{4}$

19. For each positive integer n, let $g(n) = \gcd\left(n! + 1,\ (n+1)!\right)$. What is $g(1) + g(2) + g(3) + \cdots + g(100)$? (The sum of the primes less than 100 is 1060.)

(A) 1060 **(B)** 1134 **(C)** 1161 **(D)** 1234 **(E)** 1235

20. An *inversion* in a list of numbers is a pair of positions $i < j$ whose entries appear in the wrong order, that is, the entry in position i is larger than the entry in position j. How many of the $8! = 40320$ orderings of $1, 2, 3, \ldots, 8$ have at most 3 inversions?

(A) 111 **(B)** 112 **(C)** 120 **(D)** 174 **(E)** 285

21. A sequence is defined by $x_1 = \sqrt{3}$ and $x_{n+1} = x_n + \sqrt{x_n^2 + 1}$ for $n \geq 1$. What is the greatest integer less than or equal to x_{10}?

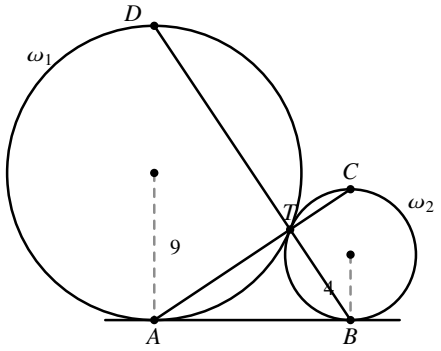
(A) 488 **(B)** 976 **(C)** 977 **(D)** 978 **(E)** 1955

22. Four married couples (eight people) hold a gift exchange. The eight names are put in a hat and each person draws one name at random, without replacement, so all 8! assignments are equally likely. What is the probability that nobody draws their own name and nobody draws the name of their own spouse?

(A) $\frac{11}{105}$ **(B)** $\frac{4457}{40320}$ **(C)** $\frac{37}{315}$ **(D)** $\frac{33}{280}$ **(E)** $\frac{2119}{5760}$

23. Circles ω_1 and ω_2, of radii 9 and 4, are externally tangent at the point T. A line tangent to both circles touches ω_1 at A and ω_2 at B (so it is a common tangent that does not pass through T). The line AT meets ω_2 again at C, and the line BT meets ω_1 again at D. What is the area of triangle TCD?

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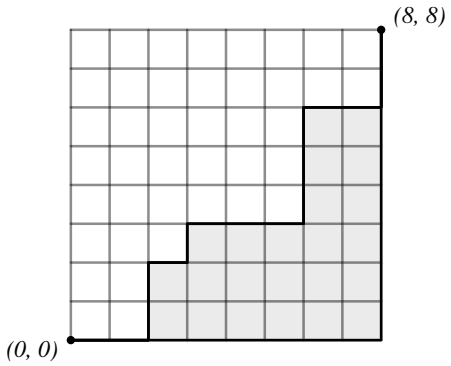


(A) $\frac{144}{13}$ **(B)** $\frac{432}{13}$ **(C)** 36 **(D)** 48 **(E)** 78

24. For how many integers n with $1 \leq n \leq 1000$ is $n!$ divisible by n^4? (You may use that there are 168 primes less than 1000, 95 primes less than 500, and 67 primes less than 334.)

(A) 665 **(B)** 668 **(C)** 671 **(D)** 734 **(E)** 735

25. A lattice path from $(0, 0)$ to $(8, 8)$ consists of 16 unit steps, each one unit right or one unit up. For how many of the $\binom{16}{8} = 12870$ such paths is the area of the region under the path, above the x-axis and to the left of the line $x = 8$, an even number?



(A) 6365 **(B)** 6400 **(C)** 6435 **(D)** 6470 **(E)** 6505

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Answer Key — AMC 12 Practice Contest 8

1. (B) $\frac{3}{4}$

2. (C) 16

3. (E) $\frac{5\pi}{2}$

4. (D) $2 - \sqrt{2}$

5. (B) 79

6. (E) $\frac{48}{125}$

7. (D) 23

8. (A) 1012

9. (A) $\frac{7\sqrt{2}}{2}$

10. (C) 210

11. (C) 30

12. (D) $8 + 8\sqrt{5}$

13. (A) $9\sqrt{15}$

14. (E) $(-\sqrt{3}, -1)$

15. (A) $\frac{\pi}{6}$

16. (E) 29525

17. (B) $\sqrt{145}$

18. (C) $\frac{3\sqrt{3}}{4}\sqrt[3]{4}$

19. (E) 1235

20. (A) 111

21. (C) 977

22. (D) $\frac{33}{280}$

23. (B) $\frac{432}{13}$

24. (B) 668

25. (D) 6470

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Solutions — AMC 12 Practice Contest 8

1. Answer: (B) $\frac{3}{4}$. The idea. The largest value $3\sin x + 4\cos x$ can ever take is $\sqrt{3^2+4^2} = 5$. So the equation says the expression is at its maximum, and the maximum is reached at only one direction of $(\cos x, \sin x)$.

Work it out. Think of $3\sin x + 4\cos x$ as the dot product of $(4,3)$ with the unit vector $(\cos x, \sin x)$. A dot product with a unit vector is at most the length of the other vector, $\sqrt{16+9} = 5$, with equality exactly when the unit vector points the same way as $(4,3)$: $(\cos x, \sin x) = \left(\frac{4}{5}, \frac{3}{5}\right)$. Hence $\tan x = \dfrac{3/5}{4/5} = \dfrac{3}{4}$.

Check. Algebraically: $3\sin x = 5 - 4\cos x$; squaring, $9 - 9\cos^2 x = 25 - 40\cos x + 16\cos^2 x$, so $25\cos^2 x - 40\cos x + 16 = 0$, i.e. $(5\cos x - 4)^2 = 0$. So $\cos x = \frac{4}{5}$ and $\sin x = \frac{5-16/5}{3} = \frac{3}{5}$, giving $\tan x = \frac{3}{4}$ again. (The trap $\frac{4}{3}$ swaps the roles of sine and cosine.)

2. Answer: (C) 16. The idea. Everything is a logarithm of x in base 2 or base 4, and $\log_4 x = \frac{1}{2}\log_2 x$. So one unknown, $t = \log_2 x$, controls the whole equation.

Work it out. With $t = \log_2 x$ we have $\log_4 x = t/2$, and $\log_4 t = \frac{1}{2}\log_2 t$. The equation becomes $\log_2\left(\frac{t}{2}\right) + \frac{1}{2}\log_2 t = 2$, that is $\log_2 t - 1 + \frac{1}{2}\log_2 t = 2$, so $\frac{3}{2}\log_2 t = 3$ and $\log_2 t = 2$. Hence $t = 4$ and $x = 2^4 = 16$.

Check. $\log_4 16 = 2$ and $\log_2 2 = 1$; $\log_2 16 = 4$ and $\log_4 4 = 1$. The sum is $1 + 1 = 2$. The answer 256 comes from stopping at $\log_2 x = 8$ after mixing up the two bases, and 4 from reporting t instead of x.

3. Answer: (E) $\frac{5\pi}{2}$. The idea. Write $\cos 2x$ in terms of $\sin x$ alone, using $\cos 2x = 1 - 2\sin^2 x$. The equation becomes a quadratic in $s = \sin x$; then read off every angle in $[0, 2\pi)$ with each sine value.

Work it out. $\sin x = 1 - 2\sin^2 x$ gives $2s^2 + s - 1 = 0$, i.e. $(2s-1)(s+1) = 0$.

$s = \frac{1}{2}$: $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

$s = -1$: only $x = \frac{3\pi}{2}$ (sine reaches -1 just once per turn).

The sum is $\frac{\pi}{6} + \frac{5\pi}{6} + \frac{3\pi}{2} = \pi + \frac{3\pi}{2} = \frac{5\pi}{2}$.

Check. At $x = \frac{3\pi}{2}$: $\sin x = -1$ and $\cos 3\pi = -1$. At $x = \frac{\pi}{6}$: $\sin x = \frac{1}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$. Keeping only the root $s = \frac{1}{2}$ gives the trap π; forgetting the second angle $\frac{5\pi}{6}$ gives $\frac{\pi}{6} + \frac{3\pi}{2} = \frac{5\pi}{3}$.

4. Answer: (D) $2 - \sqrt{2}$. The idea. A circle tangent to the left and bottom sides has its centre at the same distance r from both, so the centre is (r, r), on the diagonal from $(0,0)$ to $(1,1)$. The other centre is $(1-s, 1-s)$, on the same diagonal. Tangency then becomes one equation along that diagonal, and it only involves $r + s$.

Work it out. Place the square with corners $(0,0)$ and $(1,1)$. The distance between the centres (r,r) and $(1-s,1-s)$ is $\sqrt{2}\,(1-r-s)$. External tangency means this distance is $r+s$. Writing $S = r + s$: $\sqrt{2}(1-S) = S$, so $S = \dfrac{\sqrt{2}}{1+\sqrt{2}} = \sqrt{2}(\sqrt{2}-1) = 2 - \sqrt{2}$.

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Check. Along the diagonal of length $\sqrt{2}$, the pieces are: corner to first centre $r\sqrt{2}$, then r, then s, then $s\sqrt{2}$ to the far corner. So $(\sqrt{2}+1)(r+s) = \sqrt{2}$, the same equation. For example $r = s = 1-\frac{\sqrt{2}}{2}$ gives two equal circles that fit, and their radii add to $2 - \sqrt{2} \approx 0.586$.

5. Answer: (B) 79**. The idea.** Split the range into the intervals $[n, n+1)$. On each one $\lfloor x \rfloor = n$ is fixed, so the equation is linear in the fractional part and has at most one solution; it has one exactly when that fractional part is really less than 1.

Work it out. Write $x = n + f$ with $n = \lfloor x \rfloor$ an integer from 0 to 99 and $0 \le f < 1$. The equation is $nf = 20$. For $n = 0$ there is no solution. For $n \ge 1$ we need $f = \frac{20}{n}$, and this is a valid fractional part exactly when $\frac{20}{n} < 1$, i.e. $n \ge 21$. So each $n = 21, 22, \ldots, 99$ gives exactly one solution $x = n + \frac{20}{n}$, and these all lie below 100. That is $99 - 21 + 1 = 79$ solutions.

Check. $n = 20$ would need $f = 1$, which is not allowed ($x = 21$ has $\lfloor x \rfloor = 21$ and $\{x\} = 0$); counting it gives the trap 80. The largest solution is $99 + \frac{20}{99} < 100$, and the smallest is $21 + \frac{20}{21}$, with $21 \cdot \frac{20}{21} = 20$.

6. Answer: (E) $\frac{48}{125}$**. The idea.** If exactly one box is empty, five balls fill the other four boxes with none empty, so exactly one box holds two balls and the other three hold one each. Build such an outcome step by step.

Work it out. There are $5^5 = 3125$ equally likely outcomes. For a good outcome: choose the empty box (5 ways); choose the box holding two balls from the remaining four (4 ways); choose which two balls go into it ($\binom{5}{2} = 10$ ways); place the other three balls, one each, in the remaining three boxes ($3! = 6$ ways). That is $5 \cdot 4 \cdot 10 \cdot 6 = 1200$ outcomes, and the probability is $\dfrac{1200}{3125} = \dfrac{48}{125}$.

Check. Count by the number of occupied boxes: all five occupied in $5! = 120$ ways, exactly four occupied in 1200 ways, so at least one box is empty in $3125 - 120 = 3005$ ways, which is more than 1200, as it should be, since two or more empty boxes is also possible. (Forgetting to choose the doubled box gives $\frac{12}{125}$; forgetting to choose which box is empty gives $\frac{48}{625}$.)

7. Answer: (D) 23**. The idea.** Add 4 to both sides to complete the product: $n+4 = ab+2a+2b+4 = (a+2)(b+2)$. Since $a, b \ge 1$, the two factors are at least 3. So n can be written in the form exactly when $m = n+4$ is a product of two integers that are both at least 3.

Work it out. As n runs from 1 to 50, $m = n+4$ runs from 5 to 54. A number m fails exactly when every factorisation $m = xy$ uses a factor 1 or 2. That happens for:

primes m: 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, that is 14 numbers;

twice a prime, $m = 2p$ with $p \ge 3$ (its only factorisations are $1 \cdot 2p$ and $2 \cdot p$): 6, 10, 14, 22, 26, 34, 38, 46, that is 8 numbers;

$m = 8$, whose factorisations are $1 \cdot 8$ and $2 \cdot 4$: 1 number.

Every other m works: if m has an odd prime factor p and $m \ne p, 2p$, then $m = p \cdot \frac{m}{p}$ with $\frac{m}{p} \ge 3$; otherwise m is a power of 2 that is at least 16, and $m = 4 \cdot \frac{m}{4}$. So $14 + 8 + 1 = 23$ values of n fail.

Check. The failures are $n = m - 4$: $n = 1, 2, 3, 4$ (from $m = 5, 6, 7, 8$), and indeed the smallest value of $ab + 2a + 2b$ is 5. On the other side, $n = 11$ gives $m = 15 = 3 \cdot 5$, matching $a = 1$, $b = 3$, and $n = 12$ gives $m = 16 = 4 \cdot 4$, so $a = b = 2$ works: $4+4+4 = 12$. Excluding only primes gives 14; forgetting $m = 8$ gives 22; insisting that $a \ne b$ would also lose $m = 9, 16, 25, 49$ and give 27.

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8. Answer: (A) 1012**. The idea.** The sum of the first n cubes is the square of a triangular number: $\left(\frac{n(n+1)}{2}\right)^2$. Because 2 and 1013 are primes, a prime divides a square exactly when it divides the number being squared. So we need both 2 and 1013 to divide $T = \frac{n(n+1)}{2}$.

Work it out. $1013 \mid T$: since 1013 is an odd prime, this means $1013 \mid n(n+1)$, so $n \equiv 0$ or $n \equiv -1 \pmod{1013}$. The smallest candidates are $n = 1012, 1013, 2025, 2026, \ldots$

$2 \mid T$: this means $4 \mid n(n+1)$, which happens exactly when $n \equiv 0$ or $3 \pmod 4$.

Now test in order: $1012 = 4 \cdot 253$ is a multiple of 4, so $n = 1012$ works.

Check. $T = \frac{1012 \cdot 1013}{2} = 506 \cdot 1013$, which is even and a multiple of 1013, so T^2 is divisible by $2 \cdot 1013 = 2026$. The next candidate 1013 fails: $T = 1013 \cdot 507$ is odd, so T^2 is odd. No $n < 1012$ can work, since 1013 must divide n or $n + 1$.

9. Answer: (A) $\frac{7\sqrt{2}}{2}$**. The idea.** The diagonals of a square are perpendicular, so $\angle AOB = 90°$, just like $\angle ACB$. Two right angles on the same segment AB, seen from opposite sides, put A, C, B, O on one circle (the circle with diameter AB). Ptolemy's theorem for the cyclic quadrilateral $ACBO$ then links CO to known lengths.

Work it out. $AB = 5$, and $OA = OB = \frac{5}{\sqrt{2}}$ (half a diagonal of a square of side 5). The quadrilateral $ACBO$ is cyclic with diagonals AB and CO, so Ptolemy gives

$AB \cdot CO = AC \cdot BO + CB \cdot AO = (3 + 4) \cdot \frac{5}{\sqrt{2}}$.

So $5\,CO = \frac{35}{\sqrt{2}}$ and $CO = \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$.

Check. Coordinates: $C = (0, 0)$, $A = (3, 0)$, $B = (0, 4)$. The midpoint of AB is $\left(\frac{3}{2}, 2\right)$ and half of $\overrightarrow{AB}$ is $\left(-\frac{3}{2}, 2\right)$; turning it by 90° away from C gives $(2, \frac{3}{2})$, so $O = \left(\frac{7}{2}, \frac{7}{2}\right)$ and $CO = \frac{7}{2}\sqrt{2}$. Note O lies on the bisector of the right angle at C, as the equal chords $OA = OB$ predict.

10. Answer: (C) 210**. The idea.** Count all handouts in which nobody is left empty, then use the symmetry between Ana and Ben: swapping their piles matches the handouts where Ana has more with those where Ben has more. Only the ties, where they have the same number, break the symmetry, so count those separately.

Work it out. By inclusion–exclusion, the handouts that give everyone at least one book number $3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 729 - 192 + 3 = 540$. Ties: Ana and Ben each get one book and Cy four, in $6 \cdot 5 = 30$ ways, or all three get two books, in $\binom{6}{2}\binom{4}{2} = 90$ ways; that is 120 ties. The rest split evenly, so Ana gets more in $\frac{540-120}{2} = 210$ handouts.

Check. Directly by the numbers of books (Ana, Ben, Cy): $(2, 1, 3)$ gives $\frac{6!}{2!\,1!\,3!} = 60$, $(3, 1, 2)$ gives 60, $(4, 1, 1)$ gives 30 and $(3, 2, 1)$ gives 60, for a total of 210. The choice $270 = \frac{540}{2}$ forgets the ties.

11. Answer: (C) 30**. The idea.** Modulo 5, every nonzero residue is a power of 2: the powers $2^0, 2^1, 2^2, 2^3$ are $1, 2, 4, 3$, repeating every 4 steps. In particular $3 \equiv 2^3$, so $2^a 3^b \equiv 2^{a+3b} \pmod 5$, and the condition becomes a condition on $a + 3b$ modulo 4.

Work it out. We need $2^a 3^b \equiv -1 \equiv 4 = 2^2 \pmod 5$, that is, $a + 3b \equiv 2 \pmod 4$, or $a \equiv 2 + b \pmod 4$ (since $-3 \equiv 1$). Among $0, 1, \ldots, 10$ there are 3 numbers in each of the classes $0, 1, 2 \pmod 4$ and 2 in the class 3. So: $b \equiv 0$ (3 values) needs $a \equiv 2$ (3 values): 9 pairs; $b \equiv 1$ (3)

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needs $a \equiv 3$ (2): 6; $b \equiv 2$ (3) needs $a \equiv 0$ (3): 9; $b \equiv 3$ (2) needs $a \equiv 1$ (3): 6. The total is $9 + 6 + 9 + 6 = 30$.

Check. A few cases: $2^2 = 4$ and $3^2 = 9$ work ($a \equiv 2 + b$ holds for $(2, 0)$ and $(0, 2)$), and so does $2^3 \cdot 3 = 24$ from $(3, 1)$. A uniform guess of one quarter of 121 is about 30.25, close but not a proof; the classes are uneven, and the choice 31 or 27 comes from miscounting them.

12. Answer: (D) $8 + 8\sqrt{5}$**. The idea.** With BC fixed, the angle at A is fixed on each of the two arcs (inscribed angle theorem). By the law of sines, $AB + AC = 2R(\sin C + \sin B)$, and with $B + C$ fixed this sum of sines is largest when $B = C$, that is, when A is the midpoint of an arc. The longer arc gives the larger answer.

Work it out. The chord $BC = 8$ in a circle of radius 5 is at distance $\sqrt{25 - 16} = 3$ from the centre, and $\sin A = \frac{BC}{2R} = \frac{4}{5}$. On the major arc, A is acute with $\cos A = \frac{3}{5}$. Then

$$AB + AC = 10(\sin B + \sin C) = 20 \sin \tfrac{B+C}{2} \cos \tfrac{B-C}{2} \le 20 \cos \tfrac{A}{2},$$

with equality when $B = C$. Since $\cos^2 \frac{A}{2} = \frac{1+3/5}{2} = \frac{4}{5}$, the maximum of $AB + AC$ is $20 \cdot \frac{2}{\sqrt{5}} = 8\sqrt{5}$. On the minor arc, $\cos A = -\frac{3}{5}$ gives only $20\sqrt{1/5} = 4\sqrt{5}$. So the largest perimeter is $8 + 8\sqrt{5}$.

Check. Directly: the midpoint of the major arc is $5 + 3 = 8$ above line BC, over the midpoint of BC, so $AB = AC = \sqrt{8^2 + 4^2} = 4\sqrt{5}$, perimeter $8 + 8\sqrt{5} \approx 25.9$. This is less than 28 (two diameters, impossible since A cannot be diametrically opposite both B and C), and the minor-arc midpoint gives $AB = AC = \sqrt{2^2 + 4^2} = 2\sqrt{5}$, perimeter $8 + 4\sqrt{5}$.

13. Answer: (A) $9\sqrt{15}$**. The idea.** $AEDF$ is a parallelogram by construction, and its diagonal AD bisects the angle at A. A parallelogram whose diagonal bisects an angle is a rhombus: $\angle ADF = \angle DAE = \angle DAF$, so triangle ADF is isosceles with $FA = FD$. So we only need its side t and the angle A.

Work it out. Triangle CFD is similar to triangle CAB (since $FD \parallel AB$). The angle-bisector theorem gives $\frac{CD}{CB} = \frac{AC}{AB+AC} = \frac{10}{25}$, so $t = FD = \frac{10}{25} \cdot 15 = 6$. (In general $t = \frac{bc}{b+c}$, and the length BC does not enter.) For the angle, the law of cosines gives $\cos A = \frac{15^2+10^2-20^2}{2 \cdot 15 \cdot 10} = -\frac{1}{4}$, so $\sin A = \frac{\sqrt{15}}{4}$. The area of the rhombus is $t^2 \sin A = 36 \cdot \frac{\sqrt{15}}{4} = 9\sqrt{15}$.

Check. Compare with the whole triangle, of area $\frac{1}{2} \cdot 15 \cdot 10 \cdot \frac{\sqrt{15}}{4} = \frac{75\sqrt{15}}{4}$. The rhombus is twice triangle AEF, and $[AEF] = \frac{AE}{AB} \cdot \frac{AF}{AC} \cdot [ABC]$ because it shares angle A with triangle ABC. That gives $2 \cdot \frac{6}{15} \cdot \frac{6}{10} \cdot \frac{75\sqrt{15}}{4} = 9\sqrt{15}$ again. Forgetting the angle (area $t^2 = 36$) is the usual slip.

14. Answer: (E) $(-\sqrt{3}, -1)$**. The idea.** Two reflections in a row, across lines through the origin, make a rotation about the origin. In polar terms, reflecting across the line at angle θ sends the point at angle φ to the point at angle $2\theta - \varphi$, at the same distance from the origin. Since $\tan 15^\circ = 2 - \sqrt{3}$, the line ℓ_2 is at angle 15° and ℓ_1 is at angle 0°.

Work it out. Reflecting across ℓ_1 sends φ to $-\varphi$, and then reflecting across ℓ_2 sends $-\varphi$ to $30^\circ - (-\varphi) = \varphi + 30^\circ$. So each pair of steps (odd step, then even step) rotates the point 30° counterclockwise, and P_{2k} is P_0 rotated by $30k$ degrees. The point $P_0 = (1, \sqrt{3})$ is at distance 2 and angle 60°. Since $2026 = 2 \cdot 1013$, P_{2026} is at angle $60^\circ + 1013 \cdot 30^\circ = 60^\circ + 30390^\circ$, and $30390 = 84 \cdot 360 + 150$, so the angle is $60^\circ + 150^\circ = 210^\circ$. Hence $P_{2026} = (2\cos 210^\circ, 2 \sin 210^\circ) = (-\sqrt{3}, -1)$.

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Check. By hand: $P_1 = (1, -\sqrt{3})$ at angle $-60°$, and its mirror image across the $15°$ line is at angle $30° + 60° = 90°$, so $P_2 = (0, 2)$, which is P_0 turned by $30°$. One reflection too many gives $P_{2027} = (-\sqrt{3}, 1)$; turning the wrong way gives $(0, -2)$; $(-2, 0)$ is P_{2024}; and turning $30°$ at every single step gives $(2, 0)$.

15. Answer: (A) $\frac{\pi}{6}$. The idea. Since $\alpha - \beta$ lies in $(0, \frac{\pi}{2})$, where tangent is increasing, maximising $\alpha - \beta$ is the same as maximising $\tan(\alpha - \beta)$. That becomes a one-variable expression in $t = \tan\beta$, and AM-GM bounds it.

Work it out. With $t = \tan\beta > 0$ and $\tan\alpha = 3t$, $\tan(\alpha - \beta) = \frac{3t - t}{1 + 3t \cdot t} = \frac{2t}{1 + 3t^2}$. By AM-GM, $1 + 3t^2 \geq 2\sqrt{3}\, t$, so $\tan(\alpha - \beta) \leq \frac{2t}{2\sqrt{3}\, t} = \frac{1}{\sqrt{3}}$. Equality holds when $3t^2 = 1$, that is $t = \frac{1}{\sqrt{3}}$: $\beta = \frac{\pi}{6}$ and $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$. The maximum of $\alpha - \beta$ is $\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

Check. Other values fall short. For example $\beta = \frac{\pi}{4}$ gives $\tan\alpha = 3$, and $\alpha - \beta = \arctan 3 - \frac{\pi}{4} = \arctan\frac{3-1}{1+3} = \arctan\frac{1}{2} < \arctan\frac{1}{\sqrt{3}}$.

16. Answer: (E) 29525. The idea. A pair $A \subseteq B$ is the same as giving each element one of three labels: in A, in B but not A, or outside B. That makes 3^{10} pairs in all. We want the pairs where the middle label is used an even number of times. Weight the middle label by x and use the standard even-part trick.

Work it out. The generating function is $(2+x)^{10}$: each element contributes 1 (in A), x (in $B \setminus A$) or 1 (outside B). The sum of the even-power coefficients is $\frac{f(1)+f(-1)}{2} = \frac{3^{10}+1^{10}}{2} = \frac{59049+1}{2} = 29525$.

Check. Try one element: the pairs are $(\varnothing, \varnothing)$, $(\{1\}, \{1\})$ and $(\varnothing, \{1\})$, and the first two have an even gap, which matches $\frac{3+1}{2} = 2$. The odd case gets the remaining 29524, and exactly half of the 59049 is impossible because 59049 is odd. Counting only the pairs with $A = B$ gives $2^{10} = 1024$.

17. Answer: (B) $\sqrt{145}$. The idea. The incircle of a triangle touches a side at a distance from a vertex equal to the semiperimeter minus the opposite side. Writing that distance from A along AX in both small triangles, and setting the two equal, turns the tangency condition into a single linear equation for BX.

Work it out. In triangle ABX, the incircle touches AX at distance $\frac{AB+AX-BX}{2}$ from A. In triangle ACX the distance is $\frac{AC+AX-CX}{2}$. Setting them equal gives $BX - CX = AB - AC = -2$, and with $BX + CX = 14$ we get $BX = 6$, $CX = 8$. (So X is exactly where the incircle of ABC touches BC.) Place $B = (0, 0)$ and $C = (14, 0)$. Then $A = (5, 12)$, since $5^2 + 12^2 = 13^2$ and $9^2 + 12^2 = 15^2$. With $X = (6, 0)$, $AX^2 = 1^2 + 12^2 = 145$, so $AX = \sqrt{145}$.

Check. Stewart's theorem on cevian AX with $BX = 6$, $XC = 8$: $13^2 \cdot 8 + 15^2 \cdot 6 = 14(AX^2 + 48)$, so $1352 + 1350 = 2702 = 14AX^2 + 672$ and $AX^2 = 145$. Swapping the roles ($BX = 8$) would give $\sqrt{153}$, and the midpoint would give the median $2\sqrt{37}$.

18. Answer: (C) $\frac{3\sqrt{3}}{4}\sqrt[3]{4}$. The idea. The vertices of an equilateral triangle with centre c are $c+\rho$, $c+\rho\omega$, $c+\rho\omega^2$ for some complex ρ, where $\omega = e^{2\pi i/3}$. Those are exactly the roots of $(z-c)^3 = \rho^3$. So the cubic must have the form $(z-c)^3 - d$, and matching coefficients pins down everything.

Work it out. The centre is the average of the roots, $c = \frac{6}{3} = 2$. Now $(z-2)^3 - d = z^3 - 6z^2 + 12z - 8 - d$. Matching the constant term gives $-8 - d = -10$, so $d = 2$ (and $k = 12$). The roots are $2 + \sqrt[3]{2}\,\omega^j$, so the circumradius is $R = \sqrt[3]{2}$. An equilateral triangle with circumradius R has side $R\sqrt{3}$ and area $\frac{\sqrt{3}}{4} \cdot 3R^2 = \frac{3\sqrt{3}}{4}R^2 = \frac{3\sqrt{3}}{4}\sqrt[3]{4}$.

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Check. A triangle $z_1z_2z_3$ is equilateral exactly when $z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$, i.e. $e_1^2 = 3e_2$. That gives $36 = 3k$, so $k = 12$ as found. The roots of $z^3 - 6z^2 + 12z - 10$ are $2 + \sqrt[3]{2} \approx 3.26$ and $2 - \frac{\sqrt[3]{2}}{2} \pm \frac{\sqrt{3}\sqrt[3]{2}}{2}i$, whose side length $\sqrt{3}\sqrt[3]{2} \approx 2.18$ gives area ≈ 2.06.

19. Answer: (E) 1235**. The idea.** Any common divisor of $n! + 1$ and $n!$ divides 1, so $n! + 1$ shares no factor with $n!$. Since $(n+1)! = n!\,(n+1)$, the gcd reduces to $g(n) = \gcd(n! + 1,\ n + 1)$. Then Wilson's theorem decides everything.

Work it out. If $n + 1 = p$ is prime, Wilson's theorem says $(p - 1)! \equiv -1 \pmod{p}$, so p divides $n! + 1$ and $g(n) = n + 1 = p$. If $n + 1$ is composite, every prime q dividing $n + 1$ satisfies $q \le n$, so $q \mid n!$ and $q \nmid n! + 1$. Hence $g(n) = 1$. As n runs from 1 to 100, $n + 1$ runs over $2, 3, \ldots, 101$. That range holds 26 primes (the 25 primes below 100, plus 101), with sum $1060 + 101 = 1161$. It also holds $100 - 26 = 74$ composites, each contributing 1. The total is $1161 + 74 = 1235$.

Check. Small cases: $g(1) = \gcd(2, 2) = 2$, $g(2) = \gcd(3, 6) = 3$, $g(3) = \gcd(7, 24) = 1$, $g(4) = \gcd(25, 120) = 5$. These match $n + 1 = 2, 3$ (prime), 4 (composite) and 5 (prime). Forgetting the prime 101 gives $1060 + 74 = 1134$, and an off-by-one in the count of composites gives 1234.

20. Answer: (A) 111**. The idea.** Build the ordering by inserting $1, 2, \ldots, 8$ one at a time. When k is inserted into the list of $1, \ldots, k - 1$, the number of smaller entries to its right can be anything from 0 to $k - 1$, and those are exactly the new inversions it creates. So an ordering corresponds to a sequence $(c_1, \ldots, c_8)$ with $0 \le c_k \le k - 1$, and the number of inversions is $c_1 + \cdots + c_8$. We count such sequences with sum at most 3.

Work it out. $c_1 = 0$ is forced. For a sum at most 3, the only constraint that ever bites is $c_2 \le 1$ ($c_3 \le 2$ matters only if $c_3 = 3$, and $c_k \le k - 1$ is automatic for $k \ge 4$). Seven free variables $c_2, \ldots, c_8$:

Sum 0: 1 way.

Sum 1: 7 ways.

Sum 2: $\binom{8}{2} = 28$ compositions, minus the one with $c_2 = 2$, giving 27.

Sum 3: $\binom{9}{3} = 84$, minus those with $c_2 \ge 2$ ($c_2 = 2$ with one more unit elsewhere: 6; $c_2 = 3$: 1) and the one with $c_3 = 3$: $84 - 7 - 1 = 76$.

Total: $1 + 7 + 27 + 76 = 111$.

Check. These are the coefficients of $\prod_{k=1}^{8}(1 + x + \cdots + x^{k-1})$. Its x^1 coefficient 7 counts the 7 adjacent swaps of the sorted list, as it should. Forgetting the cap $c_3 \le 2$ gives 112, forgetting both caps gives $1 + 7 + 28 + 84 = 120$, and 174 is the number of orderings with exactly 4 inversions.

21. Answer: (C) 977**. The idea.** Expressions $x + \sqrt{x^2 + 1}$ appear naturally for cotangents: if $x = \cot\theta$ with $0 < \theta < \frac{\pi}{2}$, then $\sqrt{x^2 + 1} = \csc\theta$ and $\cot\theta + \csc\theta = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$. So each step halves the angle.

Work it out. $x_1 = \sqrt{3} = \cot\frac{\pi}{6}$. After nine halvings, $x_{10} = \cot\frac{\pi}{6\cdot 2^9} = \cot\frac{\pi}{3072}$. For small $t > 0$, $\frac{1}{t} - \frac{t}{3} < \cot t < \frac{1}{t}$. With $t = \frac{\pi}{3072}$, $\frac{1}{t} = \frac{3072}{\pi} \approx 977.848$ and $\frac{t}{3} \approx 0.0003$. So $977.84 < x_{10} < 977.85$, and $\lfloor x_{10} \rfloor = 977$.

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Check. $x_2 = \sqrt{3}+2 = \cot 15^\circ$ (indeed $\tan 15^\circ = 2-\sqrt{3}$), confirming one halving per step. The value $\frac{3072}{\pi}$ is not close to an integer, so the crude bound $\frac{1}{t}$ already settles the floor. Miscounting the halvings by one gives ≈ 488.9 or ≈ 1955.7.

22. Answer: (D) $\frac{33}{280}$**. The idea.** Use inclusion–exclusion on the 16 forbidden events "person X draws name Y", where Y is X or X's spouse. We need, for each k, the number r_k of ways to choose k forbidden events that can happen together, that is, no giver and no name used twice. Forbidden events of different couples never clash. So r_k is the x^k coefficient of $q(x)^4$, where $q(x)$ counts choices inside one couple.

Work it out. Inside a couple $\{H, W\}$ the forbidden draws are $H{\to}H$, $H{\to}W$, $W{\to}W$, $W{\to}H$. There is 1 way to choose none, 4 ways to choose one, and 2 compatible ways to choose two: both draw themselves, or they draw each other. So $q(x) = 1+4x+2x^2$ and $q(x)^4 = 1+16x+104x^2+352x^3+664x^4+704x^5+416x^6+128x^7+16x^8$. Each chosen set of k events leaves $(8-k)!$ ways to finish, so the number of good assignments is

$$\textstyle\sum_k (-1)^k r_k (8-k)! = 40320 - 16\cdot 5040 + 104\cdot 720 - 352\cdot 120 + 664\cdot 24 - 704\cdot 6 + 416\cdot 2 - 128 + 16$$

$$= 40320 - 80640 + 74880 - 42240 + 15936 - 4224 + 832 - 128 + 16 = 4752.$$

The probability is $\frac{4752}{40320} = \frac{33}{280}$.

Check. Try one couple: $q(x) = 1+4x+2x^2$ gives $2! - 4\cdot 1! + 2\cdot 0! = 0$, correct, since two people cannot avoid both names. Two couples: $q^2 = 1+8x+20x^2+16x^3+4x^4$ gives $24-48+40-16+4 = 4$. Directly, each person of couple 1 must draw from couple 2 and vice versa, which is $2\cdot 2 = 4$ ways. Dropping the $2x^2$ term gives $\frac{11}{105}$, keeping only one of the two compatible pairs ($q = 1+4x+x^2$) gives $\frac{4457}{40320}$, losing the final +16 gives $\frac{37}{315}$, and ignoring spouses gives the derangement probability $\frac{14833}{40320}$.

23. Answer: (B) $\frac{432}{13}$**. The idea.** The tangent line to both circles at T meets $\overline{AB}$ at a point equally far from A, B and T (equal tangent lengths), so $\angle ATB = 90^\circ$. That right angle makes BC a diameter of ω_2 and AD a diameter of ω_1, both perpendicular to AB. So $ABCD$ is a trapezoid whose diagonals cross at T, and the area splits in a known way.

Work it out. The common tangent has length $AB = \sqrt{(9+4)^2 - (9-4)^2} = 12$. Since $\angle ATB = 90^\circ$, also $\angle BTC = 90^\circ$ (as C is on line AT), so $\overline{BC}$ is a diameter of ω_2: $BC = 8$, perpendicular to AB (a radius to a point of tangency). Likewise $\angle ATD = 90^\circ$, so $AD = 18$ is a diameter of ω_1, also perpendicular to AB. Thus $ABCD$ is a trapezoid with parallel sides $AD = 18$ and $BC = 8$ and height 12, of area $\frac{18+8}{2}\cdot 12 = 156$, and its diagonals AC and BD meet at T. The diagonals cut each other in the ratio $AD : BC = 9 : 4$, so the four triangles at T have areas in the ratio $[TBC] : [TAB] : [TCD] : [TAD] = 16 : 36 : 36 : 81$. With $16+36+36+81 = 169$, we get $[TCD] = \frac{36}{169}\cdot 156 = \frac{432}{13}$.

Check. Coordinates: $A = (0,0)$, $B = (12,0)$, centres $(0,9)$ and $(12,4)$, and $T = (0,9) + \frac{9}{13}(12,-5) = \left(\frac{108}{13}, \frac{72}{13}\right)$, which lies on the line from A to $C = (12,8)$. With $D = (0,18)$ the shoelace formula gives $\frac{1}{2}\left|\frac{108}{13}(8-18) + 12\left(18 - \frac{72}{13}\right)\right| = \frac{1}{2}\cdot\frac{864}{13} = \frac{432}{13}$. The choice 78 is half the trapezoid, and 48 is triangle ABC.

24. Answer: (B) 668**. The idea.** Since $n! = n\cdot(n-1)!$, the question is whether n^3 divides $(n-1)!$. For each prime p dividing n, counting the multiples of p below n decides this (write $v_p(m)$ for the exponent of p in m). If p divides n only once, the multiples $p, 2p, \ldots$ below n must supply

three factors of p, and that fails only when n is p, $2p$ or $3p$. Primes dividing n more than once fail only for a few small n.

Work it out. Write $n = kp$ for a prime p. If $k \le 3$ (and $p \ge 3$, or $n \in \{2, 4\}$), then $n - 1 < p^2$, so the only multiples of p up to $n - 1$ are $p, \dots, (k-1)p$ and $v_p((n-1)!) = k - 1 \le 2 < 3$: such n fail. So every n of the form p, $2p$ or $3p$ fails. There are 168 numbers p, 95 numbers $2p \le 1000$ (including 4) and 67 numbers $3p \le 1000$ (including 6 and 9); only $6 = 2 \cdot 3 = 3 \cdot 2$ is counted twice, so these give $168 + 95 + 67 - 1 = 329$ failures. For any other n, a prime p dividing n exactly once has $n = kp$ with $k \ge 4$, so $p, 2p, 3p$ lie below n and supply the three factors of p. Only primes with $p^2 \mid n$ can still cause trouble.

So let $p^2 \mid n$. For $p = 2$: $n = 8$ has $v_2(7!) = 4 < 9$ and $n = 16$ has $v_2(15!) = 11 < 12$, so both fail, while $n = 12, 20, 24, 32, \dots$ all have plenty of factors of 2 (e.g. $v_2(11!) = 8 \ge 6$, $v_2(31!) = 26 \ge 15$). For $p = 3$: $n = 18$ has $v_3(17!) = 6 \ge 6$ and $n = 27$ has $v_3(26!) = 10 \ge 9$, fine. For $p = 5$: $n = 25$ has $v_5(24!) = 4 < 6$, a failure, but 50 and 75 are fine. For $p = 7$: $v_7(48!) = 6 \ge 6$, so 49 works, and for $p \ge 11$, $v_p((p^2-1)!) = p - 1 \ge 10$. So the extra failures are exactly 8, 16 and 25.

In all, $329 + 3 = 332$ values of n fail and $1000 - 332 = 668$ succeed.

Check. Small cases: $n = 1$ works, $2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ all fail (primes, $2p$, $3p$ and 8), and $n = 12$ is the first success after 1: $12^4 = 20736$ divides $12! = 479001600$, with quotient 23100. Forgetting the overlap at 6 gives 667; forgetting 8, 16, 25 gives 671; and 735 is the count for $n^3 \mid n!$.

25. Answer: (D) 6470**. The idea.** Each right step adds a column of width 1 whose height is the number of up steps taken before it, so the area is the sum of those heights. Now group the 16 steps into 8 consecutive pairs (steps 1 and 2, steps 3 and 4, and so on) and look for the first pair that is mixed, one right and one up. Swapping the two steps of that pair changes the area by exactly 1, and it pairs the paths off.

Work it out. Swapping inside the first mixed pair moves its right step from just before its up step to just after it, or back. That changes that right step's height by 1 and leaves every other height alone, so the area changes parity. Doing it twice returns the original path, and the first mixed pair stays the first mixed pair, so the paths with at least one mixed pair split into matched couples, one of even area and one of odd area.

The paths with no mixed pair are built from 8 pairs, each two rights or two ups, with 4 of each kind: $\binom{8}{4} = 70$ paths. In such a path the up steps come in pairs, so every right step has an even height and the area is even.

So the even paths number $\frac{12870-70}{2} + 70 = 6400 + 70 = 6470$.

Check. For paths from $(0, 0)$ to $(2, 2)$ the areas of the six paths are $0, 1, 2, 2, 3, 4$ (for RRUU, RURU, RUUR, URRU, URUR, UURR), so 4 are even. The same argument gives $\frac{6-2}{2} + \binom{2}{1} = 4$. Half of all paths would be 6435, and adding the 70 unmatched paths without halving the rest gives 6505.

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